OSCILLATION OF SECOND ORDER NONLINEAR MIXED NEUTRAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

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Abstract. In this work, some new oscillation criteria are established for a second-order nonlinear mixed neutral differential equation with distributed deviating arguments. Several examples are also provided to illustrate these results.

1. Introduction

In this work, we are concerned with oscillatory behavior of a second-order nonlinear neutral differential equation of the form

$$\begin{aligned}&\left(r(t)\left\{[x(t)+p_1(t)x(t-\sigma_1)+p_2(t)x(t+\sigma_2)]'\right\}^{\gamma}\right)'\\ &+\int_a^b q_1(t,\xi)x^{\gamma}(t-\xi)\mathrm{d}\xi+\int_a^b q_2(t,\xi)x^{\gamma}(t+\xi)\mathrm{d}\xi=0,\end{aligned}\tag{1.1}$$

where $t \geq t_0 > 0$ and $\gamma \geq 1$ is the quotient of odd positive integers. Throughout, we will assume that:

(H_1) r, $p_i \in \mathrm{C}(\mathbb{I},\mathbb{R})$, $r(t) > 0$, and $0 \leq p_i(t) \leq a_i$ for $i = 1, 2$, $\mathbb{I} = [t_0,\infty)$, where a_i are constants;

(H_2) $q_i \in \mathrm{C}(\mathbb{I} \times [a, b], [0,\infty))$ and $q_i(t, \xi)$ is not eventually zero on any half line $[t_\mu,\infty) \times [a, b]$, $t_\mu \geq t_0$, for $i = 1, 2$;

(H_3) $\sigma_i \geq 0$ are constants for $i = 1, 2$, and the integral of equation (1.1) is in the sense of Riemann–Stieltijes.

We set $z(t) := x(t) + p_1(t)x(t - \sigma_1) + p_2(t)x(t + \sigma_2)$. By a solution of (1.1) we mean a nontrivial real-valued function x which has the properties $z \in \mathrm{C}^1([T_x,\infty),\mathbb{R})$ and $r(z')^{\gamma} \in \mathrm{C}^1([T_x,\infty),\mathbb{R})$ for some $T_x \geq t_0$ and satisfying (1.1) on $[T_x,\infty)$. We restrict our attention to those solutions x of equation (1.1) which exist on some half linear $[T_x,\infty)$ and satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq T_x$. As is customary, a solution of (1.1) is called oscillatory if it has arbitrarily

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large zeros on $[t_0, \infty)$; otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory.

Neutral functional differential equations have numerous applications in electric networks. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines which rise in high speed computers where the lossless transmission lines are used to interconnect switching circuits; see [19].

Recently, many results on oscillation of nonneutral differential equations and neutral functional differential equations have been established. We refer the reader to [1–18, 30] and [20–29, 31–38], and the references cited therein. Philos [27] established some Philos-type oscillation criteria for a second-order linear differential equation

$$(r(t)x'(t))' + q(t)x(t) = 0.$$

In [1,2,14,30], the authors gave some sufficient conditions for oscillation of all solutions of a second-order half-linear differential equation

$$(r(t)|x'(t)|^{\gamma-1}x'(t))' + q(t)|x(\tau(t))|^{\gamma-1}x(\tau(t)) = 0$$

by using the Riccati substitution technique. Džurina [10] presented some sufficient conditions for oscillation of a second-order differential equation with mixed arguments

$$(r(t)x'(t))' + p(t)x(\tau(t)) + q(t)x(\sigma(t)) = 0.$$

Some oscillation criteria for the following second-order neutral differential equation

$$(r(t)|z'(t)|^{\gamma-1}z'(t))' + q(t)|x(\sigma(t))|^{\gamma-1}x(\sigma(t)) = 0,$$

where $z := x + px \circ \tau$ were obtained by several authors. Džurina et al. [12] established some criteria for the following mixed neutral equation

$$(x(t) + p_1x(t-\tau_1) + p_2x(t+\tau_2))'' = q_1(t)x(t-\sigma_1) + q_2(t)x(t+\sigma_2),$$

where q_1 and q_2 are nonnegative real-valued functions. Grace [16] obtained some theorems for an odd-order neutral differential equation

$$(x(t) + p_1x(t-\tau_1) + p_2x(t+\tau_2))^{(n)} = q_1x(t-\sigma_1) + q_2x(t+\sigma_2).$$

Wang [32] studied a second-order differential equation

$$(r(t)(x(t) + p(t)x(t-\tau))')' + \int_a^b q(t,\xi)x(g(t,\xi))\mathrm{d}\sigma(\xi) = 0$$

in the case

$$\int_{t_0}^{\infty} \frac{\mathrm{d}t}{r(t)} = \infty.$$

Yan [36] considered an even-order mixed neutral differential equation

$$(x(t) - c_1 x(t-h_1) - c_2 x(t+h_2))^{(n)} + qx(t-g_1) + px(t+g_2) = 0,$$

where c_1 and c_2 are nonnegative, p and q are positive real numbers. Yu and Fu [37] considered a second-order differential equation

$$(x(t) + p(t)x(t-\tau))'' + \int_a^b q(t,\xi)x(g(t,\xi))\mathrm{d}\sigma(\xi) = 0.$$

Thandapani and Piramanantham [31], Xu and Weng [35], Zhao and Meng [38] examined an equation

$$(r(t)(x(t) + p(t)x(t-\tau))')' + \int_a^b q(t,\xi)f\big(x(g(t,\xi))\big)\mathrm{d}\sigma(\xi) = 0.$$

As yet, there are few results on oscillation of mixed neutral differential equations with distributed deviating arguments. Candan [5] considered an odd-order mixed neutral differential equation with distributed deviating arguments

$$[x(t) \pm ax(t \pm h) \pm bx(t \pm g)]^{(n)} = p\int_c^d x(t-\xi)\mathrm{d}\xi + q\int_c^d x(t+\xi)\mathrm{d}\xi,$$

where a, h, b, g, p, c, d, and q are constants and $0 < c < d$. Candan [6] examined an even-order equation

$$[x(t) + \lambda ax(t+\alpha h) + \mu bx(t+\beta g)]^{(n)} = p\int_c^d x(t-\xi)\mathrm{d}\xi + q\int_c^d x(t+\xi)\mathrm{d}\xi.$$

Candan and Dahiya [7] studied the following equation

$$\left[x(t) + h\int_a^b x(t-\xi)\mathrm{d}\xi + g\int_a^b x(t+\xi)\mathrm{d}\xi\right]^{(n)} = p\int_c^d x(t-\nu)\mathrm{d}\nu + q\int_c^d x(t+\nu)\mathrm{d}\nu$$

and

$$\left[x(t) + h\int_a^b x(t-\xi)\mathrm{d}\xi + g\int_a^b x(t+\xi)\mathrm{d}\xi\right]^{(n)} = px(t-\tau) + qx(t+\nu).$$

Candan and Dahiya [8] investigated the following equation

$$[x(t) + \lambda ax(t+\alpha h) + \mu bx(t+\beta g)]^{(n)} + p\int_c^d x(t-\xi)\mathrm{d}\xi + q\int_c^d x(t+\xi)\mathrm{d}\xi = 0.$$

Motivated by the above work, the objective of this paper is to study oscillation problem of (1.1) in the cases

$$\int_{t_0}^{\infty} \frac{\mathrm{d}t}{r^{1/\gamma}(t)} = \infty \tag{1.2}$$

and

$$\int_{t_0}^{\infty} \frac{\mathrm{d}t}{r^{1/\gamma}(t)} < \infty. \tag{1.3}$$

The organization of this paper is as follows: In Sect. 2, by using Riccati substitution technique, some oscillation criteria are obtained for (1.1). In Sect. 3, three examples are included to illustrate the main results.

In what follows, all functional inequalities without specifying its domain of validity are assumed to hold for all sufficiently large t.

2. Main results

In order to prove main theorems, we need the following auxiliary result.

Lemma 2.1 (See [4, Lemma 2.5]). Assume $\gamma \geq 1$, x_1 and $x_2 \in \mathbb{R}$. If $x_1 \geq 0$ and $x_2 \geq 0$, then

$$x_1{}^{\gamma} + x_2{}^{\gamma} \geq \frac{1}{2^{\gamma-1}}(x_1 + x_2)^{\gamma}.$$

Below, we use the notation

$$\begin{aligned} \tilde{Q}(t) &:= \int_a^b Q(t,\xi)\mathrm{d}\xi, \quad Q(t,\xi) := Q_1(t,\xi) + Q_2(t,\xi), \\ Q_1(t,\xi) &:= \min\{q_1(t,\xi), q_1(t-\sigma_1,\xi), q_1(t+\sigma_2,\xi)\}, \\ Q_2(t,\xi) &:= \min\{q_2(t,\xi), q_2(t-\sigma_1,\xi), q_2(t+\sigma_2,\xi)\}, \\ (\rho'(t))_+ &:= \max\{0, \rho'(t)\}, \quad \delta(t) := \int_{t+b}^{\infty} \frac{\mathrm{d}s}{r^{1/\gamma}(s)}, \end{aligned}$$

and

$$\zeta(t) := \delta(t+\sigma_2) \quad \text{for} \quad (t,\xi) \in \mathbb{I} \times [a,b].$$

Theorem 2.2. *Suppose* (1.2) *holds and* $a + b \geq 0$*,* $b \geq \sigma_1$*. Assume also that there exists* $\rho \in \mathrm{C}^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\rho(s)\frac{\tilde{Q}(s)}{(2^{\gamma-1})^2} - \frac{[1 + a_1{}^{\gamma} + a_2{}^{\gamma}/2^{\gamma-1}]r(s-b)((\rho'(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\rho^{\gamma}(s)}\right]\mathrm{d}s = \infty. \tag{2.1}$$

Then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(t-\sigma_1) > 0$, $x(t+\sigma_2) > 0$, $x(t-\xi) > 0$, and $x(t+\xi) > 0$ for all $t \geq t_1$, $\xi \in [a,b]$. Then $z(t) > 0$ for $t \geq t_1$. In view of (1.1), we have

$$\begin{aligned}(r(t)(z'(t))^\gamma)' = & - \int_a^b q_1(t,\xi)x^\gamma(t-\xi)\mathrm{d}\xi \\ & - \int_a^b q_2(t,\xi)x^\gamma(t+\xi)\mathrm{d}\xi \leq 0, \quad t \geq t_1.\end{aligned} \tag{2.2}$$

Thus, $r(z')^\gamma$ is nonincreasing. By virtue of (1.2), there exists a $t_2 \geq t_1$ such that

$$z'(t-\sigma_1) > 0 \quad \text{for} \quad t \geq t_2. \tag{2.3}$$

Using (1.1), for all sufficiently large t, we obtain

$$\begin{aligned}(r(t)(z'(t))^\gamma)' \quad & + \int_a^b q_1(t,\xi)x^\gamma(t-\xi)\mathrm{d}\xi + \int_a^b q_2(t,\xi)x^\gamma(t+\xi)\mathrm{d}\xi \\ & + {a_1}^\gamma(r(t-\sigma_1)(z'(t-\sigma_1))^\gamma)' \\ & + {a_1}^\gamma \int_a^b q_1(t-\sigma_1,\xi)x^\gamma(t-\sigma_1-\xi)\mathrm{d}\xi \\ & + {a_1}^\gamma \int_a^b q_2(t-\sigma_1,\xi)x^\gamma(t-\sigma_1+\xi)\mathrm{d}\xi \\ & + \frac{{a_2}^\gamma}{2^{\gamma-1}}(r(t+\sigma_2)(z'(t+\sigma_2))^\gamma)' \\ & + \frac{{a_2}^\gamma}{2^{\gamma-1}} \int_a^b q_1(t+\sigma_2,\xi)x^\gamma(t+\sigma_2-\xi)\mathrm{d}\xi \\ & + \frac{{a_2}^\gamma}{2^{\gamma-1}} \int_a^b q_2(t+\sigma_2,\xi)x^\gamma(t+\sigma_2+\xi)\mathrm{d}\xi = 0.\end{aligned} \tag{2.4}$$

It follows from Lemma 2.1 and the definition of z that

$$\begin{aligned}
&q_1(t,\xi)x^\gamma(t-\xi)+{a_1}^\gamma q_1(t-\sigma_1,\xi)x^\gamma(t-\sigma_1-\xi)\\
&+\frac{{a_2}^\gamma}{2^{\gamma-1}}q_1(t+\sigma_2,\xi)x^\gamma(t+\sigma_2-\xi)\\
&\geq Q_1(t,\xi)\left[x^\gamma(t-\xi)+{a_1}^\gamma x^\gamma(t-\sigma_1-\xi)+\frac{{a_2}^\gamma}{2^{\gamma-1}}x^\gamma(t+\sigma_2-\xi)\right]\\
&\geq \frac{Q_1(t,\xi)}{2^{\gamma-1}}\left[[x(t-\xi)+a_1x(t-\sigma_1-\xi)]^\gamma+{a_2}^\gamma x^\gamma(t+\sigma_2-\xi)\right]\\
&\geq \frac{Q_1(t,\xi)}{(2^{\gamma-1})^2}\left[x(t-\xi)+a_1x(t-\sigma_1-\xi)+a_2x(t+\sigma_2-\xi)\right]^\gamma\\
(2.5)\ &\geq \frac{Q_1(t,\xi)}{(2^{\gamma-1})^2}z^\gamma(t-\xi).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&q_2(t,\xi)x^\gamma(t+\xi)+{a_1}^\gamma q_2(t-\sigma_1,\xi)x^\gamma(t-\sigma_1+\xi)\\
&+\frac{{a_2}^\gamma}{2^{\gamma-1}}q_2(t+\sigma_2,\xi)x^\gamma(t+\sigma_2+\xi)\geq\frac{Q_2(t,\xi)}{(2^{\gamma-1})^2}z^\gamma(t+\xi).
\end{aligned}\tag{2.6}$$

Hence by (2.4), (2.5), and (2.6), we find

$$\begin{aligned}
&(r(t)(z'(t))^\gamma)'+{a_1}^\gamma(r(t-\sigma_1)(z'(t-\sigma_1))^\gamma)'\\
&+\frac{{a_2}^\gamma}{2^{\gamma-1}}(r(t+\sigma_2)(z'(t+\sigma_2))^\gamma)'\\
&+\frac{1}{(2^{\gamma-1})^2}\int_a^b\left[Q_1(t,\xi)z^\gamma(t-\xi)+Q_2(t,\xi)z^\gamma(t+\xi)\right]\mathrm{d}\xi\leq 0.
\end{aligned}\tag{2.7}$$

From $z'>0$ and $a+b\geq 0$, we obtain

$$\begin{aligned}
&(r(t)(z'(t))^\gamma)'+{a_1}^\gamma(r(t-\sigma_1)(z'(t-\sigma_1))^\gamma)'\\
&+\frac{{a_2}^\gamma}{2^{\gamma-1}}(r(t+\sigma_2)(z'(t+\sigma_2))^\gamma)'+\frac{\tilde{Q}(t)}{(2^{\gamma-1})^2}z^\gamma(t-b)\leq 0.
\end{aligned}\tag{2.8}$$

Using the Riccati transformation

$$\omega_1(t):=\rho(t)\frac{r(t)(z'(t))^\gamma}{z^\gamma(t-b)},\quad t\geq t_2.\tag{2.9}$$

Then $\omega_1(t)>0$ for $t\geq t_2$. Differentiating (2.9), we obtain

$$\begin{aligned}
\omega_1'(t)\ &=\ \rho'(t)\frac{r(t)(z'(t))^\gamma}{z^\gamma(t-b)}+\rho(t)\frac{(r(t)(z'(t))^\gamma)'}{z^\gamma(t-b)}\\
&\quad -\gamma\rho(t)\frac{r(t)(z'(t))^\gamma z'(t-b)}{z^{\gamma+1}(t-b)}.
\end{aligned}\tag{2.10}$$

By virtue of (2.2), we have $r(t-b)(z'(t-b))^\gamma \geq r(t)(z'(t))^\gamma$. Thus, we get by (2.9) and (2.10) that

$$\omega_1'(t) \leq \frac{(\rho'(t))_+}{\rho(t)}\omega_1(t)+\rho(t)\frac{(r(t)(z'(t))^\gamma)'}{z^\gamma(t-b)}-\gamma\frac{(\omega_1(t))^{(\gamma+1)/\gamma}}{\rho^{1/\gamma}(t)r^{1/\gamma}(t-b)}. \tag{2.11}$$

Next, define function ω_2 by

$$\omega_2(t) := \rho(t)\frac{r(t-\sigma_1)(z'(t-\sigma_1))^\gamma}{z^\gamma(t-b)}, \quad t \geq t_2. \tag{2.12}$$

Then $\omega_2(t) > 0$ for $t \geq t_2$. Differentiating (2.12), we see that

$$\begin{aligned} \omega_2'(t) &= \rho'(t)\frac{r(t-\sigma_1)(z'(t-\sigma_1))^\gamma}{z^\gamma(t-b)} + \rho(t)\frac{(r(t-\sigma_1)(z'(t-\sigma_1))^\gamma)'}{z^\gamma(t-b)} \\ &\quad -\gamma\rho(t)\frac{r(t-\sigma_1)(z'(t-\sigma_1))^\gamma z'(t-b)}{z^{\gamma+1}(t-b)}. \end{aligned} \tag{2.13}$$

Note that $b \geq \sigma_1$. In view of (2.2), we have $r(t-b)(z'(t-b))^\gamma \geq r(t-\sigma_1)(z'(t-\sigma_1))^\gamma$. Hence by (2.12) and (2.13), we have

$$\begin{aligned} \omega_2'(t) &\leq \frac{(\rho'(t))_+}{\rho(t)}\omega_2(t) + \rho(t)\frac{(r(t-\sigma_1)(z'(t-\sigma_1))^\gamma)'}{z^\gamma(t-b)} \\ &\quad -\gamma\frac{(\omega_2(t))^{(\gamma+1)/\gamma}}{\rho^{1/\gamma}(t)r^{1/\gamma}(t-b)}. \end{aligned} \tag{2.14}$$

Below, we define another function ω_3 by

$$\omega_3(t) := \rho(t)\frac{r(t+\sigma_2)(z'(t+\sigma_2))^\gamma}{z^\gamma(t-b)}, \quad t \geq t_2. \tag{2.15}$$

Then $\omega_3(t) > 0$ for $t \geq t_2$. Differentiating (2.15), we obtain

$$\begin{aligned} \omega_3'(t) &= \rho'(t)\frac{r(t+\sigma_2)(z'(t+\sigma_2))^\gamma}{z^\gamma(t-b)} + \rho(t)\frac{(r(t+\sigma_2)(z'(t+\sigma_2))^\gamma)'}{z^\gamma(t-b)} \\ &\quad -\gamma\rho(t)\frac{r(t+\sigma_2)(z'(t+\sigma_2))^\gamma z'(t-b)}{z^{\gamma+1}(t-b)}. \end{aligned} \tag{2.16}$$

From (2.2), we have $r(t-b)(z'(t-b))^\gamma \geq r(t+\sigma_2)(z'(t+\sigma_2))^\gamma$. Then, we have by (2.15) and (2.16) that

$$\begin{aligned} \omega_3'(t) &\leq \frac{(\rho'(t))_+}{\rho(t)}\omega_3(t) + \rho(t)\frac{(r(t+\sigma_2)(z'(t+\sigma_2))^\gamma)'}{z^\gamma(t-b)} \\ &\quad -\gamma\frac{(\omega_3(t))^{(\gamma+1)/\gamma}}{\rho^{1/\gamma}(t)r^{1/\gamma}(t-b)}. \end{aligned} \tag{2.17}$$

Therefore, (2.11), (2.14), and (2.17) imply that

$$\omega_1'(t) + {a_1}^\gamma\omega_2'(t) + \frac{{a_2}^\gamma}{2^{\gamma-1}}\omega_3'(t)$$

$$\leq \rho(t)\left[\frac{(r(t)(z'(t))^{\gamma})' + a_1{}^{\gamma}(r(t-\sigma_1)(z'(t-\sigma_1))^{\gamma})' + \frac{a_2{}^{\gamma}}{2^{\gamma-1}}(r(t+\sigma_2)(z'(t+\sigma_2))^{\gamma})'}{z^{\gamma}(t-b)}\right]$$

$$+\frac{(\rho'(t))_+}{\rho(t)}\omega_1(t) - \gamma\frac{(\omega_1(t))^{(\gamma+1)/\gamma}}{\rho^{1/\gamma}(t)r^{1/\gamma}(t-b)} + a_1{}^{\gamma}\frac{(\rho'(t))_+}{\rho(t)}\omega_2(t) - \gamma a_1{}^{\gamma}\frac{(\omega_2(t))^{(\gamma+1)/\gamma}}{\rho^{1/\gamma}(t)r^{1/\gamma}(t-b)}$$

$$+\frac{a_2{}^{\gamma}}{2^{\gamma-1}}\frac{(\rho'(t))_+}{\rho(t)}\omega_3(t) - \gamma\frac{a_2{}^{\gamma}}{2^{\gamma-1}}\frac{(\omega_3(t))^{(\gamma+1)/\gamma}}{\rho^{1/\gamma}(t)r^{1/\gamma}(t-b)}. \tag{2.18}$$

Thus, we have by (2.8) and (2.18) that

$$\begin{aligned} & \omega_1'(t) + a_1{}^{\gamma}\omega_2'(t) + \frac{a_2{}^{\gamma}}{2^{\gamma-1}}\omega_3'(t) \\ \leq\ & -\rho(t)\frac{\tilde{Q}(t)}{(2^{\gamma-1})^2} + \left[\frac{(\rho'(t))_+}{\rho(t)}\omega_1(t) - \gamma\frac{(\omega_1(t))^{(\gamma+1)/\gamma}}{\rho^{1/\gamma}(t)r^{1/\gamma}(t-b)}\right] \\ & + a_1{}^{\gamma}\left[\frac{(\rho'(t))_+}{\rho(t)}\omega_2(t) - \gamma\frac{(\omega_2(t))^{(\gamma+1)/\gamma}}{\rho^{1/\gamma}(t)r^{1/\gamma}(t-b)}\right] \\ & + \frac{a_2{}^{\gamma}}{2^{\gamma-1}}\left[\frac{(\rho'(t))_+}{\rho(t)}\omega_3(t) - \gamma\frac{(\omega_3(t))^{(\gamma+1)/\gamma}}{\rho^{1/\gamma}(t)r^{1/\gamma}(t-b)}\right]. \end{aligned} \tag{2.19}$$

Then, using (2.19) and inequality

$$Au - Bu^{(\gamma+1)/\gamma} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{A^{\gamma+1}}{B^{\gamma}}, \quad B > 0, \tag{2.20}$$

we find that

$$\begin{aligned} \omega_1'(t) + a_1{}^{\gamma}\omega_2'(t) + \frac{a_2{}^{\gamma}}{2^{\gamma-1}}\omega_3'(t) \ \leq\ & -\rho(t)\frac{\tilde{Q}(t)}{(2^{\gamma-1})^2} \\ & + \frac{[1 + a_1{}^{\gamma} + a_2{}^{\gamma}/2^{\gamma-1}]r(t-b)((\rho'(t))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\rho^{\gamma}(t)}. \end{aligned}$$

Integrating the above inequality from t_2 to t, we obtain

$$\begin{aligned} & \int_{t_2}^{t}\left[\rho(s)\frac{\tilde{Q}(s)}{(2^{\gamma-1})^2} - \frac{[1 + a_1{}^{\gamma} + a_2{}^{\gamma}/2^{\gamma-1}]r(s-b)((\rho'(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\rho^{\gamma}(s)}\right]\mathrm{d}s \\ & \leq \omega_1(t_2) + a_1{}^{\gamma}\omega_2(t_2) + \frac{a_2{}^{\gamma}}{2^{\gamma-1}}\omega_3(t_2), \end{aligned}$$

which contradicts (2.1). The proof is complete. □

As an immediate consequence of Theorem 2.2 we get the following result.

Corollary 2.3. Let assumption (2.1) in Theorem 2.2 be replaced by

$$\limsup_{t\to\infty}\int_{t_0}^{t}\rho(s)\tilde{Q}(s)\mathrm{d}s=\infty$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{r(s-b)\left((\rho'(s))_+\right)^{\gamma+1}}{\rho^{\gamma}(s)}\mathrm{d}s<\infty.$$

Then (1.1) is oscillatory.

From Theorem 2.2 by choosing the function ρ appropriately, one can obtain various classes of different sufficient conditions for oscillation of (1.1). For instance, if we define function ρ by $\rho(t)=1$ and $\rho(t)=t$, respectively, then one has the following results.

Corollary 2.4. Assume (1.2) holds and $a+b\geq 0$, $b\geq\sigma_1$. If

$$\int_{t_0}^{\infty}\tilde{Q}(s)\mathrm{d}s=\infty, \tag{2.21}$$

then (1.1) is oscillatory.

Corollary 2.5. Suppose (1.2) holds and $a+b\geq 0$, $b\geq\sigma_1$. If

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[s\frac{\tilde{Q}(s)}{(2^{\gamma-1})^2}-\frac{[1+{a_1}^{\gamma}+{a_2}^{\gamma}/2^{\gamma-1}]r(s-b)}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right]\mathrm{d}s=\infty, \tag{2.22}$$

then (1.1) is oscillatory.

Theorem 2.6. *Assume* (1.2) *holds and* $a+b\leq 0$, $-a\geq\sigma_1$. *Suppose further that there exists* $\rho\in \mathrm{C}^1([t_0,\infty),(0,\infty))$ *such that*

$$\begin{aligned}\limsup_{t\to\infty}\int_{t_0}^{t}\Bigg[&\rho(s)\frac{\tilde{Q}(s)}{(2^{\gamma-1})^2}\\ &-\frac{[1+{a_1}^{\gamma}+{a_2}^{\gamma}/2^{\gamma-1}]r(s+a)((\rho'(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\rho^{\gamma}(s)}\Bigg]\mathrm{d}s=\infty.\end{aligned} \tag{2.23}$$

Then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1\geq t_0$ such that $x(t)>0$, $x(t-\sigma_1)>0$, $x(t+\sigma_2)>0$, $x(t-\xi)>0$, and $x(t+\xi)>0$ for all $t\geq t_1$, $\xi\in[a,b]$. Then $z(t)>0$ for $t\geq t_1$. Proceeding as in the proof of Theorem 2.2, we have (2.2)–(2.7). By $z'>0$ and $a+b\leq 0$, we obtain

$$\begin{aligned}&(r(t)(z'(t))^{\gamma})'+{a_1}^{\gamma}(r(t-\sigma_1)(z'(t-\sigma_1))^{\gamma})'\\ &+\frac{{a_2}^{\gamma}}{2^{\gamma-1}}(r(t+\sigma_2)(z'(t+\sigma_2))^{\gamma})'+\frac{\tilde{Q}(t)}{(2^{\gamma-1})^2}z^{\gamma}(t+a)\leq 0.\end{aligned} \tag{2.24}$$

Define the functions ω_1, ω_2, and ω_3 by

$$\omega_1(t) := \rho(t)\frac{r(t)(z'(t))^\gamma}{z^\gamma(t-(-a))},$$

$$\omega_2(t) := \rho(t)\frac{r(t-\sigma_1)(z'(t-\sigma_1))^\gamma}{z^\gamma(t-(-a))},$$

and

$$\omega_3(t) := \rho(t)\frac{r(t+\sigma_2)(z'(t+\sigma_2))^\gamma}{z^\gamma(t-(-a))},$$

respectively. The rest of the proof is similar to that of Theorem 2.2. This completes the proof. □

Theorem 2.7. *Suppose* (1.2) *holds and* $a+b \ge 0$, $\sigma_1 \ge b$. *Assume also that there exists* $\rho \in \mathrm{C}^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(s)\frac{\tilde{Q}(s)}{(2^{\gamma-1})^2} - \frac{[1+a_1{}^\gamma+a_2{}^\gamma/2^{\gamma-1}]r(s-\sigma_1)((\rho'(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\rho^\gamma(s)}\right]\mathrm{d}s=\infty. \tag{2.25}$$

Then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$, $x(t-\sigma_1) > 0$, $x(t+\sigma_2) > 0$, $x(t-\xi) > 0$, and $x(t+\xi) > 0$ for all $t \ge t_1$, $\xi \in [a,b]$. Then $z(t) > 0$ for $t \ge t_1$. Proceeding as in the proof of Theorem 2.2, we get (2.2)–(2.7). In view of $z' > 0$ and $a+b \ge 0$, we obtain (2.8). Define the functions ω_1, ω_2, and ω_3 by

$$\omega_1(t) := \rho(t)\frac{r(t)(z'(t))^\gamma}{z^\gamma(t-\sigma_1)},$$

$$\omega_2(t) := \rho(t)\frac{r(t-\sigma_1)(z'(t-\sigma_1))^\gamma}{z^\gamma(t-\sigma_1)},$$

and

$$\omega_3(t) := \rho(t)\frac{r(t+\sigma_2)(z'(t+\sigma_2))^\gamma}{z^\gamma(t-\sigma_1)},$$

respectively. The rest of the proof is similar to that of Theorem 2.2. This completes the proof. □

Theorem 2.8. *Suppose* (1.2) *holds and* $a+b\leq 0$, *and* $\sigma_1\geq -a$. *Assume further that there exists* $\rho\in \mathrm{C}^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(s)\frac{\tilde{Q}(s)}{(2^{\gamma-1})^2} - \frac{[1+a_1{}^{\gamma}+a_2{}^{\gamma}/2^{\gamma-1}]r(s-\sigma_1)((\rho'(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\rho^{\gamma}(s)}\right]\mathrm{d}s=\infty. \tag{2.26}$$

Then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1\geq t_0$ such that $x(t)>0$, $x(t-\sigma_1)>0$, $x(t+\xi)>0$, $x(t-\xi)>0$, and $x(t+\xi)>0$ for all $t\geq t_1$, $\xi\in[a,b]$. Then $z(t)>0$ for $t\geq t_1$. Proceeding as in the proof of Theorem 2.2, we get (2.2)–(2.7). From $z'>0$ and $a+b\leq 0$, we obtain (2.24). Define the functions ω_1, ω_2, and ω_3 as in Theorem 2.7, the remainder of the proof is similar to that of Theorem 2.2. This completes the proof. □

Remark 2.9. From Theorem 2.6–Theorem 2.8, one can obtain some oscillation criteria for (1.1) by choosing different ρ. The details are left to the reader.

Now we establish some oscillation results for (1.1) in the case where (1.3) holds.

Theorem 2.10. *Suppose* (1.3) *holds and* $a+b\geq 0$, $b\geq\sigma_1$. *Assume further that there exists* $\rho\in \mathrm{C}^1([t_0,\infty),(0,\infty))$ *such that* (2.1) *holds. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\zeta^{\gamma}(s)\frac{\tilde{Q}(s)}{(2^{\gamma-1})^2} - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{(1+a_1{}^{\gamma})r(s+b)+\frac{a_2{}^{\gamma}}{2^{\gamma-1}}r(s+\sigma_2+b)}{r^{(\gamma+1)/\gamma}(s+\sigma_2+b)\zeta(s)}\right]\mathrm{d}s=\infty, \tag{2.27}$$

then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1\geq t_0$ such that $x(t)>0$, $x(t-\sigma_1)>0$, $x(t+\sigma_2)>0$, $x(t-\xi)>0$, and $x(t+\xi)>0$ for all $t\geq t_1$, $\xi\in[a,b]$. Then $z(t)>0$ for $t\geq t_1$. In view of (1.1), we obtain that (2.2) holds. From (2.2), we see that $r(z')^{\gamma}$ is nonincreasing and there exist two possible cases for the sign of z'. Assume first that $z'(t-\sigma_1)>0$ for $t\geq t_2\geq t_1$. Then we have that (2.8) holds. Proceeding as in the proof of Theorem 2.2, we can obtain a contradiction to (2.1). Suppose now that $z'(t-\sigma_1) < 0$ for $t \geq t_2 \geq t_1$. We also have (2.7). From $z' < 0$ and $a + b \geq 0$, we have

$$\begin{aligned} &(r(t)(z'(t))^\gamma)' + {a_1}^\gamma (r(t-\sigma_1)(z'(t-\sigma_1))^\gamma)' \\ &+ \frac{{a_2}^\gamma}{2^{\gamma-1}}(r(t+\sigma_2)(z'(t+\sigma_2))^\gamma)' + \frac{\tilde{Q}(t)}{(2^{\gamma-1})^2} z^\gamma(t+b) \leq 0. \end{aligned} \tag{2.28}$$

Define function ω_1 by

$$\omega_1(t) := \frac{r(t)(z'(t))^\gamma}{z^\gamma(t+b)}, \quad t \geq t_2. \tag{2.29}$$

Then $\omega_1(t) < 0$ for $t \geq t_2$. Noting that $r(z')^\gamma$ is nonincreasing, we have

$$z'(s) \leq \frac{r^{1/\gamma}(t) z'(t)}{r^{1/\gamma}(s)}, \quad s \geq t \geq t_2.$$

Integrating this from $t + b$ to l, we obtain

$$z(l) \leq z(t+b) + r^{1/\gamma}(t) z'(t) \int_{t+b}^{l} \frac{\mathrm{d}s}{r^{1/\gamma}(s)}, \quad l \geq t+b.$$

Note that $\lim_{l\to\infty} z(l) \geq 0$. Letting $l \to \infty$ in the above inequality, we have

$$0 \leq z(t+b) + r^{1/\gamma}(t) z'(t) \delta(t), \quad t \geq t_2.$$

Therefore,

$$\frac{r^{1/\gamma}(t) z'(t)}{z(t+b)} \delta(t) \geq -1, \quad t \geq t_2.$$

From (2.29), we have

$$-1 \leq \omega_1(t) \delta^\gamma(t) \leq 0, \quad t \geq t_2. \tag{2.30}$$

By virtue of (2.2), we obtain $z'(t+b) \leq r^{1/\gamma}(t) z'(t)/r^{1/\gamma}(t+b)$. Differentiating (2.29), we get

$$\omega_1'(t) \leq \frac{(r(t)(z'(t))^\gamma)'}{z^\gamma(t+b)} - \gamma \frac{(\omega_1(t))^{(\gamma+1)/\gamma}}{r^{1/\gamma}(t+b)}. \tag{2.31}$$

Next, we introduce another function

$$\omega_2(t) := \frac{r(t-\sigma_1)(z'(t-\sigma_1))^\gamma}{z^\gamma(t+b)}, \quad t \geq t_2. \tag{2.32}$$

Then $\omega_2(t) < 0$ for $t \geq t_2$. Noting that $r(z')^\gamma$ is nonincreasing for $t \geq t_1$, we get $r(t-\sigma_1)(z'(t-\sigma_1))^\gamma \geq r(t)(z'(t))^\gamma$ for $t \geq t_2$. Thus $\omega_2(t) \geq \omega_1(t)$ for $t \geq t_2$. By (2.30), we obtain

$$-1 \leq \omega_2(t) \delta^\gamma(t) \leq 0, \quad t \geq t_2. \tag{2.33}$$

It follows from (2.2) that $z'(t+b) \le r^{1/\gamma}(t-\sigma_1)z'(t-\sigma_1)/r^{1/\gamma}(t+b)$. Differentiating (2.32), we have

$$\omega_2'(t) \le \frac{(r(t-\sigma_1)(z'(t-\sigma_1))^\gamma)'}{z^\gamma(t+b)} - \gamma\frac{(\omega_2(t))^{(\gamma+1)/\gamma}}{r^\gamma(t+b)}. \tag{2.34}$$

Similarly, we introduce substitution

$$\omega_3(t) := \frac{r(t+\sigma_2)(z'(t+\sigma_2))^\gamma}{z^\gamma(t+\sigma_2+b)}, \quad t \ge t_2. \tag{2.35}$$

Then $\omega_3(t) < 0$ for $t \ge t_2$. By the definition of ω_1 and (2.30), we find that $\omega_3(t) = \omega_1(t+\sigma_2)$ and

$$-1 \le \omega_3(t)\delta^\gamma(t+\sigma_2) \le 0, \quad t \ge t_2. \tag{2.36}$$

In view of (2.2), we have $z'(t+\sigma_2+b) \le r^{1/\gamma}(t+\sigma_2)z'(t+\sigma_2)/r^{1/\gamma}(t+\sigma_2+b)$. Differentiating (2.35), we get

$$\begin{aligned}\omega_3'(t) &\le \frac{(r(t+\sigma_2)(z'(t+\sigma_2))^\gamma)'}{z^\gamma(t+\sigma_2+b)} - \gamma\frac{(\omega_3(t))^{(\gamma+1)/\gamma}}{r^{1/\gamma}(t+\sigma_2+b)}\\ &\le \frac{(r(t+\sigma_2)(z'(t+\sigma_2))^\gamma)'}{z^\gamma(t+b)} - \gamma\frac{(\omega_3(t))^{(\gamma+1)/\gamma}}{r^{1/\gamma}(t+\sigma_2+b)}.\end{aligned} \tag{2.37}$$

Note that $\delta(t) \ge \delta(t+\sigma_2)$. Then, we have

$$-1 \le \omega_1(t)\delta^\gamma(t+\sigma_2) \le 0, \quad t \ge t_2 \tag{2.38}$$

and

$$-1 \le \omega_2(t)\delta^\gamma(t+\sigma_2) \le 0, \quad t \ge t_2. \tag{2.39}$$

From (2.31), (2.34), and (2.37), we obtain

$$\begin{aligned}&\omega_1'(t) + a_1{}^\gamma\omega_2'(t) + \frac{a_2{}^\gamma}{2^{\gamma-1}}\omega_3'(t)\\ \le\ &\frac{(r(t)(z'(t))^\gamma)' + a_1{}^\gamma(r(t-\sigma_1)(z'(t-\sigma_1))^\gamma)' + \frac{a_2{}^\gamma}{2^{\gamma-1}}(r(t+\sigma_2)(z'(t+\sigma_2))^\gamma)'}{z^\gamma(t+b)}\\ &-\gamma\frac{(\omega_1(t))^{(\gamma+1)/\gamma}}{r^{1/\gamma}(t+b)} - \gamma a_1{}^\gamma\frac{(\omega_2(t))^{(\gamma+1)/\gamma}}{r^{1/\gamma}(t+b)} - \gamma\frac{a_2{}^\gamma}{2^{\gamma-1}}\frac{(\omega_3(t))^{(\gamma+1)/\gamma}}{r^{1/\gamma}(t+\sigma_2+b)}.\end{aligned} \tag{2.40}$$

Therefore, we have by (2.28) and (2.40) that

$$\begin{aligned}&\omega_1'(t) + a_1{}^\gamma\omega_2'(t) + \frac{a_2{}^\gamma}{2^{\gamma-1}}\omega_3'(t)\\ \le\ & -\frac{\tilde{Q}(t)}{(2^{\gamma-1})^2} - \gamma\frac{(\omega_1(t))^{(\gamma+1)/\gamma}}{r^{1/\gamma}(t+b)}\\ &-\gamma a_1{}^\gamma\frac{(\omega_2(t))^{(\gamma+1)/\gamma}}{r^{1/\gamma}(t+b)} - \gamma\frac{a_2{}^\gamma}{2^{\gamma-1}}\frac{(\omega_3(t))^{(\gamma+1)/\gamma}}{r^{1/\gamma}(t+\sigma_2+b)}.\end{aligned} \tag{2.41}$$

Multiplying (2.41) by $\zeta^\gamma(t)$, and integrating the resulting inequality on $[t_2, t]$ yields

$$\begin{aligned}
\zeta^\gamma(t)\omega_1(t) &- \zeta^\gamma(t_2)\omega_1(t_2) + \gamma\int_{t_2}^t \frac{\zeta^{\gamma-1}(s)\omega_1(s)}{r^{1/\gamma}(s+\sigma_2+b)}\mathrm{d}s \\
&+ \gamma\int_{t_2}^t \frac{\zeta^\gamma(s)(\omega_1(s))^{(\gamma+1)/\gamma}}{r^{1/\gamma}(s+b)}\mathrm{d}s + {a_1}^\gamma\zeta^\gamma(t)\omega_2(t) - {a_1}^\gamma\zeta(t_2)\omega_2(t_2) \\
&+ \gamma {a_1}^\gamma\int_{t_2}^t \frac{\zeta^{\gamma-1}(s)\omega_2(s)}{r^{1/\gamma}(s+\sigma_2+b)}\mathrm{d}s + \gamma {a_1}^\gamma\int_{t_2}^t \frac{\zeta^\gamma(s)(\omega_2(s))^{(\gamma+1)/\gamma}}{r^{1/\gamma}(s+b)}\mathrm{d}s \\
&+ \frac{{a_2}^\gamma}{2^{\gamma-1}}\zeta^\gamma(t)\omega_3(t) - \frac{{a_2}^\gamma}{2^{\gamma-1}}\zeta^\gamma(t_2)\omega_3(t_2) + \gamma\frac{{a_2}^\gamma}{2^{\gamma-1}}\int_{t_2}^t \frac{\zeta^{\gamma-1}(s)\omega_3(s)}{r^{1/\gamma}(s+\sigma_2+b)}\mathrm{d}s \\
&+ \gamma\frac{{a_2}^\gamma}{2^{\gamma-1}}\int_{t_2}^t \frac{\zeta^\gamma(s)(\omega_3(s))^{(\gamma+1)/\gamma}}{r^{1/\gamma}(s+\sigma_2+b)}\mathrm{d}s + \int_{t_2}^t \zeta^\gamma(s)\frac{\tilde{Q}(s)}{(2^{\gamma-1})^2}\mathrm{d}s \le 0.
\end{aligned}$$

From the above inequality and (2.20), we obtain

$$\begin{aligned}
&\int_{t_2}^t \left[\zeta^\gamma(s)\frac{\tilde{Q}(s)}{(2^{\gamma-1})^2} - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{(1+{a_1}^\gamma)r(s+b) + \frac{{a_2}^\gamma}{2^{\gamma-1}}r(s+\sigma_2+b)}{r^{(\gamma+1)/\gamma}(s+\sigma_2+b)\zeta(s)}\right]\mathrm{d}s \\
&\le -[\zeta^\gamma(t)\omega_1(t) + {a_1}^\gamma\zeta^\gamma(t)\omega_2(t) + \frac{{a_2}^\gamma}{2^{\gamma-1}}\zeta^\gamma(t)\omega_3(t)] \le 1 + {a_1}^\gamma + \frac{{a_2}^\gamma}{2^{\gamma-1}}
\end{aligned}$$

due to (2.36), (2.38), and (2.39). This contradicts (2.27) and finishes the proof. □

Theorem 2.11. *Suppose* (1.3) *holds and* $a + b \le 0$, $-a \ge \sigma_1$. *Assume also that there exists* $\rho \in \mathrm{C}^1([t_0,\infty),(0,\infty))$ *such that* (2.23) *holds. If*

$$\begin{aligned}
(2.42)\quad &\limsup_{t\to\infty}\int_{t_0}^t \left[\zeta^\gamma(s)\frac{\tilde{Q}(s)}{(2^{\gamma-1})^2}\right. \\
&\left. - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{(1+{a_1}^\gamma)r(s-a) + \frac{{a_2}^\gamma}{2^{\gamma-1}}r(s+\sigma_2-a)}{r^{(\gamma+1)/\gamma}(s+\sigma_2-a)\zeta(s)}\right]\mathrm{d}s = \infty,
\end{aligned}$$

then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$, $x(t-\sigma_1) > 0$, $x(t+\sigma_2) > 0$, $x(t-\xi) > 0$, and $x(t+\xi) > 0$ for all $t \ge t_1$, $\xi \in [a, b]$. Then $z(t) > 0$ for $t \ge t_1$. In view of (1.1), we obtain that (2.2) holds. From (2.2), we see that $r(z')^\gamma$ is nonincreasing and there exist two possible cases for the sign of z'. Assume first that $z'(t) > 0$, $z'(t-\sigma_1) > 0$, and $z'(t+\sigma_2) > 0$ for $t \ge t_2 \ge t_1$. Then we have that (2.24) holds. Proceeding as in the proof of Theorem 2.6, we can obtain a contradiction to (2.23). Suppose now that $z'(t) < 0$, $z'(t-\sigma_1) < 0$, and $z'(t+\sigma_2) < 0$ for $t \geq t_2 \geq t_1$. We have (2.7). From $z' < 0$ and $a+b \leq 0$, we have

$$\begin{aligned}&(r(t)(z'(t))^\gamma)' + a_1{}^\gamma(r(t-\sigma_1)(z'(t-\sigma_1))^\gamma)'\\&+\frac{a_2{}^\gamma}{2^{\gamma-1}}(r(t+\sigma_2)(z'(t+\sigma_2))^\gamma)' + \frac{\tilde{Q}(t)}{(2^{\gamma-1})^2}z^\gamma(t-a) \leq 0.\end{aligned}$$

Define the functions ω_1, ω_2, and ω_3 by

$$\omega_1(t) := \rho(t)\frac{r(t)(z'(t))^\gamma}{z^\gamma(t+(-a))},$$

$$\omega_2(t) := \rho(t)\frac{r(t-\sigma_1)(z'(t-\sigma_1))^\gamma}{z^\gamma(t+(-a))},$$

and

$$\omega_3(t) := \rho(t)\frac{r(t+\sigma_2)(z'(t+\sigma_2))^\gamma}{z^\gamma(t+(-a))},$$

respectively. The rest of the proof is similar to that of Theorem 2.10. This completes the proof. □

Similar as in the proof of Theorem 2.7 and Theorem 2.10, we give the following criterion for oscillation of (1.1) when conditions (1.3) and $\sigma_1 \geq b$ are satisfied.

Theorem 2.12. *Suppose* (1.3) *holds and* $a+b \geq 0$, $\sigma_1 \geq b$. *Assume also that there exists* $\rho \in \mathrm{C}^1([t_0,\infty),(0,\infty))$ *such that* (2.25) *holds. If* (2.27) *holds, then* (1.1) *is oscillatory.*

Below, similar to the proof of Theorem 2.8 and Theorem 2.11, we present the following criterion for oscillation of (1.1) under the assumptions that (1.3) and $\sigma_1 \geq -a$ hold.

Theorem 2.13. *Suppose* (1.3) *holds and* $a+b \leq 0$, $\sigma_1 \geq -a$. *Assume further that there exists* $\rho \in \mathrm{C}^1([t_0,\infty),(0,\infty))$ *such that* (2.26) *holds. If* (2.42) *holds, then* (1.1) *is oscillatory.*

3. Applications

In the following, we give three examples to illustrate the main results.

Example 3.1. For $t \geq 1$ and $\gamma \geq 1$, consider an equation

$$\begin{aligned}&\big(r(t)\left\{[x(t)+p_1(t)x(t-1)+p_2(t)x(t+\sigma_2)]'\right\}^\gamma\big)'\\&+\int_1^2\frac{\xi}{t}x^\gamma(t-\xi)\mathrm{d}\xi + \int_1^2\frac{\xi}{t}x^\gamma(t+\xi)\mathrm{d}\xi = 0.\end{aligned} \tag{3.1}$$

Assume that (1.2) holds. Let $t_0 = 1$, $\sigma_1 = 1$, $0 \le p_i(t) \le a_i$, $a = 1$, $b = 2$, and $q_i(t, \xi) = \xi/t$ for $i = 1, 2$. Hence by Corollary 2.4, every solution of (3.1) is oscillatory.

Example 3.2. For $t \ge 1$ and $\gamma \ge 1$, consider an equation

$$\begin{aligned}&\left(r(t)\left\{[x(t) + p_1(t)x(t-3) + p_2(t)x(t+\sigma_2)]'\right\}^{\gamma}\right)' \\ &+ \int_{-2}^{-1} \frac{\xi+3}{t} x^{\gamma}(t-\xi)\mathrm{d}\xi + \int_{-2}^{-1} \frac{\xi+3}{t} x^{\gamma}(t+\xi)\mathrm{d}\xi = 0. \qquad (3.2)\end{aligned}$$

Suppose that (1.2) holds. Let $t_0 = 1$, $\sigma_1 = 3$, $0 \le p_i(t) \le a_i$, $a = -2$, $b = -1$, $q_i(t, \xi) = (\xi+3)/t$ for $i = 1, 2$, and $\rho(t) = 1$. Thus by Theorem 2.8, every solution of (3.2) is oscillatory.

Example 3.3. For $t \ge 1$, consider an equation

$$\begin{aligned}&\left(t^2\left[x(t) + x(t-2\pi) + x\left(t + \frac{5\pi}{2}\right)\right]'\right)' \\ &+ \frac{(2-\sqrt{3})t^2 + (4\sqrt{3}+2)t}{\sqrt{3}} \int_{2\pi}^{7\pi/3} x(t-\xi)\mathrm{d}\xi \\ &+ \frac{(2+\sqrt{3})t^2 + (2-4\sqrt{3})t}{\sqrt{3}} \int_{2\pi}^{7\pi/3} x(t+\xi)\mathrm{d}\xi = 0. \qquad (3.3)\end{aligned}$$

One can easily see that condition (1.3) holds. Let $r(t) = t^2$, $p_1(t) = p_2(t) = 1$, $a = 2\pi$, $b = 7\pi/3$, $q_1(t, \xi) = [(2-\sqrt{3})t^2 + (4\sqrt{3}+2)t]/\sqrt{3}$, and $q_2(t, \xi) = [(2+\sqrt{3})t^2 + (2-4\sqrt{3})t]/\sqrt{3}$. It is not difficult to verify that all conditions of Theorem 2.10 hold, and hence every solution of (3.3) is oscillatory. $x(t) = \sin t$ is such a solution.

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