Bifurcation Analysis of a Population Dynamics in a Critical State [1]

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Abstract: A stage-structured population model *in a critical state* is studied in this paper. By analyzing the corresponding characteristic equations, the local stability of the equilibria is discussed. Hopf bifurcations occurring at the positive equilibrium under some conditions are demonstrated. Taking time delay τ as bifurcating parameter, the direction and stability of Hopf bifurcation are carried out. The global continuation of periodic solutions bifurcating from the equilibrium is investigated. Finally, an example and numerical simulations are given.

Keywords: Bifurcation; Delay; Periodic solutions; Stability;

2000 Mathematics Subject Classification: 34K18; 34C23; 92D25

1 Introduction

1.1. Motivations. It is known that Hopf bifurcation of delayed differential systems has been widely studied (e.g. see [1] and references cited therein). In particular, many authors successfully applied Hopf bifurcation theorem to study the delayed population dynamical systems. For examples, one can refer to [2–28]. However, most of the works only considered the *standard (non-critical) cases*. To illustrate this clearly the characteristic equation should be mentioned. It is known that distribution of the roots of the characteristic equation plays important role in the bifurcation analysis. As for two dimensional system, one can consider the second degree transcendental polynomial equation

$$\lambda^2 + p\lambda + r + (s\lambda + q)e^{-\lambda\tau} = 0, \tag{1.1}$$

where p, q, r, s are real numbers. When $\tau = 0$, Eq. (1.1) becomes

$$\lambda^2 + (p + s)\lambda + (r + q) = 0. \tag{1.2}$$

Most of the above mentioned works studied the distribution of roots under the *standard assumption*

$$p + s > 0, \quad \text{and} \quad q + r > 0.$$

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In this situation, all roots of Eq. (1.2) have negative real parts. Thus, it follows from [30], that if transcendental equation (1.1) has no purely imaginary roots, all roots of Eq. (1.1) have negative real parts for all $\tau \geq 0$.

However, to the best of our knowledge, there is no paper considering the transcendental equation (1.1) in *the critical case*

$$p + s = 0.$$

In fact, in this situation, Eq. (1.2) has a pair of purely imaginary roots. So the difficulty arises when determining the distribution of roots of Eq. (1.1). To show how to deal with the critical situation we discuss the bifurcation of a plant-hare dynamical system *in critical state*.

1.2. Model Formulation. It is well known that the classical Lotka-Voterra population models (competitive model, predator-prey model) have been well investigated. However, in the real world, almost all animals can be classified into juvenile and adult according to the age-structure. Juvenile and adult have different ability of consumption, mating, reproducing, attacking prey, etc. So the stage-structured factor is absolutely necessary for studying the population dynamics. To fit the more realistic environment, stage-structured population growth is introduced in population models. These models assume an average age to maturity which appears as a constant time delay reflecting a delayed birth of immature and a reduced survival of immature to their maturity. For example, some authors have studied a single-species population growth with various stages of life history (see [31–34]). Much research has been devoted the models concerning the predator-prey system with stage-structure (see [35–49]). Most of the works focused on the predator-prey system with *stage structure for the predator.* Recently, the authors of [29] considered the influence of *a stage structure for prey* (plant) in a plant-herbivore dynamic system. In the model, it is assumed that the plant can produce toxic substance to protect themselves when they are immature, while the plant toxicity can disappear when they grow into the mature plant. In this situation, the herbivores (predators) choose the mature plants to eat and avoid the immature ones. The model is described as follows:

$$\begin{cases} \dot{H}(t) &=& c_M b_M M(t)H(t) - d_H H(t), \\ \dot{J}(t) &=& \tilde{r}J(t) + \alpha M(t)H(t) - rM(t-\tau) - \alpha b_M M(t-\tau)H(t-\tau) - d_J J(t), \\ \dot{M}(t) &=& rM(t-\tau) + \alpha b_M M(t-\tau)H(t-\tau) - b_M M(t)H(t) - d_M M(t) - \beta M^2(t), \end{cases} \tag{1.3}$$

where $H(t)$ is the population density of herbivore (predator species), $J(t)$ and $M(t)$ denote the densities of the immature and mature plants (prey species), respectively; d_H denotes the death rate of herbivore; $\tilde{r}$ is the intrinsic growth rate of the immature plant; d_J and d_M denote the death rate of the immature and mature plant, respectively; τ represents a constant time to maturity; r denotes the rate of immature plant becoming into mature plant; b_M denotes the coefficient in herbivore eating immature plant; αb_M denotes the rate of mature plant which has been eaten at time $t-\tau$ becoming into new mature plant; c_M denotes the rate of conversing mature plant into new herbivore; β is the intra-specific competition rate of the mature plant. In [29], the parameter β is chosen as $\beta > 0$, by the assumption that mature plants share the nutrients in an closed environment. But in a good situation, there are enough nutrients. In this circumstance, the parameter can be chosen as $\beta = 0$ and system (1.3) reduces to

$$\begin{cases} \dot{H}(t) &= c_M b_M M(t)H(t) - d_H H(t), \\ \dot{J}(t) &= \tilde{r}J(t) + \alpha M(t)H(t) - rM(t-\tau) - \alpha b_M M(t-\tau)H(t-\tau) - d_J J(t), \\ \dot{M}(t) &= rM(t-\tau) + \alpha b_M M(t-\tau)H(t-\tau) - b_M M(t)H(t) - d_M M(t). \end{cases} \tag{1.4}$$

System (1.4) is supplemented with the initial conditions of the form

$$\begin{cases} H(s) &= \phi_1(s),\ \phi_1(s) > 0,\ s \in [-\tau, 0], \\ J(s) &= \phi_2(s),\ \phi_2(s) > 0,\ s \in [-\tau, 0], \\ M(s) &= \phi_3(s),\ \phi_3(s) > 0,\ s \in [-\tau, 0], \end{cases}$$

where $\phi = (\phi_1, \phi_2, \phi_3) \in \mathcal{C}([-\tau, 0], \mathbb{R}^3_+), i = 1, 2, 3$ and $\mathcal{C}$ denotes the set of all continuous functions from $[-\tau, 0]$ into $\mathbb{R}^3_+ = \{(H, J, M) : H > 0, J > 0, M > 0\}$. Note that the first equation and the third equation of (1.4) can be separated from the whole system. Consider the following subsystem of (1.4).

$$\begin{cases} \dot{H}(t) &= c_M b_M M(t)H(t) - d_H H(t), \\ \dot{M}(t) &= rM(t-\tau) + \alpha b_M M(t-\tau)H(t-\tau) - b_M M(t)H(t) - d_M M(t). \end{cases} \tag{1.5}$$

A preliminary result on the positivity of solutions is proved similarly as Lemma 1.1 in [29].

Lemma 1.1. *All solutions of system (1.5) are positive.*

1.3. Comparison with the previous work. In [29], the authors studied the Hopf bifurcation of system (1.3) under the assumption that $\beta > 0$. In fact, when $\beta = 0$, the results in [29] cannot be valid. Why? Because the distribution of the roots of characteristic equation is completely different. The essential reason is that $\beta > 0$ is equivalent to the *standard assumption* $p + s > 0$ in Eq. (1.2), while $\beta = 0$ corresponds to the *critical case* $p + s = 0$ in Eq. (1.2). Moreover, it is very interesting to study the global continuation of periodic solutions bifurcating from the equilibrium in a critical state. The method used in the proof of Lemma 4.2 is nontrivial and interesting.

1.4. Outline of this work. Model analysis and bifurcation of periodic solution are carried out in Section 2. Section 3 is devoted to examining the direction and the stability of Hopf bifurcation. An interesting theorem for the global continuation of periodic solution from the equilibrium is given . Finally, an example and numerical simulations are given to show the feasibility of our results.

2 Model analysis and bifurcation of periodic solution

In this section we determine the local stability of the equilibria (boundary equilibrium and positive equilibrium).

Letting

$$\begin{cases} \tilde{x}(t) &= H(t), \\ \tilde{y}(t) &= c_M b_M M(t), \end{cases} \tag{2.1}$$

then system (1.5) can be rewritten as

$$\begin{cases} \dot{\tilde{x}}(t) &= \tilde{x}(t)\tilde{y}(t) - a\tilde{x}(t), \\ \dot{\tilde{y}}(t) &= b\tilde{y}(t-\tau) + c\tilde{x}(t-\tau)\tilde{y}(t-\tau) - d\tilde{x}(t)\tilde{y}(t) - e\tilde{y}(t), \end{cases} \tag{2.2}$$

where $a = d_H, b = r, c = b_M\alpha, d = b_M, e = d_M$. From biological point of view, the parameters a, b, c, d, e are all positive. If $\tilde{E} = (\tilde{x}, \tilde{y})$ denotes the equilibrium of system (2.2), it must satisfy the algebraic equations

$$\begin{cases} \tilde{x}\tilde{y} - a\tilde{x} = 0, \\ b\tilde{y} + c\tilde{x}\tilde{y} - d\tilde{x}\tilde{y} - e\tilde{y} = 0. \end{cases} \tag{2.3}$$

Clearly, $E_0 = (0,0)$ is an equilibrium. If $d \neq c$ and $b \neq e$, then there are no other equilibria lying on the x-axis or on the y-axis. If $d = c$ and $b \neq e$, system (2.2) has only one equilibrium E_0.

It is easy to see that there is a unique positive equilibrium, $E_* = (\frac{b-e}{d-c}, a)$, if $d \neq c$ and $\frac{b-e}{d-c} > 0$. So we summarize as follows.

Lemma 2.1. *(i) If $d = c$ and $b \neq e$, then system (2.2) has only one equilibrium E_0.*

(ii) If $d \neq c$ and $\frac{b-e}{d-c} > 0$, then system (2.2) has a unique positive equilibrium E_*.

To precisely describe the stability of the equilibrium $\tilde{E}$, we need the following definition.

Definition 2.2. The equilibrium $\tilde{E}$ of system (2.2) is called *conditionally stable* (asymptotically stable on the delays) if it is asymptotically stable for some τ_j in some intervals, but not necessarily for all delays $\tau_j \geq 0, (1 \leq j \leq m)$. The equilibrium $\tilde{E}$ of system (2.2) is called *absolutely stable* (asymptotically stable independent of the delays) if it is asymptotically stable for all $\tau_j \geq 0, (1 \leq j \leq m)$.

To determine the local stability of the equilibria E_0 and E_*, we need to linearize (2.2) around $\tilde{E} = (\tilde{x}, \tilde{y})$ ($\tilde{E}$ may be E_0 or E_*). To do so, letting $x(t) = \tilde{x}(t) - \tilde{x}$ and $y(t) = \tilde{y}(t) - \tilde{y}$, (2.2) can be transformed into

$$\begin{cases} \dot{x}(t) &= (\tilde{y} - a)x(t) + \tilde{x}y(t) + x(t)y(t), \\ \dot{y}(t) &= -d\tilde{y}x(t) - (d\tilde{x} + e)y(t) + (b + c\tilde{x})y(t-\tau) + c\tilde{y}x(t-\tau) \\ & \quad -dx(t)y(t) + cx(t-\tau)y(t-\tau). \end{cases} \tag{2.4}$$

The linear part of system (2.4) is

$$\begin{cases} \dot{x}(t) &= (\tilde{y} - a)x(t) + \tilde{x}y(t), \\ \dot{y}(t) &= -d\tilde{y}x(t) - (d\tilde{x} + e)y(t) + (b + c\tilde{x})y(t-\tau) + c\tilde{y}x(t-\tau). \end{cases} \tag{2.5}$$

The corresponding characteristic equation (in the unknown λ) is

$$\det\begin{pmatrix} \lambda-(\tilde{y}-a) & -\tilde{x} \\ d\tilde{y}-c\tilde{y}e^{-\lambda\tau} & \lambda+(d\tilde{x}+e)-(b+c\tilde{x})e^{-\lambda\tau} \end{pmatrix}=0,$$

that is,

$$\lambda^2+p\lambda+r+(s\lambda+q)e^{-\lambda\tau}=0, \tag{2.6}$$

where $p=d\tilde{x}+e-\tilde{y}+a,\ r=d\tilde{x}\tilde{y}-(d\tilde{x}+e)(\tilde{y}-a),\ s=-(b+c\tilde{x}),\ q=-[c\tilde{x}\tilde{y}-(b+c\tilde{x})(\tilde{y}-a)]$.

For $\tilde{E}=E_0$, we have the following theorem.

Theorem 2.3. *(i) If $b>e$, then the equilibrium point $E_0=(0,0)$ is unstable (saddle).*

(ii) If $b<e$, then the equilibrium point $E_0=(0,0)$ is asymptotically stable.

Proof. For $\tilde{E}=E_0$, (2.6) reduces to

$$\lambda^2+(e+a)\lambda+ea-(b\lambda+ab)e^{-\lambda\tau}=0,$$

or

$$(\lambda+a)(\lambda+e-be^{-\lambda\tau})=0.$$

Obviously, $\lambda_1=-a$ is a negative root. Let

$$\tilde{F}(\lambda)=\lambda+e-be^{-\lambda\tau}.$$

(i) If $b>e$, it is easy to verify that $\tilde{F}(0)=e-b<0$ and $\tilde{F}(+\infty)=+\infty$. Thus, there exists a positive real number $\lambda_2>0$ such that $\tilde{F}(\lambda_2)=0$. Therefore, E_0 is unstable if $b>e$.

(ii) Now we consider the case $b<e$. Note that $\lambda_1=-a$ is a negative root. To show the asymptotic stability of E_0 in this case, it suffices to prove that all the roots of $\tilde{F}(\lambda)=0$ have negative real parts. In fact, when $\tau=0$, we see that $\tilde{F}(\lambda)=0$ reduces to $\lambda+e=0$. That is, $\lambda=-e$ is the unique negative root. So by Rouché's theorem (see Dieudonné [30], Theorem 9.17.4), we only need to prove that $\tilde{F}(\lambda)=0$ does not have any purely imaginary roots. To this end, assume for the contrary that $i\omega$ is a purely imaginary root of $\tilde{F}(\lambda)=0$. Rewrite $\tilde{F}(\lambda)=0$ in terms of its real and imaginary part as

$$\begin{cases} e-b\cos(\omega\tau) &=& 0, \\ \omega+b\sin(\omega\tau) &=& 0, \end{cases}$$

which implies

$$\omega^2=b^2-e^2<0.$$

This is a contradiction. Therefore, if $b<e$, E_0 is asymptotically stable.

For $\tilde{E}=E_*$, replacing (x,y) with $(x_*,y_*)=(\frac{b-e}{d-c},a)$ in (2.6), we have

$$\lambda^2+p\lambda+r+(s\lambda+q)e^{-\lambda\tau}=0, \tag{2.7}$$

where $p = dx_* + e = \frac{bd-ce}{d-c}$, $r = dx_* y_* = \frac{(b-e)da}{d-c}$, $s = -(b + cx_*) = -\frac{bd-ce}{d-c}$, $q = -cx_* y_* = -\frac{(b-e)ca}{d-c}$. Note that $p + s = 0$ in this case. So the system is in a *critical state* just as we point out in the Introduction.

When $\tau = 0$, Eq. (2.7) reduces to

$$\lambda^2 + (p+s)\lambda + (q+r) = 0, \quad \text{or} \quad \lambda^2 + (q+r) = 0. \tag{2.8}$$

Notice that $p + s = 0$, which is different from the standard assumption $p + s > 0$.

To determine the local stability of the positive equilibrium E_* more precisely, we proceed in three steps.

Step 1. The first step is to find all possible purely imaginary roots of the characteristic equation (2.7). To this end, we let $\lambda = \alpha + i\omega,\ \alpha, \omega \in \mathbb{R}$, and rewrite (2.7) in terms of its real and imaginary arts as

$$\begin{cases} \alpha^2 - \omega^2 + p\alpha + r &= e^{-\alpha\tau}[-(s\alpha + q)\cos(\omega\tau) - s\omega\sin(\omega\tau)], \\ 2\alpha\omega + p\omega &= e^{-\alpha\tau}[-s\omega\cos(\omega\tau) + (s\alpha + q)\sin(\omega\tau)],. \end{cases} \tag{2.9}$$

When $\alpha = 0$, (2.9) reduces to

$$\begin{cases} -\omega^2 + r &= -q\cos(\omega\tau) - s\omega\sin(\omega\tau), \\ p\omega &= -s\omega\cos(\omega\tau) + q\sin(\omega\tau). \end{cases} \tag{2.10}$$

It follows by taking the sum of squares that

$$\omega^4 - (s^2 - p^2 + 2r)\omega^2 + (r^2 - q^2) = 0. \tag{2.11}$$

Note that for $\tilde{E} = E_*$, $p + s = 0$, which implies that

$$-(s^2 - p^2 + 2r) = -2r < 0.$$

Hence, the two roots of Eq. (2.11) can be expressed as follows:

$$\omega_+^2 = r - q, \quad \text{and} \quad \omega_-^2 = r + q. \tag{2.12}$$

Thus, if $d < c$, then

$$r^2 - q^2 < 0,$$

so that Eq. (2.11) has a positive root ω_+^2.

And if $d > c$, then

$$r^2 - q^2 > 0,$$

so that Eq. (2.11) has two positive roots $\omega_\pm^2$.

In both cases, the characteristic equation (2.7) has purely imaginary roots when τ takes certain values. These critical values $\tau_j^{\pm}$ of τ can be determined from system (2.10) and given by

$$\left\{\begin{array}{lll} \tau_j^{+} & = & \frac{\pi}{\omega_+} + \frac{2j\pi}{\omega_+},\ j = 0, 1, 2, \cdots, \\ \tau_j^{-} & = & \frac{1}{\omega_-} \arccos\left\{\frac{q^2 - p^2(r+q)}{q^2 + p^2(r+q)}\right\} + \frac{2j\pi}{\omega_-},\ j = 0, 1, 2, \cdots. \end{array}\right. \tag{2.13}$$

Step 2. Denote by

$$\lambda_j^{\pm}(\tau) = \alpha_j^{\pm}(\tau) + \omega_j^{\pm}(\tau),\ j = 0, 1, 2, \cdots,$$

the root of Eq. (2.7) satisfying

$$\alpha_j^{\pm}(\tau_j^{\pm}) = 0, \quad \omega_j^{\pm}(\tau_j^{\pm}) = \omega_{\pm}. \tag{2.14}$$

This step is to verify that the following transversality conditions hold:

$$\frac{d}{d\tau} Re\lambda_j^{+}(\tau_j^{+}) > 0, \quad \frac{d}{d\tau} Re\lambda_j^{-}(\tau_j^{-}) < 0. \tag{2.15}$$

To see this, differentiating characteristic equation (2.7) with respect to τ, we have

$$[2\lambda + p]\frac{d\lambda}{d\tau} = -s\frac{d\lambda}{d\tau} - \tau e^{-\lambda\tau}[-s\lambda - q]\frac{d\lambda}{d\tau} - \lambda e^{-\lambda\tau}[-s\lambda - q].$$

Noticing that $p = -s$, it follows that

$$\Big\{\tau e^{-\lambda\tau}[-s\lambda - q] + 2\lambda\Big\}\frac{d\lambda}{d\tau} = -\lambda e^{-\lambda\tau}[-s\lambda - q],$$

which implies

$$\begin{array}{rcl} \Big(\dfrac{d\lambda}{d\tau}\Big)^{-1} & = & \dfrac{\tau e^{-\lambda\tau}[-s\lambda - q] + 2\lambda}{-\lambda e^{-\lambda\tau}[-s\lambda - q]} \\ & = & \dfrac{2}{-e^{-\lambda\tau}[-s\lambda - q]} - \dfrac{\tau}{\lambda} \\ & = & \dfrac{2}{-(\lambda^2 + p\lambda + r)} - \dfrac{\tau}{\lambda}. \end{array} \tag{2.16}$$

Here we have used Eq. (2.7) again.

Thus, in view of (2.14) and (2.12), it follows from (2.16) that

$$\begin{array}{rcl} \mathrm{sign}\Big\{\dfrac{d}{d\tau}(Re\lambda)\Big\}_{\tau=\tau_j^+} & = & \mathrm{sign}\Big\{Re\Big(\dfrac{d\lambda}{d\tau}\Big)^{-1}\Big\}_{\tau=\tau_j^+} \\ & = & \mathrm{sign}\Big\{Re\Big[\dfrac{2}{-(\lambda^2 + p\lambda + r)} - \dfrac{\tau}{\lambda}\Big]_{\tau=\tau_j^+}\Big\} \\ & = & \mathrm{sign}\Big\{Re\Big[\dfrac{2}{-[(i\omega_+)^2 + ip\omega_+ + r]} - \dfrac{\tau_j^+}{i\omega_+}\Big]\Big\} \\ & = & \mathrm{sign}\Big\{\dfrac{2(\omega_+^2 - r)}{(\omega_+^2 - r)^2 + p^2\omega_+^2}\Big\} \\ & = & \mathrm{sign}\Big\{\dfrac{2(r - q - r)}{q^2 + p^2\omega_+^2}\Big\} = \mathrm{sign}\{\dfrac{-2q}{q^2 + p^2\omega_+^2}\} = 1. \end{array}$$

Similarly,

$$\text{sign}\Big\{\frac{d}{d\tau}(Re\lambda)\Big\}_{\tau=\tau_j^-} = \text{sign}\Big\{\frac{2q}{q^2+p^2\omega_-^2}\Big\} = -1.$$

Thus, the assertion (2.15) holds.

Step 3. To assure the existence of a positive equilibrium, $d \neq c$ is indispensable. So we shall study the distribution of roots of Eq. (2.7) in two cases ($d < c$ or $d > c$):

Case 1: $d < c$, which implies that $q + r < 0$. Setting

$$G(\lambda) = \lambda^2 + p\lambda + r + (s\lambda + q)e^{-\lambda\tau},$$

it follows from Eq. (2.7) that

$$G(0) = r + q < 0,$$

and

$$G(+\infty) = +\infty.$$

Thus, there exists a $\lambda_0 > 0$ such that $G(\lambda_0) = 0$. Consequently, the positive equilibrium E_* is unstable.

Case 2: $d > c$, which implies that $q + r > 0$. Then it follows from Eq. (2.8) that $\lambda(0) = \pm i\sqrt{q+r}$ for $\tau = 0$, which implies $Re\lambda(0) = 0$. On the other hand, it is easy to see from *Step 2* that $\frac{d}{d\tau}Re\lambda(0) > 0$. We claim that $Re\lambda(\tau) > 0$ for all $\tau \in (0, \tau_0^+)$. If this claim is not true, there exists a $\delta(0 < \delta < \tau_0^+ < \tau_0^-)$ such that $Re\lambda(\tau) > 0$ for $\tau \in [0, \delta)$ and $Re\lambda(\delta) = 0$. This implies that $\lambda(\delta)$ is a pure imaginary root of Eq. (2.7) and $\frac{d}{d\tau}Re\lambda(\delta) < 0$. But from *Step 1*, there is no other $\tau(\tau \geq 0)$ except $\tau_j^\pm(j = 1, 2, \cdots)$ such that Eq. (2.7) has a pair of purely imaginary roots. And it follows from *Step 2* that τ_0^- is the minim number such that $\frac{d}{d\tau}Re\lambda(\tau_j^-) < 0$. This implies $\delta = \tau_0^-$, which contradicts to the assumption of $\delta(0 < \delta < \tau_0^+ < \tau_0^-)$. Therefore, the real parts of all roots of Eq. (2.7) are positive for all $\tau \in (0, \tau_0^+)$.

In view of $\frac{d}{d\tau}Re\lambda(\tau_0^+) > 0$ and $\frac{d}{d\tau}Re\lambda(\tau_0^-) < 0$, by an analogous argument for the intervals (τ_0^+, τ_0^-), we see that the real parts of all roots of Eq. (2.7) are negative for all $\tau \in (\tau_0^+, \tau_0^-)$.

Similarly, we conclude that for the intervals (τ_1^+, τ_1^-), (τ_2^+, τ_2^-), $\cdots$, $\tau \in (\tau_k^+, \tau_k^-)$, the real parts of all roots of Eq. (2.7) are negative.

Similarly, for the intervals $\tau \in (\tau_0^-, \tau_1^+)$, $\tau \in (\tau_1^-, \tau_2^+)$, $\cdots$, $\tau \in (\tau_{k-1}^-, \tau_k^+)$ and $\tau > \tau_k^-$, the real parts of the roots of Eq. (2.7) are positive.

From the above analysis, it follows that $\tau_j^\pm$ are bifurcation values. Thus, we have the following theorem about the distribution of the characteristic roots of Eq. (2.7).

Theorem 2.4. *Let $\omega_\pm$ and $\tau_j^\pm$ be defined by (2.12) and (2.13), respectively.*

(i) If $d < c$ and $b < e$, then Eq. (2.7) has at least one root with positive real part for all $\tau \geq 0$.

(ii) If $d > c$ and $b > e$, then there is a positive integer k such that there are switches from instability to stability, that is, when

$$\tau \in (0, \tau_0^+), (\tau_0^-, \tau_1^+), (\tau_1^-, \tau_2^+), \cdots, (\tau_{k-1}^-, \tau_k^+) \ \textit{and} \ \tau > \tau_k^-,$$

Eq. (2.7) has a root with positive real part, and when

$$\tau \in (\tau_0^+, \tau_0^-), (\tau_1^+, \tau_1^-), \cdots, (\tau_k^+, \tau_k^-),$$

all roots of Eq. (2.7) have negative real parts.

Remark 2.5. In the non-critical case (see e.g. [2,3,10,11,26], let $\bar{\omega}_\pm$ and $\bar{\tau}_j^\pm$ be defined by

$$\bar{\omega}_\pm^2 = \frac{1}{2}(s^2 - p^2 + 2r) \pm \frac{1}{2}[(s^2 - p^2 + 2r)^2 - 4(r^2 - q^2)]^{\frac{1}{2}},$$

and

$$\bar{\tau}_j^\pm = \frac{1}{\bar{\omega}_\pm} \arccos\left\{\frac{q(\bar{\omega}_\pm^2 - r) - ps\bar{\omega}_\pm^2}{s^2\bar{\omega}_\pm^2 + q^2}\right\} + \frac{2j\pi}{\bar{\omega}_\pm}, \ j = 0, 1, 2, \cdots.$$

Then there is a positive integer k such that there are switches from stability to instability, that is, when

$$\tau \in [0, \bar{\tau}_0^+), (\bar{\tau}_0^-, \bar{\tau}_1^+), (\bar{\tau}_1^-, \bar{\tau}_2^+), \cdots, (\bar{\tau}_{k-1}^-, \bar{\tau}_k^+),$$

all roots of Eq. (2.7) have negative real parts, and when

$$\tau \in [\bar{\tau}_0^+, \bar{\tau}_0^-), (\bar{\tau}_1^+, \bar{\tau}_1^-), \cdots, (\bar{\tau}_{k-1}^+, \bar{\tau}_{k-1}^-) \ \text{and} \ \tau > \bar{\tau}_k^+,$$

Eq. (2.7) has at least one root with positive real part.

We see that there is a big difference between the critical case and the non-critical one. In particular, the intervals for stability and instability are completely different. In the critical case, there are switches from instability to stability, the intervals $(0, \tau_0^+), (\tau_0^-, \tau_1^+), (\tau_1^-, \tau_2^+)$, $\cdots, (\tau_{k-1}^-, \tau_k^+)$ are unstable. On the contrary, in the non-critical case, there are switches from stability to instability, the intervals $[0, \bar{\tau}_0^+), (\bar{\tau}_0^-, \bar{\tau}_1^+), (\bar{\tau}_1^-, \bar{\tau}_2^+), \cdots, (\bar{\tau}_{k-1}^-, \bar{\tau}_k^+)$ are locally asymptotically stable. The reason lies in the fact that in the critical case, all roots of Eq. (2.8) are pairs of purely imaginary roots for $\tau = 0$, while in the non-critical case, all roots of Eq. (2.8) have negative real parts for $\tau = 0$.

Thus for $\tilde{E} = E_*$, we have the following main results on the local stability and bifurcation of the positive equilibrium E_*.

Theorem 2.6. *Let $\omega_\pm$ and $\tau_j^\pm$ be defined by (2.12) and (2.13), respectively.*

(i) If $d < c$ and $b < e$, then E_* is unstable.

(ii) If $d > c$ and $b > e$, then there is a positive integer k such that the equilibrium E_* switches k times from instability to stability, that is, when $\tau \in (0, \tau_0^+), (\tau_1^-, \tau_1^+), (\tau_2^-, \tau_2^+), \cdots, (\tau_{k-1}^-, \tau_{k-1}^+)$ and $\tau > \tau_k^-$, the positive equilibrium E_* of (2.2) is unstable; when $\tau \in (\tau_0^+, \tau_1^-), (\tau_1^+, \tau_2^-), \cdots, (\tau_{k-1}^+, \tau_k^-)$, the positive equilibrium E_* of (2.2) is asymptotically stable.

(iii) If all the conditions as stated in (ii) hold, then system (2.2) undergoes a Hopf bifurcation at the equilibrium when $\tau = \tau_j^+ (j = 0, 1, \cdots, k-1)$ and $\tau = \tau_j^- (j = 1, 2, \cdots, k)$.

3 Direction and stability of Hopf bifurcation

In the previous section, some sufficient conditions are obtained to guarantee that system (2.2) undergoes Hopf bifurcation at the positive equilibrium E_* when $\tau = \tau_j^+ (j = 0, 1, \cdots, k-1)$ and $\tau = \tau_j^- (j = 1, 2, \cdots, k)$. This section is to derive the explicit formulas determining the direction, stability and period of periodic solutions bifurcating from equilibrium E_* at these critical values of τ. The method is based on the normal form and the center manifold theory developed by Hassard *et al.* [1]. Without loss of generality, denote any one of these critical values $\tau = \tau_j^+ (j = 0, 1, \cdots, k-1)$ and $\tau = \tau_j^- (j = 1, 2, \cdots, k)$ by $\tilde{\tau}$, at which Eq. (2.7) has a pair of purely imaginary roots $i\omega$ and system (2.2) undergoes a Hopf bifurcation from E_*.

Let $u_1(t) = x(\tau t) - x_*$, $u_2(t) = y(\tau t) - y_*$ and $\tau = \tilde{\tau} + \mu$, $\mu \in R$. Then $\mu = 0$ is the Hopf bifurcation value of system (2.2) and system (2.2) can be rewritten as

$$\begin{cases} \dot{u}_1(t) &= \tau[x_* u_2(t) + u_1(t) u_2(t)], \\ \dot{u}_2(t) &= \tau[-dy_* u_1(t) - (dx_* + e) u_2(t) + (b + cx_*) u_2(t-1) + cy_* u_1(t-1) \\ & \qquad -du_1(t) u_2(t) + cu_1(t-1) u_2(t-1)] \end{cases} \tag{3.1}$$

Thus, we can work in the fixed phase space $\mathcal{C} = \mathcal{C}([-1, 0], \mathbb{R}^2)$, which does not depend on the delay τ. In space $\mathcal{C} = \mathcal{C}([-1, 0], \mathbb{R}^2)$, system (3.1) is transformed into a FDE as

$$\dot{u}(t) = L_\mu(u_t) + f(\mu, u_t), \tag{3.2}$$

where $u = (u_1, u_2)^T$, $u_t(\theta) = u(t + \theta) \in \mathcal{C}$, and L_μ are given respectively by

$$L_\mu(\phi) = (\tilde{\tau} + \mu) \begin{pmatrix} x_* \phi_2(0) \\ -dy_* \phi_1(0) - (dx_* + e)\phi_2(0) + (b + cx_*)\phi_2(-1) + cy_* \phi_1(-1) \end{pmatrix}, \tag{3.3}$$

and

$$f(\mu, \phi) = (\tilde{\tau} + \mu) \begin{pmatrix} \phi_1(0)\phi_2(0) \\ -d\phi_1(0)\phi_2(0) + c\phi_1(-1)\phi_2(-1) \end{pmatrix}, \tag{3.4}$$

where $\phi = (\phi_1, \phi_2) \in \mathcal{C}$.

By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\mu, \theta)$ in $\theta \in [-1, 0]$ such that

$$L_\mu(\phi) = \int_{-1}^{0} d\eta(\mu, \theta)\phi(\theta), \quad \text{for } \phi \in \mathcal{C},$$

where bounded variation functions $\eta(\mu, \theta)$ can be chosen as

$$\eta(\mu, \theta) = (\tilde{\tau} + \mu) \begin{pmatrix} 0 & x_* \\ -dy_* & -(dx_* + e) \end{pmatrix} \delta(\theta) - (\tilde{\tau} + \mu) \begin{pmatrix} 0 & 0 \\ cy_* & b + cx_* \end{pmatrix} \delta(\theta + 1), \tag{3.5}$$

where δ is the Dirac function. For $\phi \in \mathcal{C}^1([-1, 0], \mathbb{R}^2)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} d\eta(\mu, s)\phi(s), & \theta = 0, \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \phi), & \theta = 0. \end{cases}$$

Then system (3.2) is equivalent to

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{3.6}$$

where $u = (u_1, u_2)^T$, $u_t(\theta) = u(t+\theta), \theta \in [-1, 0]$.

For $\psi \in \mathcal{C}^1([0, 1], (\mathbb{R}^2)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^{0} d\eta(0, t)\phi(-t), & s = 0, \end{cases}$$

and a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta) d\eta(\theta)\phi(\xi)d\xi, \tag{3.7}$$

where $\eta(\theta) = \eta(0, \theta)$. Then $A(0)$ and A^* are adjoint operators. In addition, from Section 2 we know that $\pm i\tilde{\tau}\omega$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. We first need to compute the eigenvector of $A(0)$ and A^* corresponding to $i\tilde{\tau}\omega$ and $-i\tilde{\tau}\omega$, respectively.

To this end, suppose that $q(\theta) = (1, \alpha_0)^T e^{i\omega\tilde{\tau}\theta}$ is the eigenvector of $A(0)$ corresponding to $i\tilde{\tau}\omega$, then $A(0)q(\theta) = i\omega\tilde{\tau}q(\theta)$. It follows from the definition of $A(0)$, (3.3) and (3.5) that

$$\tilde{\tau} \begin{pmatrix} i\omega & -x_* \\ dy_* - cy_* e^{-i\omega\tilde{\tau}} & i\omega + (dx_* + e) - (b + cx_*)e^{-i\omega\tilde{\tau}} \end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, we can choose

$$q(0) = (1, \alpha_0)^T = (1, \frac{i\omega}{x_*})^T.$$

Similarly, let $q^*(s) = D(\beta_0, 1)^T = D(\beta_0, 1)e^{i\omega\tilde{\tau}s}$ is the eigenvector of A^* corresponding to $-i\tilde{\tau}\omega$, we can compute

$$q^*(s) = D(\beta_0, 1)^T = D(\beta_0, 1)e^{i\omega\tilde{\tau}s} = D(\frac{-i\omega + (dx_* + e) - (b + cx_*)e^{i\omega\tilde{\tau}}}{x_*}, 1)e^{i\omega\tilde{\tau}s}.$$

In order to assure $\langle q^*(s), q(\theta) \rangle = 1$, we need to determine the value of D. From (3.7), we have

$$\begin{aligned} \langle q^*(s), q(\theta) \rangle &= \bar{D}(\bar{\beta}_0, 1)(1, \alpha_0)^T - \int_{-1}^{0} \int_{\xi=0}^{\theta} \bar{D}(\bar{\beta}_0, 1)e^{-i\omega\tilde{\tau}(\xi-\theta)} d\eta(\theta)(1, \alpha_0)^T e^{i\omega\tilde{\tau}\xi} d\xi, \\ &= \bar{D}\Big\{\bar{\beta}_0 + \alpha_0 - \int_{-1}^{0} (\bar{\beta}_0, 1)\theta e^{i\omega\tilde{\tau}\theta} d\eta(\theta)(1, \alpha_0)^T\Big\} \\ &= \bar{D}\Big\{\bar{\beta}_0 + \alpha_0 + \tilde{\tau}(\bar{\beta}_0, 1)\begin{pmatrix} 0 & 0 \\ cy_* & b + cx_* \end{pmatrix}(1, \alpha_0)^T e^{-i\omega\tilde{\tau}}\Big\} \\ &= \bar{D}\Big\{\bar{\beta}_0 + \alpha_0 + \tilde{\tau}[cy_* + \alpha_0(b + cx_*)]e^{-i\omega\tilde{\tau}}\Big\}. \end{aligned}$$

Thus, we can choose D as

$$D = \frac{1}{\beta_0 + \bar{\alpha}_0 + \tilde{\tau}[cy_* + \bar{\alpha}_0(b + cx_*)]e^{i\omega\tilde{\tau}}}.$$

It is also easy to verify that $\langle q^*(s), \bar{q}(\theta)\rangle = 0$. Now we compute the coordinates to describe the center manifold $\mathbf{C}_0$ at $\mu = 0$. Let u_t be the solution of (3.6) when $\mu = 0$. Define

$$z(t) = \langle q^*, u_t\rangle, \quad W(t,\theta) = u_t(\theta) - 2Re\{z(t)q(\theta)\}. \tag{3.8}$$

On the center manifold $\mathbf{C}_0$, we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta),$$

where

$$W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots, \tag{3.9}$$

z and $\bar{z}$ are local coordinates for center manifold $\mathbf{C}_0$ in the direction of q^* and $\bar{q}^*$. Note that W is real if u_t is real. We only consider real solutions. For solution $u_t \in \mathbf{C}_0$ of (3.6), since $\mu = 0$, we have

$$\begin{aligned} \dot{z}(t) &= i\omega\tilde{\tau}z + \langle q^*(\theta), f(0, W(z,\bar{z},\theta) + 2Re\{zq(\theta)\})\rangle \\ &= i\omega\tilde{\tau}z + \bar{q}^*(0)f(0, W(z,\bar{z},0) + 2Re\{zq(0)\}) := i\omega\tilde{\tau}z + \bar{q}^*(0)f_0(z,\bar{z}). \end{aligned}$$

We rewrite the equation as

$$\dot{z}(t) = i\omega\tilde{\tau}z(t) + g(z,\bar{z}),$$

where

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{3.10}$$

It follows from (3.8) and (3.9) that

$$\begin{aligned} u_t(\theta) &= W(t,\theta) + 2Re\{z(t)q(\theta)\} \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + (1,\alpha_0)^T e^{i\omega\tilde{\tau}\theta}z + (1,\bar{\alpha}_0)^T e^{-i\omega\tilde{\tau}\theta}\bar{z} + \cdots, \end{aligned} \tag{3.11}$$

which together with (3.4) gives

$$
\begin{aligned}
g(z,\bar z) &= \bar q^*(0) f_0(z,z) = \bar q^*(0) f(0,u_t) = \bar D(\bar\beta_0,1)\tilde\tau \begin{pmatrix} u_1(0)u_2(0) \\ -du_1(0)u_2(0)+cu_1(-1)u_2(-1) \end{pmatrix} \\
&= \tilde\tau\bar D\bar\beta_0 u_1(0)u_2(0) + \tilde\tau\bar D(-du_1(0)u_2(0)+cu_1(-1)u_2(-1)) \\
&= \tilde\tau\bar D\bar\beta_0\Big[z+\bar z+W_{20}^1(0)\frac{z^2}{2}+W_{11}^1(0)z\bar z+W_{02}^1(0)\frac{\bar z^2}{2}+\cdots\Big] \\
&\qquad \times\Big[\alpha_0 z+\bar\alpha_0\bar z+W_{20}^2(0)\frac{z^2}{2}+W_{11}^2(0)z\bar z+W_{02}^2(0)\frac{\bar z^2}{2}+\cdots\Big] \\
&\quad +\tilde\tau\bar D\Big\{-d\Big[z+\bar z+W_{20}^1(0)\frac{z^2}{2}+W_{11}^1(0)z\bar z+W_{02}^1(0)\frac{\bar z^2}{2}+\cdots\Big] \\
&\qquad \times\Big[\alpha_0 z+\bar\alpha_0\bar z+W_{20}^2(0)\frac{z^2}{2}+W_{11}^2(0)z\bar z+W_{02}^2(0)\frac{\bar z^2}{2}+\cdots\Big] \\
&\qquad +c\Big[e^{-i\omega\tilde\tau}z+e^{i\omega\tilde\tau}\bar z+W_{20}^1(-1)\frac{z^2}{2}+W_{11}^1(-1)z\bar z+W_{02}^1(-1)\frac{\bar z^2}{2}+\cdots\Big] \\
&\qquad \times\Big[\alpha_0 e^{-i\omega\tilde\tau}z+\bar\alpha_0 e^{i\omega\tilde\tau}\bar z+W_{20}^2(-1)\frac{z^2}{2}+W_{11}^2(-1)z\bar z+W_{02}^2(-1)\frac{\bar z^2}{2}+\cdots\Big]\Big\} \\
&= \Big[2\tilde\tau\bar D\bar\beta_0\alpha_0-2\tilde\tau\bar D d\alpha_0+2\tilde\tau\bar D c\alpha_0 e^{-2i\omega\tilde\tau}\Big]\frac{z^2}{2} \\
&\quad +\Big[2\tilde\tau\bar D\bar\beta_0 Re\{\alpha_0\}-2\tilde\tau\bar D d Re\{\alpha_0\}+2\tilde\tau\bar D c Re\{\alpha_0\}\Big]z\bar z \\
&\quad +\Big[2\tilde\tau\bar D\bar\beta_0\bar\alpha_0-2\tilde\tau\bar D d\bar\alpha_0+2\tilde\tau\bar D c\bar\alpha_0 e^{2i\omega\tilde\tau}\Big]\frac{\bar z^2}{2} \\
&\quad +\Big[\tilde\tau\bar D\bar\beta_0\Big(\bar\alpha_0 W_{20}^1(0)+W_{20}^2(0)+2W_{11}^1(0)+2W_{11}^2(0)\Big) \\
&\quad -\tilde\tau\bar D d\Big(\bar\alpha_0 W_{20}^1(0)+W_{20}^2(0)+2W_{11}^1(0)+2W_{11}^2(0)\Big) \\
&\quad +\tilde\tau\bar D c\Big(\bar\alpha_0 e^{-i\omega\tilde\tau}W_{20}^1(-1)+e^{i\omega\tilde\tau}W_{20}^2(-1)+2\alpha_0 e^{-i\omega\tilde\tau}W_{11}^1(-1)+2e^{-i\omega\tilde\tau}W_{11}^2(-1)\Big)\Big]\frac{z^2\bar z}{2} \\
&= 2\tilde\tau\bar D\alpha_0\Big(\bar\beta_0-d+ce^{-2i\omega\tilde\tau}\Big)\frac{z^2}{2}+2\tilde\tau\bar D Re\{\alpha_0\}\Big(\bar\beta_0-d+c\Big)z\bar z \\
&\quad +2\tilde\tau\bar D\bar\alpha_0\Big(\bar\beta_0-d+ce^{2i\omega\tilde\tau}\Big)\frac{\bar z^2}{2} \\
&\quad +\tilde\tau\bar D\Big[(\bar\beta_0-d)\Big(\bar\alpha_0 W_{20}^1(0)+W_{20}^2(0)+2W_{11}^1(0)+2W_{11}^2(0)\Big) \\
&\quad +c\Big(\bar\alpha_0 e^{-i\omega\tilde\tau}W_{20}^1(-1)+e^{i\omega\tilde\tau}W_{20}^2(-1)+2\alpha_0 e^{-i\omega\tilde\tau}W_{11}^1(-1)+2e^{-i\omega\tilde\tau}W_{11}^2(-1)\Big)\Big]\frac{z^2\bar z}{2}.
\end{aligned}
\tag{3.12}
$$

Comparing the coefficients with (3.10), we have

$$
\begin{aligned}
g_{20} &= 2\tilde\tau\bar D\alpha_0[\bar\beta_0-d+ce^{-2i\omega\tilde\tau}] \\
g_{11} &= 2\tilde\tau\bar D Re\{\alpha_0\}[\bar\beta_0-d+c] \\
g_{02} &= 2\tilde\tau\bar D\bar\alpha_0[\bar\beta_0-d+ce^{2i\omega\tilde\tau}] \\
g_{21} &= \tilde\tau\bar D\Big[(\bar\beta_0-d)\Big(\bar\alpha_0 W_{20}^1(0)+W_{20}^2(0)+2W_{11}^1(0)+2W_{11}^2(0)\Big) \\
&\quad +c\Big(\bar\alpha_0 e^{-i\omega\tilde\tau}W_{20}^1(-1)+e^{i\omega\tilde\tau}W_{20}^2(-1)+2\alpha_0 e^{-i\omega\tilde\tau}W_{11}^1(-1)+2e^{-i\omega\tilde\tau}W_{11}^2(-1)\Big)\Big].
\end{aligned}
\tag{3.13}
$$

Since there are $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}, we still need to compute them.

From (3.6) and (3.8), we have

$$\dot{W} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = \begin{cases} AW - 2Re\{\bar{q}^*(0)f_0 q(\theta)\}, & \theta \in [-1, 0), \\ AW - 2Re\{\bar{q}^*(0)f_0 q(0)\} + f_0, & \theta = 0. \end{cases} := AW + H(z, \bar{z}, \theta), \quad (3.14)$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots . \quad (3.15)$$

Substituting the corresponding series into (3.14) and comparing the coefficients, we obtain

$$(A - 2i\omega\tilde{\tau})W_{20} = -H_{20}(\theta), \quad AW_{11} = -H_{11}(\theta), \cdots . \quad (3.16)$$

From (3.14), it is easy to see that for $\theta \in [-1, 0)$,

$$H(z, \bar{z}, \theta) = -\bar{q}^*(0)f_0 q(\theta) - q^*(0)\bar{f}_0 \bar{q}(\theta) = -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta). \quad (3.17)$$

Comparing the coefficients with (3.15) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \quad (3.18)$$

and

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \quad (3.19)$$

It follows from (3.16),(3.18) and the definition of A that

$$\dot{W}_{20}(\theta) = 2i\omega\tilde{\tau}W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta),$$

Note that $q(\theta) = (1, \alpha_0)^T e^{i\omega\tilde{\tau}\theta}$. Hence

$$W_{20}(\theta) = \frac{ig_{20}}{\omega\tilde{\tau}}q(0)e^{i\omega\tilde{\tau}\theta} + \frac{i\bar{g}_{02}}{3\omega\tilde{\tau}}\bar{q}(0)e^{-i\omega\tilde{\tau}\theta} + E_1 e^{2i\omega\tilde{\tau}\theta}, \quad (3.20)$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}) \in \mathbb{R}^2$ is a constant vector.

Similarly, it follows from (3.16) and (3.19) that

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega\tilde{\tau}}q(0)e^{i\omega\tilde{\tau}\theta} + \frac{i\bar{g}_{11}}{\omega\tilde{\tau}}\bar{q}(0)e^{-i\omega\tilde{\tau}\theta} + E_2, \quad (3.21)$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}) \in \mathbb{R}^2$ is also a constant vector.

In what follows, we shall seek appropriate E_1 and E_2. From the definition A and (3.16), we obtain

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega\tilde{\tau}W_{20}(0) - H_{20}(\theta), \quad (3.22)$$

and

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(\theta), \quad (3.23)$$

where $\eta(\theta) = \eta(0, \theta)$. By (3.14), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tilde{\tau}\begin{pmatrix} \alpha_0 \\ -d\alpha_0 + c\alpha_0 e^{-2i\omega\tilde{\tau}} \end{pmatrix}, \quad (3.24)$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2\tilde{\tau}\begin{pmatrix} Re\{\alpha_0\} \\ -dRe\{\alpha_0\} + cRe\{\alpha_0 e^{-2i\omega\tilde{\tau}}\} \end{pmatrix}. \tag{3.25}$$

Note that

$$\Big(i\omega\tilde{\tau}I - \int_{-1}^{0} e^{i\omega\tilde{\tau}\theta} d\eta(\theta)\Big)q(0) = 0,$$

and

$$\Big(-i\omega\tilde{\tau}I - \int_{-1}^{0} e^{-i\omega\tilde{\tau}\theta} d\eta(\theta)\Big)\bar{q}(0) = 0.$$

Substituting (3.20) and (3.24) into (3.22), we obtain

$$\Big(2i\omega\tilde{\tau}I - \int_{-1}^{0} e^{2i\omega\tilde{\tau}\theta} d\eta(\theta)\Big)E_1 = 2\tilde{\tau}\begin{pmatrix} \alpha_0 \\ -d\alpha_0 + c\alpha_0 e^{-2i\omega\tilde{\tau}} \end{pmatrix},$$

which leads to

$$\begin{pmatrix} 2i\omega & -x_* \\ dy_* - cy_* e^{-2i\omega\tilde{\tau}} & 2i\omega + (dx_* + e) - (b + cx_*)e^{-2i\omega\tilde{\tau}} \end{pmatrix} E_1 = 2\tilde{\tau}\begin{pmatrix} \alpha_0 \\ -d\alpha_0 + c\alpha_0 e^{-2i\omega\tilde{\tau}} \end{pmatrix}. \tag{3.26}$$

Solving Eq. (3.26), we have

$$E_1^{(1)} = \frac{2}{|A_1|}\det\begin{pmatrix} \alpha_0 & -x_* \\ -d\alpha_0 + c\alpha_0 e^{-2i\omega\tilde{\tau}} & 2i\omega + (dx_* + e) - (b + cx_*)e^{-2i\omega\tilde{\tau}} \end{pmatrix},$$

$$E_1^{(2)} = \frac{2}{|A_1|}\det\begin{pmatrix} 2i\omega & \alpha_0 \\ dy_* - cy_* e^{-2i\omega\tilde{\tau}} & -d\alpha_0 + c\alpha_0 e^{-2i\omega\tilde{\tau}} \end{pmatrix},$$

where

$$|A_1| = \det\begin{pmatrix} 2i\omega & -x_* \\ dy_* - cy_* e^{-2i\omega\tilde{\tau}} & 2i\omega + (dx_* + e) - (b + cx_*)e^{-2i\omega\tilde{\tau}} \end{pmatrix}.$$

Similarly, substituting (3.21) and (3.25) into (3.23) gives

$$\begin{pmatrix} 0 & -x_* \\ dy_* - cy_* & (dx_* + e) - (b + cx_*) \end{pmatrix} E_2 = 2\begin{pmatrix} Re\{\alpha_0\} \\ -dRe\{\alpha_0\} + cRe\{\alpha_0\} \end{pmatrix}. \tag{3.27}$$

Solving Eq. (3.27), we have

$$E_2^{(1)} = \frac{2}{|A_2|}\det\begin{pmatrix} Re\{\alpha_0\} & -x_* \\ -dRe\{\alpha_0\} + cRe\{\alpha_0\} & (dx_* + e) - (b + cx_*) \end{pmatrix},$$

$$E_2^{(2)} = \frac{2}{|A_2|}\det\begin{pmatrix} 0 & Re\{\alpha_0\} \\ dy_* - cy_* & -dRe\{\alpha_0\} + cRe\{\alpha_0\} \end{pmatrix},$$

where

$$|A_2| = \det\begin{pmatrix} 0 & -x_* \\ dy_* - cy_* & (dx_* + e) - (b + cx_*) \end{pmatrix}.$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (3.20) and (3.21). Furthermore, g_{21} in (3.13) can be expressed by the parameters and delay. Therefore, we can compute the following values:

$$
\begin{aligned}
c_1(0) &= \frac{i}{2\omega\tilde{\tau}}\Big[g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\Big] + \frac{g_{21}}{2}, \\
\mu_2 &= -\frac{Re\{c_1(0)\}}{Re\{\lambda'(\tilde{\tau})\}}, \\
\beta_2 &= 2Re\{c_1(0)\}, \\
T_2 &= -\frac{Im\{c_1(0)\} + \mu_2 Im\{\lambda'(\tilde{\tau})\}}{\omega\tilde{\tau}},
\end{aligned}
\tag{3.28}
$$

which determine the quantities of bifurcating periodic solutions at the critical value $\tilde{\tau}$, i.e., μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0(\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tilde{\tau}(\tau < \tilde{\tau})$; β_2 determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions in the center manifold are stable (unstable) if $\beta_2 < 0(\beta_2 > 0)$; and T_2 determines the period of the bifurcating periodic solutions: the period increase (decrease) if $T_2 > 0(T_2 < 0)$. Further, it follows from (2.15) and (3.28) that the following results about the direction of the Hopf bifurcations hold.

Theorem 3.1. *Assume that $d > c$ and $b > e$ hold. Then the Hopf bifurcations of (2.2) at E_* and $\tau = \tau_j^+$ are supercritical (respectively subcritical) if $Re(c_1(0)) < 0$ (respectively $Re(c_1(0)) > 0$). However, the directions of the Hopf bifurcations (2.2) at E_* and $\tau = \tau_j^-$ is $\tau < \tau_j^-$ (respectively $\tau > \tau_j^-$) if $Re(c_1(0)) < 0$ (respectively $Re(c_1(0)) > 0$).*

Theorem 3.2. *If $d < c, b < e$ and $\tau = \tau_0^+$ hold, or $d > c, b > e$ and $\tau = \tau_j^+, (j = 0, 1, \cdots, k)$ and $\tau = \tau_j^-, (j = 0, 1, \cdots, k-1)$, then the bifurcating periodic solution is stable if $Re(c_1(0)) < 0$ and unstable if $Re(c_1(0)) > 0$.*

4 Global existence of periodic solutions

This section studies the global continuation of periodic solutions bifurcating from the point (E_*, τ_j^+) of system (2.2). To this end, we need some preliminary results. Recall that system (1.5) can be transformed into (2.2) by (2.1). Moreover, by (2.1) and (1.4), the initial conditions of (2.2) take the form

$$
\left\{
\begin{aligned}
\tilde{x}(s) &= \tilde{\phi}_1(s), \ \tilde{\phi}_1(s) > 0, \ s \in [-\tau, 0], \\
\tilde{y}(s) &= \tilde{\phi}_2(s), \ \tilde{\phi}_2(s) > 0, \ s \in [-\tau, 0],
\end{aligned}
\right.
\tag{4.1}
$$

where $\tilde{\phi} = (\tilde{\phi}_1, \tilde{\phi}_2) \in \mathcal{C}([-\tau, 0], \mathbb{R}_+^2), i = 1, 2$. Thus, we have

Lemma 4.1. *Solutions of system (2.2) with initial conditions (4.1) are positive for all $t \geq 0$.*

Proof. Let $(\tilde{x}(t), \tilde{y}(t))$ be a solution of system (2.2) with initial conditions (4.1). Let us consider $\tilde{y}(t)$ for $t \in [0, \tau]$. It follows from the second equation of system (2.2) that

$$\begin{aligned} \dot{\tilde{y}}(t) &= b\tilde{\phi}_2(t-\tau) + c\tilde{\phi}_1(t-\tau)\tilde{\phi}_2(t-\tau) - d\tilde{x}(t)\tilde{y}(t) - e\tilde{y}(t) \\ &\geq -d\tilde{x}(t)\tilde{y}(t) - e\tilde{y}(t), \text{ for } t \in [0, \tau]. \end{aligned}$$

By comparison argument, it follows that

$$\tilde{y}(t) \geq \tilde{y}(0) \exp\{\int_0^t [-d\tilde{x}(s) - e] ds\} > 0, \text{ for } t \in [0, \tau].$$

In a similar way we treat the intervals $[\tau, 2\tau], [2\tau, 3\tau], \cdots, [n\tau, (n+1)\tau], n \in \mathbb{N}$, thus, $\tilde{y}(t) \geq 0$ for $t \geq 0$. On the other hand, it follows from the first equation of system (2.2) that

$$\tilde{x}(t) = \tilde{x}(0) \exp\{\int_0^t [\tilde{y}(s) - a] ds\} > 0, \text{ for } t \geq 0.$$

This completes the proof.

Lemma 4.2. *If $d \neq c$ and $(b-e)/(d-c) > 0$, then all the nontrivial p-periodic solutions of system (2.2) are uniformly bounded, where p is an arbitrary bounded constant.*

Proof. Let $\tilde{x}(t) = e^{v_1(t)}, \tilde{y}(t) = e^{v_2(t)}$, then system (2.2) can be rewritten as

$$\begin{cases} \dot{v}_1(t) = -a + e^{v_2(t)}, \\ \dot{v}_2(t) = be^{v_2(t-\tau)-v_2(t)} + ce^{v_1(t-\tau)+v_2(t-\tau)-v_2(t)} - de^{v_1(t)} - e. \end{cases} \tag{4.2}$$

To prove that all nontrivial periodic solutions of system (2.2) are uniformly bounded, it suffices to prove that all periodic solutions of system (4.2) are uniformly bounded. To achieve this goal, let $v(t) = (v_1(t), v_2(t))$ be any p-periodic solution of system (4.2). Note that $\int_0^p \dot{v}_i(t) dt = 0, i = 1, 2$ due to the p-periodicity of $v(t)$. Integrating (4.2) over $[0, p]$ gives

$$\int_0^p e^{v_2(t)} dt = ap, \tag{4.3}$$

and

$$\int_0^p \Big[be^{v_2(t-\tau)-v_2(t)} + ce^{v_1(t-\tau)+v_2(t-\tau)-v_2(t)}\Big] dt = d\int_0^p e^{v_1(t)} dt + ep. \tag{4.4}$$

It follows from the first equation of system (4.2) and (4.3) that

$$\int_0^p \Big|\dot{v}_1(t)\Big| dt \leq \Big|\int_0^p e^{v_2(t)} dt\Big| + ap = 2ap. \tag{4.5}$$

Similarly, it follows from the second equation of system (4.2) and (4.4) that

$$\begin{aligned} \int_0^p \big|\dot{v}_2(t)\big| dt &\leq \Big|\int_0^p \Big[be^{v_2(t-\tau)-v_2(t)} + ce^{v_1(t-\tau)+v_2(t-\tau)-v_2(t)}\Big] dt\Big| + \Big|d\int_0^p e^{v_1(t)} dt\Big| + ep \\ &\leq 2\Big(d\int_0^p e^{v_1(t)} dt + ep\Big). \end{aligned} \tag{4.6}$$

Noticing that $\int_0^p e^{v_2(t)}\dot{v}_2(t)dt = 0$. Multiplying (4.2) by $e^{v_2(t)}$ on both sides of the second equation of (4.2) and integrating over $[0,p]$, we have

$$b\int_0^p e^{v_2(t-\tau)}dt + c\int_0^p e^{v_1(t-\tau)+v_2(t-\tau)}dt = d\int_0^p e^{v_1(t)+v_2(t)}dt + e\int_0^p e^{v_2(t)}dt. \tag{4.7}$$

By the p-periodicity of $v(t)$, it is easy to show that

$$\int_0^p e^{v_2(t-\tau)}dt = \int_0^p e^{v_2(t)}dt, \quad \int_0^p e^{v_1(t-\tau)+v_2(t-\tau)}dt = \int_0^p e^{v_1(t)+v_2(t)}dt. \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$b\int_0^p e^{v_2(t)}dt + c\int_0^p e^{v_1(t)+v_2(t)}dt = d\int_0^p e^{v_1(t)+v_2(t)}dt + e\int_0^p e^{v_2(t)}dt,$$

which implies

$$(d-c)\int_0^p e^{v_1(t)} \cdot e^{v_2(t)}dt = (b-e)\int_0^p e^{v_2(t)}dt. \tag{4.9}$$

Since $v(t) = (v_1(t), v_2(t))$ is p-periodic, there exist $\xi_i, \eta_i \in [0,p], (i = 1,2)$ such that

$$v_i(\xi_i) = \min_{t\in[0,p]} v_i(t), \quad v_i(\eta_i) = \max_{t\in[0,p]} v_i(t), \quad i = 1,2. \tag{4.10}$$

It follows from (4.9) and (4.10) that

$$(d-c)e^{v_1(\xi_1)}\int_0^p e^{v_2(t)}dt \le (b-e)\int_0^p e^{v_2(t)}dt.$$

Since $d \neq c$ and $(b-e)/(d-c) > 0$, it follows that

$$v_1(\xi_1) \le \ln\Big\{\frac{b-e}{d-c}\Big\}. \tag{4.11}$$

Similarly, we have

$$v_1(\eta_1) \ge \ln\Big\{\frac{b-e}{d-c}\Big\}. \tag{4.12}$$

Thus, it follows from (4.5) and (4.11) that

$$v_1(t) \le v_1(\xi_1) + \int_0^p \Big|\dot{v}_1(t)\Big|dt \le \ln\Big\{\frac{b-e}{d-c}\Big\} + 2ap, \tag{4.13}$$

and

$$v_1(t) \ge v_1(\eta_1) + \int_0^p \Big|\dot{v}_1(t)\Big|dt \ge \ln\Big\{\frac{b-e}{d-c}\Big\} - 2ap. \tag{4.14}$$

From (4.13) and (4.14), we obtain

$$\Big|v_1(t)\Big| \le \max\Big\{\ln\Big\{\Big|\frac{b-e}{d-c}\Big\} + 2ap\Big|, \Big|\frac{b-e}{d-c}\Big\} - 2ap\Big|\Big\} := B_1. \tag{4.15}$$

On the other hand, it follows from (4.6) and (4.15) that

$$\int_0^p \big|\dot{v}_2(t)\big|dt \le 2\Big(d\int_0^p e^{v_1(t)}dt + ep\Big) \le 2(de^{B_1}+e)p. \tag{4.16}$$

Moreover, from (4.3) and (4.10), it is easy to see that

$$e^{v_1(\xi_2)}p \le ap, \text{ or } v_1(\xi_1) \le \ln a, \tag{4.17}$$

and

$$e^{v_1(\eta_2)}p \ge ap, \text{ or } v_1(\eta_1) \ge \ln a. \tag{4.18}$$

Thus, it follows from (4.16) and (4.17) that

$$v_2(t) \le v_2(\xi_2) + \int_0^p \Big|\dot{v}_2(t)\Big|dt \le \ln a + 2(de^{B_1}+e)p, \tag{4.19}$$

and

$$v_2(t) \ge v_2(\eta_1) + \int_0^p \Big|\dot{v}_2(t)\Big|dt \ge \ln a - 2(de^{B_1}+e)p. \tag{4.20}$$

From (4.19) and (4.20), we obtain

$$\Big|v_2(t)\Big| \le \max\Big\{\ln\Big\{\Big|\ln a + 2(de^{B_1}+e)p\Big|, \Big|\ln a - 2(de^{B_1}+e)p\Big|\Big\} := B_2. \tag{4.21}$$

Taking $B_0 = \max\{B_1, B_2\}$, we obtain $\|v(t)\| \le B_0$, for any p-periodic solution $v(t)$. Although the boundaries of $v(t)$ depend on the value of p, p is bounded. Therefore, all periodic solutions of (4.2) are uniformly bounded. Consequently, all periodic solution of (2.2) are uniformly bounded. This ends the proof of Lemma 4.2.

Lemma 4.3. *If $d \ne c$ and $(b-e)/(d-c) > 0$, system (2.2) has no nontrivial τ-periodic solution.*

Proof. To the contrary, suppose that system (2.2) has a τ-periodic solution. Then the system of ordinary differential equations

$$\begin{cases} \dot{x}(t) &= x(t)[y(t)-a], \\ \dot{y}(t) &= y(t)[(b-e)-(d-c)x(t)], \end{cases} \tag{4.22}$$

has a periodic solution. System (4.22) has the same equilibria as system (2.2), which are

$$z_0 = (0,0), \;\; z_* = (x_*, y_*) = (\frac{b-e}{d-c}, a).$$

Note that x-axis and y-axis are the invariable manifold of system (4.22) and the orbits of system (4.22) do not intersect each other. Thus, there are no solutions crossing the coordinate axes. On the other hand, note the fact that if system (4.22) has a periodic solution, then there must be an equilibrium in its interior, and that z_0 is located on the origin. Thus, we conclude that the periodic orbit of system (4.22) must lie in the first quadrant. It is well known that the positive equilibrium z_* is globally asymptotically stable in the first quadrant. Thus, there is no periodic orbit in the first quadrant too. The above discussion means that (4.22) has no nontrivial periodic solution. It is a contradiction. Therefore, Lemma 4.3 is proved.

Below, we follow closely the notations in [50]. For simplification of notations, setting $z_t = (x_t, y_t)$, we may rewrite systems (2.2) as the following functional differential equation

$$z(t) = F(z_t, \tau, p), \tag{4.23}$$

where $z_t(\theta) = z(t+\theta) \in \mathcal{C}([-\tau, 0], \mathbb{R}^2)$. From the discussion in Section 2, we know that system (4.23) has two equilibria $z_0 = E_0(0,0)$ and $z_* = E_*(x_*, y_*)$. If $d \neq c$ and $(b-e)/(d-c) > 0$, z_* is the unique positive equilibrium. In order to apply Theorem 3.3 in [50], define

$$\begin{aligned} \mathbf{X} &= \mathcal{C}([-\tau, 0], \mathbb{R}^2), \\ \mathbf{\Sigma} &= \mathcal{C}l\{(z, \tau, p) \in \mathbf{X} \times \mathbb{R} \times \mathbb{R}^+ : z \text{ is a } p\text{-periodic solution of (4.23)}\} \\ \mathbf{N} &= \{(z, \tau, p) : F(z_t, \tau, p) = 0\}, \end{aligned}$$

and let $\ell(z, \tau_j^+, 2\pi/\omega_+)$ denote the connected component of $(z, \tau_j^+, 2\pi/\omega_+)$ in $\mathbf{\Sigma}$, τ_j^+ and ω_+ are defined by (2.12) and (2.13), respectively.

Theorem 4.4. *If $d > c$ and $b > e$, then for each $\tau > \tau_j^+$, $(j = 0, 1, 2, \cdots, k-1)$, system (2.2) has at least $k-1$ periodic solutions.*

Proof. It suffices to prove that the projection of $\ell(z, \tau_j^+, 2\pi/\omega_+)$ onto τ-space is $[\tilde{\tau}, \infty)$ for each $j \geq 1$, where $\tilde{\tau} \leq \tau_j^+$.

The characteristic matrix of (4.23) at an equilibrium $\tilde{z} = (\tilde{x}, \tilde{y}) \in \mathbb{R}^2$ takes the following form:

$$\Delta_{(\tilde{z}, \tau, p)}(\lambda) = \lambda I - DF(\tilde{z}_t, \tau, p)(e^{\lambda} I),$$

where I is an identity matrix. Consequently, the characteristic equation of (4.23) at $\tilde{z}$ is

$$\lambda^2 + \bar{p}\lambda + \bar{r} + (\bar{s}\lambda + \bar{q})e^{-\lambda\tau}, \tag{4.24}$$

where $\bar{p} = d\tilde{x} + e - \tilde{y} + a$, $\bar{r} = d\tilde{x}\tilde{y} - (d\tilde{x} + e)(\tilde{y} - a)$, $\bar{s} = -(b + c\tilde{x})$, $\bar{q} = -[c\tilde{x}\tilde{y} - (b + c\tilde{x})(\tilde{y} - a)]$. $(\tilde{z}, \tilde{\tau}, \tilde{p})$ is called a center if $F(\tilde{z}, \tilde{\tau}, \tilde{p}) = 0$ and $\det \Delta_{(\tilde{z}, \tau, \tilde{p})}(i\frac{2\pi}{p}) = 0$. A center $(\tilde{z}, \tilde{\tau}, \tilde{p})$ is said to be isolated if it is the only center in some neighborhood of $(\tilde{z}, \tilde{\tau}, \tilde{p})$.

If $\tilde{z} = z_0$, Eq. (4.24) reduces to

$$(\lambda + a)(\lambda + e - be^{-\lambda\tau}) = 0. \tag{4.25}$$

Obviously, $\lambda_1 = -a$ is a negative root of (4.25). Suppose that $i\omega$ is a purely imaginary root of $\tilde{F}(\lambda) = 0$. Rewrite $\tilde{F}(\lambda) = 0$ in terms of its real and imaginary part as

$$\left\{ \begin{aligned} e - b\cos(\omega\tau) &= 0, \\ \omega + b\sin(\omega\tau) &= 0, \end{aligned} \right. \tag{4.26}$$

which implies

$$\omega^2 = b^2 - e^2.$$

Therefore, if $b < e$, then $\omega^2 < 0$. Consequently, Eq. (4.25) has no any purely imaginary roots. Under the assumption in Theorem 4.4, system (4.23) has no any center of the form (z_0, τ, p).

On the other hand, from the discussion about the local Hopf bifurcation in Section 2, it is easy to verify that $(z_*, \tau_j^+, \frac{2\pi}{\omega_+})$ is also a isolated center. By Theorem 2.6, there exist $\varepsilon > 0$, $\delta > 0$ and a smooth curve $\lambda : (\tau_j^+ - \delta, \tau_j^+ + \delta) \to \mathcal{C}$ such that $\det(\Delta(\lambda(\tau)))) = 0,\ |\lambda(\tau) - \omega_+| < \varepsilon$ for all $\tau \in [\tau_j^+ - \delta, \tau_j^+ + \delta]$ and

$$\lambda(\tau_j^+) = i\omega_+, \quad \frac{dRe\lambda(\tau)}{d\tau}\Big|_{\tau=\tau_j^+} > 0.$$

Let

$$\Omega_{\varepsilon, 2\pi/\omega_+} = \{(\eta, p) : 0 < \eta < \varepsilon, |p - 2\pi/\omega_+| < \varepsilon\}.$$

It is easy to verify that on $[\tau_j^+ - \delta, \tau_j^+ + \delta] \times \Omega_{\varepsilon, 2\pi/\omega_+}$,

$$\det(\Delta_{(z_*, \tau, p)}(\eta + 2\pi i/p)) = 0 \text{ if and only if } \eta = 0, \tau = \tau_j^+, \text{ and } p = 2\pi/\omega_+.$$

Therefore, the hypotheses $(A_1) - (A_4)$ in [50] are satisfied. Moreover, if we define

$$H^{\pm}(z_*, \tau, 2\pi/\omega_+)(\eta, p) = \det(\Delta_{(z_*, \tau_j^+ \pm \delta, p)}(\eta + 2\pi i/p)),$$

then we have the crossing number of isolated center $(z_*, \tau_j^+, 2\pi/\omega_+)$ as follows:

$$\begin{aligned} \gamma(z_*, \tau_j^+, 2\pi/\omega_+) &= \deg_B(H^-(z_*, \tau_j^+, 2\pi/\omega_+), \Omega_{\varepsilon, 2\pi/\omega_+}) \\ &\quad - \deg_B(H^+(z_*, \tau_j^+, 2\pi/\omega_+), \Omega_{\varepsilon, 2\pi/\omega_+}) = -1. \end{aligned}$$

Thus, we have

$$\sum_{(\tilde{z}, \tilde{\tau}, \tilde{p}) \in \ell(z_*, \tau_j^+, 2\pi/\omega_+)} \gamma(\tilde{z}, \tilde{\tau}, \tilde{p}) < 0,$$

where $(\tilde{z}, \tilde{\tau}, \tilde{p})$, in fact, takes the form of $(z_*, \tau_j^+, 2\pi/\omega_+)$ $j = 0, 1, 2, \cdots$. It follows from Theorem 3.3 in [50] that the connected component $\ell(z_*, \tau_j^+, 2\pi/\omega_+)$ through $(z_*, \tau_j^+, 2\pi/\omega_+)$in $\mathbf{\Sigma}$ is unbounded. From (2.13), we have

$$\tau_j^+ = \frac{1}{\omega_+} \arccos\left\{\frac{h(q - \omega_+^2) + mg\omega_+^2}{g^2\omega_+^2 + h^2}\right\} + \frac{2j\pi}{\omega_+}, \ j = 0, 1, 2, \cdots.$$

Thus, when $j > 0$, we have $2\pi/\omega_+ < \tau_j^+$.

Now we prove that the projection of $\ell(z_*, \tau_j^+, 2\pi/\omega_+)$ onto the τ-space is $[\tilde{\tau}, \infty)$, where $\tilde{\tau} \le \tau_j^+$. Clearly, it follows from the proof of Lemma 4.3 that system (2.2) with $\tau = 0$ has no nontrivial periodic solution. Hence, the projection of $\ell(z_*, \tau_j^+, 2\pi/\omega_+)$ onto the τ-space is away from zero.

For a contradiction, we suppose that the projection of $\ell(z_*, \tau_j^+, 2\pi/\omega_+)$ onto the τ-space is bounded. This means that the projection of $\ell(z_*, \tau_j^+, 2\pi/\omega_+)$ onto the τ-space is included in interval $(0, \tau_*)$. Noticing that $2\pi/\omega_+ < \tau_j^+$ and applying Lemma 4.3, we have $0 < p < \tau_*$ for $(z(t), \tau, p)$ belonging to $\ell(z_*, \tau_j^+, 2\pi/\omega_+)$. This implies that the projection of $\ell(z_*, \tau_j^+, 2\pi/\omega_+)$ onto the p-space is bounded. Then, applying Lemma 4.2 we get that the connected component $\ell(z_*, \tau_j^+, 2\pi/\omega_+)$ is bounded. This contradiction completes the proof.

5 Example and simulations

Example 5.1 Consider the following system:

$$\begin{cases} \tilde{x}'(t) &=& \tilde{x}(t)\tilde{y}(t) - \dfrac{1}{2}\tilde{x}(t), \\ \tilde{y}'(t) &=& \tilde{y}(t-\tau) + \tilde{x}(t-\tau)\tilde{y}(t-\tau) - 2\tilde{x}(t)\tilde{y}(t) - \dfrac{1}{2}\tilde{y}(t), \end{cases} \tag{5.1}$$

Simple computation shows that system (5.1) has a positive equilibrium $E_* = (0.5, 0.5)$. And $p = 3/2$, $s = -3/2$, $r = 1/2$ and $q = -1/4$. So, it is the critical case in view of $p + s = 0$. By the formula (2.13), we can determine that $\tau_0^+ = 3.6276$, $\tau_0^- = 4.9962$, $\cdots$. Thus, the positive equilibrium E_* switches from instability to stability (see Fig. 1 and Fig 3). The bifurcation occurring at critical value τ_0^+ takes place when τ crosses τ_0^+ to the left (see Fig. 2).

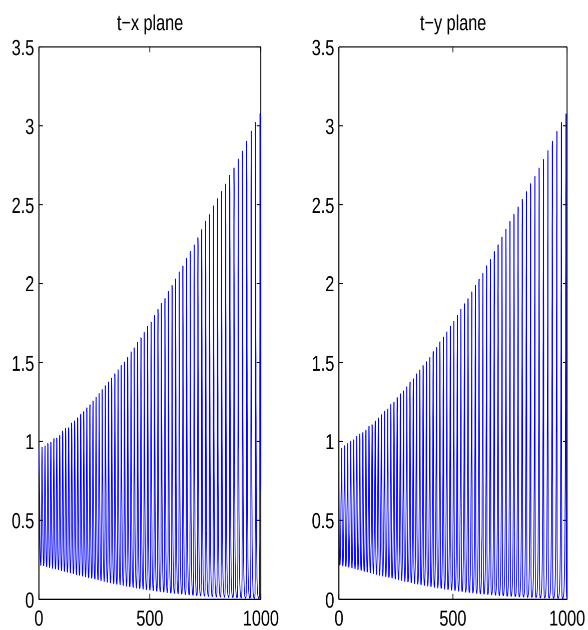

Figure 1: When $0 < \tau = 0.0001 < \tau_0^+$, the positive equilibrium E_* is unstable.

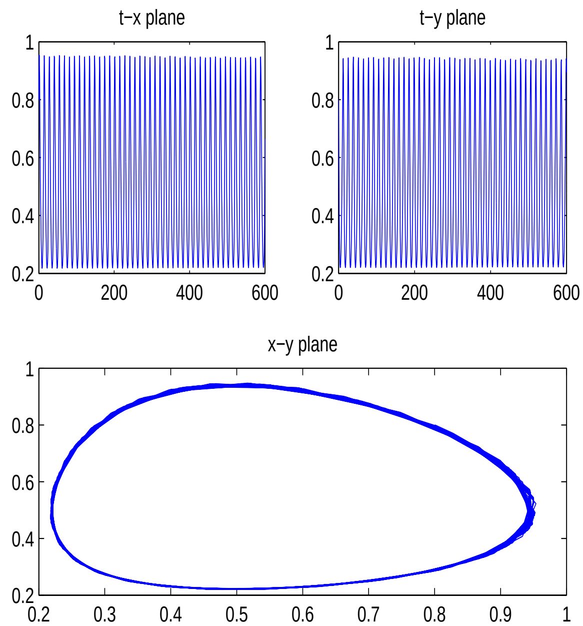

Figure 2: When $0 < \tau = 0.015 < \tau_0^+$, the bifurcating periodic solution from E_* occurs.

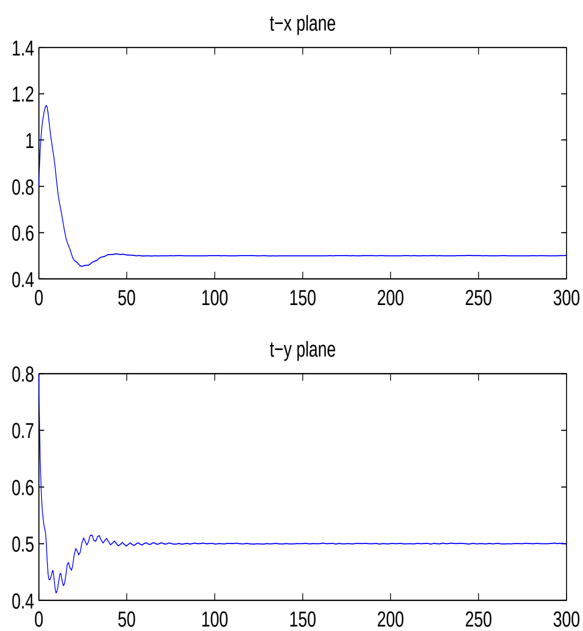

Figure 3: When $\tau = 3.8 > \tau_0^+$, the positive equilibrium E_* is asymptotically stable.

Acknowledgement

This first author was supported by NNSFC under Grants (No.11271333 and No. 10901140) and the Slovene Human Resources Development and Scholarship Fund, and the second author acknowledges the support by the Slovenian Research Agency. The authors would like to express their gratitude to the editors and anonymous referees for their valuable suggestions which helped to improve the presentation of this paper.

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