Fractional boundary value problems with integral and anti-periodic boundary conditions

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Abstract

In this paper, we consider a class of boundary value problems of fractional differential equations with integral and anti-periodic boundary conditions, which is a new type of mixed boundary condition. By using the contraction mapping principle, Krasnosel'skii fixed point theorem, and Leray-Schauder degree theory, we obtain some results of existence and uniqueness. Finally, several examples are provided for illustrating the applications of our theoretical analysis.

Keywords: existence and uniqueness; boundary value problem; fractional differential equation; functional analysis; fixed point theorem.

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1 Introduction

In this paper, we consider the following fractional boundary value problem:

$$\begin{cases} {}^{C}D^{q}x(t) = f(t, x(t)),\ t \in [0,1],\ 1 < q < 2, \\ x(1) = \mu \int_0^1 x(s)ds, \quad x'(0) + x'(1) = 0, \end{cases} \tag{1.1}$$

where ${}^{C}D^{q}$ denotes the Caputo fractional derivative of order q, f is a given continuous function, and $x \in X$, $f : [0,1] \times X \to X$. Here $(X, \|\cdot\|)$ is a Banach space and $C = C([0,1], X)$ denotes

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the Banach space of all continuous functions from $[0, 1]$ to X endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$.

The origin of fractional calculus goes back to the Marquis de L'Hôpital and Gottfried Wilhelm Leibnitz in the seventeenth century. It is an old subject, but it has gained much attention in recent half of the century. Fractional integral and differential equations, have been studied recently by many researchers [5, 6, 8, 12, 14, 18, 19, 22, 25, 26, 30]. Fractional differential equations appear in a large number of fields of science and engineering, such as viscoelasticity [20], electrochemistry [24], electromagnetism [16], biology [2, 17], optimal control [21, 32], diffusion process [11, 13, 29], economics [23] and fractional variational problems [3].

Integral and anti-periodic boundary conditions can be seen in models of a variety of physical, economic and biological processes, and they have been investigated extensively in recent years (see [7, 15, 28, 33] and related references therein for boundary value problems with integral boundary conditions, and [1, 4, 9, 31] and related references therein for boundary value problems with anti-periodic boundary conditions). Specifically, [7] considers the positive solution of a fractional boundary value problem consisting of one integral and two zero initial conditions. By constructing a proper cone, the existence of positive solution is shown. In [4], a class of high-order fractional boundary value problems consisting of four anti-periodic boundary conditions are studied. By using the topological degree theory, some existence results are obtained. On the whole, little work has been done on the fractional boundary value problems with integral or anti-periodic boundary conditions. Since these problems arise in many applications, it is important that we further examine this subject. Motivated by the above work, we study the fractional boundary value problems with integral and anti-periodic boundary conditions.

The remainder of our paper is organized as follows: In Section 2, we give definitions of fractional integral and derivative operators, prove a lemma and present some classical fixed point theorems which are very useful for verifying our main results. The main results consist of four existence theorems discussed in Section 3. To validate our theoretical results, examples are provided in Section 4. Finally, the conclusions are given in Section 5.

2 Preliminaries

To define fractional differential operator, we need the Euler's Gamma function and the Riemann-Liouville fractional integral operator given below.

Definition 2.1 (see [14, 26]) The function $\Gamma : (0, +\infty) \to \mathbb{R}$, defined by

$$\Gamma(q) := \int_0^{+\infty} t^{q-1} e^{-t} dt, \tag{2.1}$$

is called Euler's Gamma function (or Euler's integral of the second kind). Particularly, for a positive integer n, we have $\Gamma(n) = (n-1)!$.

Definition 2.2 (see [14, 26]) The Riemann-Liouville fractional integral operator of order $q \geq 0$, of a function $x(t)$ is defined as

$$I^q x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} x(s) ds, \quad q > 0, \quad t > 0, \tag{2.2}$$

provided the integral exists. Particularly, if $q = 0$, $I^0 x(t) = x(t)$.

There are several kinds of fractional derivatives, such as Riemann-Liouville, Grünwald-Letnikov and Caputo derivatives. For more details, we refer to [14, 26]. In this paper, we use a Caputo derivative, the definition of which is given below.

Definition 2.3 (see [14, 26]) The fractional derivative of order q for a continuously differentiable function $x : [0, +\infty) \to \mathbb{R}$ in the sense of Caputo is defined as

$$^C D^q x(t) = \frac{1}{\Gamma(m-q)} \int_a^t (t-s)^{m-q-1} x^{(m)}(s) ds, \tag{2.3}$$

where $m - 1 < q \leq m$, $m = [q] + 1$, and $[q]$ denotes the integer part of the real number q.

Lemma 2.1 (see [14]) Let $q > 0$, then the homogeneous fractional differential equation

$$^C D^q x(t) = 0 \tag{2.4}$$

has a unique solution given by the expression

$$x(t) = \sum_{j=0}^{[q]} c_j t^j, \tag{2.5}$$

where $c_j = x^{(j)}(0)/\Gamma(j+1)$ are the coefficients.

Lemma 2.2 (see [14]) Let $q > 0$, then the Riemann-Liouville integral and Caputo derivative have the following composite property:

$$I^q \left\{ {}^C D^q x(t) \right\} = x(t) - \sum_{j=0}^{[q]} c_j t^j, \tag{2.6}$$

where c_j are the coefficients defined in Lemma 2.1.

Consider the following fractional differential equation with integral and anti-periodic boundary conditions:

$$\begin{cases} {}^C D^q x(t) = \sigma(t), \ t \in [0,1], \ 1 < q < 2, \\ x(1) = \mu \int_0^1 x(s)ds, \quad x'(0) + x'(1) = 0. \end{cases} \tag{2.7}$$

We have the following useful lemma:

Lemma 2.3 Suppose that $\sigma \in C[0,1]$ and $\mu \neq 1$, then the problem (2.7) has a unique solution given by:

$$x(t) = \int_0^1 G(t,s)\sigma(s)ds,$$

where

$$G(t,s) = \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} \\ +\frac{\mu}{4(1-\mu)\Gamma(q+1)} \left[4(1-s)^q - 4q(1-s)^{q-1} + q(q-1)(1-s)^{q-2} \right] \\ +\frac{1}{2\Gamma(q)} \left[(1-t)(q-1)(1-s)^{q-2} - 2(1-s)^{q-1} \right], & 0 \le s \le t \le 1, \\ \frac{\mu}{4(1-\mu)\Gamma(q+1)} \left[4(1-s)^q - 4q(1-s)^{q-1} + q(q-1)(1-s)^{q-2} \right] \\ +\frac{1}{2\Gamma(q)} \left[(1-t)(q-1)(1-s)^{q-2} - 2(1-s)^{q-1} \right], & 0 \le t \le s \le 1, \end{cases} \tag{2.8}$$

is the Green's function of the problem.

Proof. According to Lemma 2.2, problem (2.7) is equivalent to the following integral equation:

$$\begin{aligned} x(t) &= I^q \sigma(t) + \sum_{j=0}^{1} \frac{x^{(j)}(0)}{j!} t^j \\ &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds + x(0) + x'(0)t. \end{aligned} \tag{2.9}$$

Differentiating Equation (2.9) with respect to t yields

$$x'(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds + x'(0).$$

Applying the integral boundary condition $x(1) = \mu\int_0^1 x(s)ds$ implies that:

$$x(1) = \mu\int_0^1 x(s)ds = \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + x(0) + x'(0),$$

that is

$$x(0) + x'(0) = \mu\int_0^1 x(s)ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds. \tag{2.10}$$

The anti-periodic boundary condition $x'(0) + x'(1) = 0$ implies that:

$$x'(0) + x'(1) = x'(0) + \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds + x'(0),$$

that is

$$2x'(0) + \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds = 0. \tag{2.11}$$

Hence by Equations (2.10) and (2.11), we have

$$\begin{cases} x(0) = \mu\int_0^1 x(s)ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \frac{1}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds, \\ x'(0) = -\frac{1}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds. \end{cases} \tag{2.12}$$

Substituting Equation (2.12) into Equation (2.9), we have

$$\begin{aligned} x(t) &= \mu\int_0^1 x(s)ds \\ &\quad + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds \\ &\quad + \frac{1}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds - \frac{t}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds. \end{aligned} \tag{2.13}$$

Now we integrate Equation (2.13) from 0 to 1 on both sides to obtain

$$\begin{aligned} \int_0^1 x(s)ds &= \mu\int_0^1 x(s)ds \\ &\quad + \int_0^1\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)dsdt - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds \\ &\quad + \frac{1}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds - \int_0^1 \frac{t}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)dsdt \\ &= \mu\int_0^1 x(s)ds + \int_0^1 \frac{(1-s)^{q}}{\Gamma(q+1)}\sigma(s)ds \end{aligned}$$

$$
-\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \frac{1}{4}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds. \tag{2.14}
$$

Therefore from above, one can immediately have

$$
\begin{aligned}
\mu\int_0^1 x(s)ds &= \frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)}\sigma(s)ds \\
&\quad -\frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \frac{\mu}{4(1-\mu)}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds. \quad (2.15)
\end{aligned}
$$

Substituting Equation (2.15) into Equation (2.13), we arrive at the following expression for solution $x(t)$:

$$
\begin{aligned}
x(t) &= \frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)}\sigma(s)ds \\
&\quad -\frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \frac{\mu}{4(1-\mu)}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds \\
&\quad +\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds \\
&\quad +\frac{1}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds - \frac{t}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds \\
&= \int_0^t \Bigg\{ \frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{\mu\left[4(1-s)^q - 4q(1-s)^{q-1} + q(q-1)(1-s)^{q-2}\right]}{4(1-\mu)\Gamma(q+1)} \\
&\quad + \frac{\left[(1-t)(q-1)(1-s)^{q-2} - 2(1-s)^{q-1}\right]}{2\Gamma(q)} \Bigg\}\sigma(s)ds \\
&\quad + \int_t^1 \Bigg\{ \frac{\mu\left[4(1-s)^q - 4q(1-s)^{q-1} + q(q-1)(1-s)^{q-2}\right]}{4(1-\mu)\Gamma(q+1)} \\
&\quad + \frac{\left[(1-t)(q-1)(1-s)^{q-2} - 2(1-s)^{q-1}\right]}{2\Gamma(q)} \Bigg\}\sigma(s)ds \\
&= \int_0^1 G(t,s)\sigma(s)ds. \quad (2.16)
\end{aligned}
$$

This completes the proof. □

Theorem 2.1 (Contraction mapping principle, see [10]) Let E be a Banach space, $D \subset E$ closed and $F : D \to D$ a strict contraction, i.e. $|Fx - Fy| \le k|x-y|$ for some $k \in (0,1)$ and all $x, y \in D$. Then F has a unique fixed point x^*. Furthermore the successive approximations $x_{n+1} = Fx_n = F^n x_0$, starting at any $x_0 \in D$, converge to x^* and satisfy $|x_n - x^*| \le (1-k)^{-1}k^n|Fx_0 - x_0|$.

Theorem 2.2 (Arzelà-Ascoli, see [10]) If a sequence $\{x_n\}$ in a compact subset of X is uniformly bounded and equicontinuous, then it has a uniformly convergent subsequence.

The following two fixed point theorems are necessary to prove the existence of solution for fractional boundary value problem (1.1).

Theorem 2.3 (see [27]) Let X be a Banach space. Assume that Ω is an open bounded subset of X with $\theta \in \Omega$ and let $T : \overline{\Omega} \to X$ be a completely continuous operator such that

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega.$$

Then T has a fixed point in $\overline{\Omega}$.

Theorem 2.4 (Krasnosel'skii, see [27]) Let $\mathbb{M}$ be a closed convex and nonempty subset of a Banach space X. Let A and B be two operators such that:

(I1) $Ax + By \in \mathbb{M}$, wherever $x, y \in \mathbb{M}$;

(I2) A is compact and continuous; and

(I3) B is a contraction mapping.

Then there exists $z \in \mathbb{M}$ such that $z = Az + Bz$.

3 Main results

In this section, first, we renew some notions. Let $C = C([0,1], \mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{0 \leq t \leq 1}\{|x(t)|\}$.

Define an operator $F : C \to C$ as

$$\begin{aligned}(Fx)(t) \;=\;& \frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} f(s,x(s))ds \\ & -\frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \frac{\mu}{4(1-\mu)}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds \\ & +\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ & +\frac{1}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds - \frac{t}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds, \quad t \in [0,1]. \end{aligned} \tag{3.1}$$

If the operator F has a fixed point, then the fixed point coincides with the solution of problem (1.1). In what follows, we first prove that the operator $F : C \to C$ is completely continuous.

Lemma 3.1 The operator $F : C \to C$ defined by Equation (3.1) is completely continuous.

Proof. Let $\Omega \subset C$ be bounded. Then for any $t \in [0,1]$ and $x \in \Omega$, since $f(t,x)$ is continuous on $[0,1] \times \mathbb{R}$, there exists a positive constant L_1 such that $|f(t,x)| \leq L_1$. Thus one can deduce that

$$
\begin{aligned}
|(Fx)(t)| &\leq \frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)}|f(s,x(s))|ds \\
&\quad + \frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds + \frac{\mu}{4(1-\mu)}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds \\
&\quad + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds \\
&\quad + \frac{1}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds + \frac{t}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds \\
&\leq L_1\left[\frac{\mu}{1-\mu}\cdot\frac{1}{\Gamma(q+2)} + \frac{1}{1-\mu}\cdot\frac{1}{\Gamma(q+1)} + \frac{2-\mu}{4(1-\mu)}\cdot\frac{1}{\Gamma(q)} + \frac{t}{2}\cdot\frac{1}{\Gamma(q)} + \frac{t^q}{\Gamma(q+1)}\right] \\
&\leq L_1\left[\frac{\mu}{(1-\mu)\Gamma(q+2)} + \frac{2-\mu}{(1-\mu)\Gamma(q+1)} + \frac{4-3\mu}{4(1-\mu)\Gamma(q)}\right] := L_2, \qquad (3.2)
\end{aligned}
$$

which implies that $\|(Fx)\| \leq L_2$. Moreover, for the derivative, we have

$$
\begin{aligned}
|(Fx)'(t)| &= \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds + \frac{1}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds \\
&\leq L_1\left[\frac{t^{q-1}}{\Gamma(q)} + \frac{1}{2\Gamma(q)}\right] \\
&\leq L_1 \cdot \frac{3}{2\Gamma(q)} := L_3. \qquad (3.3)
\end{aligned}
$$

Therefore, for all $0 \leq t_1 < t_2 \leq 1$, we have

$$|(Fx)(t_2) - (Fx)(t_1)| \leq \int_{t_1}^{t_2} |(Fx)'(s)|ds \leq L_3(t_2 - t_1),$$

which implies that the operator F is equicontinuous on $[0,1]$. Thus, by the Arzelà-Ascoli theorem, the operator $F : C \to C$ is completely continuous. $\square$

We have the following existence results.

Theorem 3.1 Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $\lim_{x\to 0} f(t,x)/x = 0$. Then problem (1.1) has at least one solution.

Proof. Since $\lim_{x\to 0} f(t,x)/x = 0$, there exist constants $d > 0$ and $d_1 > 0$ such that $|f(t,x)| \leq d_1|x|$ for all $0 < |x| < d$, where d_1 is such that

$$\max_{t\in[0,1]}\left\{\frac{\mu+q+1}{(1-\mu)\Gamma(q+2)} + \frac{2-\mu}{4(1-\mu)\Gamma(q)} + \frac{t}{2\Gamma(q)} + \frac{t^q}{\Gamma(q+1)}\right\} \cdot d_1 \leq 1. \qquad (3.4)$$

Define $\Omega_1 = \{x \in C : |x| < d\}$. Taking $x_0 \in C$ such that $|x_0| = d$, which means that $x_0 \in \partial\Omega_1$. By Lemma 3.1, we know that F is completely continuous and

$$\begin{aligned} |(Fx_0)(t)| &\leq \max_{t\in[0,1]} \left\{ \frac{\mu+q+1}{(1-\mu)\Gamma(q+2)} + \frac{2-\mu}{4(1-\mu)\Gamma(q)} + \frac{t}{2\Gamma(q)} + \frac{t^q}{\Gamma(q+1)} \right\} \cdot d_1|x_0| \\ &\leq |x_0|, \end{aligned} \tag{3.5}$$

by using Equation (3.4). Hence by Theorem 2.3, the operator F has at least one fixed point, which in turn implies that the problem (1.1) has at least one solution. □

Theorem 3.2 Let $f : [0,1] \times X \to X$ be a jointly continuous function satisfying the Lipschitz condition

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \ \forall t \in [0,1], \quad x, y \in X.$$

Then the boundary value problem (1.1) has a unique solution provided $\Delta < 1$ where

$$\Delta = 2L\left[\frac{\mu}{(1-\mu)\Gamma(q+2)} + \frac{2-\mu}{(1-\mu)\Gamma(q+1)} + \frac{4-3\mu}{4(1-\mu)\Gamma(q)}\right].$$

Proof. First, we show that F maps bounded ball to itself. Define $M = \sup_{t\in[0,1]} |f(t,0)|$, and select

$$r \geq 2M\left[\frac{\mu}{(1-\mu)\Gamma(q+2)} + \frac{2-\mu}{(1-\mu)\Gamma(q+1)} + \frac{4-3\mu}{4(1-\mu)\Gamma(q)}\right].$$

Now we define a closed ball as $B_r = \{x \in C : \|x\| \leq r\}$, then we have

$$\begin{aligned} \|(Fx)(t)\| &\leq \frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)}|f(s,x(s))|ds \\ &\quad + \frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds + \frac{\mu}{4(1-\mu)}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds \\ &\quad + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds \\ &\quad + \frac{1}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds + \frac{t}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds \\ &\leq \frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)}\left(|f(s,x(s)) - f(s,0)| + |f(s,0)|\right)ds \\ &\quad + \frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\left(|f(s,x(s)) - f(s,0)| + |f(s,0)|\right)ds \\ &\quad + \frac{\mu}{4(1-\mu)}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\left(|f(s,x(s)) - f(s,0)| + |f(s,0)|\right)ds \\ &\quad + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\left(|f(s,x(s)) - f(s,0)| + |f(s,0)|\right)ds \end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\left(|f(s,x(s))-f(s,0)|+|f(s,0)|\right)ds \\
&+\frac{1}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\left(|f(s,x(s))-f(s,0)|+|f(s,0)|\right)ds \\
&+\frac{t}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\left(|f(s,x(s))-f(s,0)|+|f(s,0)|\right)ds \\
\leq\ & (Lr+M)\left[\frac{\mu}{(1-\mu)\Gamma(q+2)}+\frac{1}{(1-\mu)\Gamma(q+1)}+\frac{2-\mu}{4(1-\mu)\Gamma(q)}+\frac{t}{2\Gamma(q)}+\frac{t^q}{\Gamma(q+1)}\right] \\
\leq\ & (Lr+M)\left[\frac{\mu}{(1-\mu)\Gamma(q+2)}+\frac{2-\mu}{(1-\mu)\Gamma(q+1)}+\frac{4-3\mu}{4(1-\mu)\Gamma(q)}\right] \\
\leq\ & r,
\end{aligned}
$$

which implies that $F(B_r)\subset B_r$. In what follows, for $x,y\in C$ and for each $t\in[0,1]$, one can obtain that

$$
\begin{aligned}
\|(Fx)(t)-(Fy)(t)\| \leq\ & \frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)}\|f(s,x(s))-f(s,y(s))\|ds \\
&+\frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\|f(s,x(s))-f(s,y(s))\|ds \\
&+\frac{\mu}{4(1-\mu)}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\|f(s,x(s))-f(s,y(s))\|ds \\
&+\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\|f(s,x(s))-f(s,y(s))\|ds \\
&+\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\|f(s,x(s))-f(s,y(s))\|ds \\
&+\frac{1}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\|f(s,x(s))-f(s,y(s))\|ds \\
&+\frac{t}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\|f(s,x(s))-f(s,y(s))\|ds \\
\leq\ & L\|x-y\|\left[\frac{\mu}{(1-\mu)\Gamma(q+2)}+\frac{2-\mu}{4(1-\mu)\Gamma(q)}+\frac{t}{2\Gamma(q)}+\frac{\frac{1}{1-\mu}+t^q}{\Gamma(q+1)}\right] \\
\leq\ & L\left[\frac{\mu}{(1-\mu)\Gamma(q+2)}+\frac{2-\mu}{(1-\mu)\Gamma(q+1)}+\frac{4-3\mu}{4(1-\mu)\Gamma(q)}\right]\|x-y\| \\
=\ & \frac{\Delta}{2}\cdot\|x-y\| \\
<\ & \|x-y\|,
\end{aligned}
$$

which implies that F is a contraction as $\Delta<1$. Therefore the conclusion of this theorem follows by the contraction mapping principle (i.e. Banach fixed point theorem). □

Remark 3.1 Since $\Delta < 1$ in Theorem 3.2, we can find positive real number $\bar{\mu} < 1$ such that $\Delta \leq \bar{\mu} < 1$. Similarly, by taking $M = \sup_{t\in[0,1]} |f(t,0)|$, and selecting

$$\bar{r} \geq \frac{M}{1-\bar{\mu}}\left[\frac{\mu}{(1-\mu)\Gamma(q+2)} + \frac{2-\mu}{(1-\mu)\Gamma(q+1)} + \frac{4-3\mu}{4(1-\mu)\Gamma(q)}\right],$$

one can easily prove the conclusion that the boundary value problem (1.1) has a unique solution which lies in a closed ball $B_{\bar{r}} = \{x \in C : \|x\| \leq \bar{r}\}$. The case of fixing $\bar{\mu} = 1/2$ is proved in Theorem 3.2.

Theorem 3.3 Assume that $f : [0,1] \times X \to X$ is a jointly continuous function and further:

(H1) $|f(t,x) - f(t,y)| \leq L|x-y|$, $\forall t \in [0,1]$, $x, y \in X$;

(H2) $|f(t,x)| \leq \lambda(t)$, $\forall (t,x) \in [0,1] \times X$, and $\lambda \in L^1([0,1], \mathbb{R}^+)$.

If

$$L\left[\frac{\mu}{(1-\mu)\Gamma(q+2)} + \frac{1}{(1-\mu)\Gamma(q+1)} + \frac{4-3\mu}{4(1-\mu)\Gamma(q)}\right] < 1,$$

then the nonlinear boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. Let

$$r \geq \|\lambda\|_{L^1}\left[\frac{\mu}{(1-\mu)\Gamma(q+2)} + \frac{2-\mu}{(1-\mu)\Gamma(q+1)} + \frac{4-3\mu}{4(1-\mu)\Gamma(q)}\right],$$

and consider $B_r = \{x \in C : \|x\| \leq r\}$. Here we define the operators Φ and Ψ on B_r as

$$\begin{aligned}
(\Phi x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds \\
(\Psi x)(t) &= \frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} f(s,x(s))ds - \frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\
&\quad + \frac{\mu}{4(1-\mu)}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\
&\quad + \frac{1}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds - \frac{t}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds.
\end{aligned}$$

For $x_1, x_2 \in B_r$, simple computation yields

$$\begin{aligned}
\|(\Phi x_1)(t) + (\Psi x_2)(t)\| &\leq \|\lambda\|_{L^1}\left[\frac{\mu}{(1-\mu)\Gamma(q+2)} + \frac{2-\mu}{(1-\mu)\Gamma(q+1)} + \frac{4-3\mu}{4(1-\mu)\Gamma(q)}\right] \\
&\leq r,
\end{aligned}$$

thus, $\Phi x_1 + \Psi x_2 \in B_r$. Clearly, it follows from the condition (H1) that Ψ is a contraction mapping for

$$L\left[\frac{\mu}{(1-\mu)\Gamma(q+2)} + \frac{1}{(1-\mu)\Gamma(q+1)} + \frac{4-3\mu}{4(1-\mu)\Gamma(q)}\right] < 1.$$

Since f is continuous, Φ is also continuous. Moreover, Φ is uniformly bounded on B_r as

$$\|\Phi u\| \leq \frac{t^q}{\Gamma(q+1)}\|\lambda\|_{L^1} \leq \frac{1}{\Gamma(q+1)}\|\lambda\|_{L^1}.$$

In what follows we prove the compactness of the operator Φ. Let $\mathbb{S} = [0,1] \times B_r$, and define $f_{\max} = \sup_{(t,x)\in\mathbb{S}} |f(t,x)|$, then we have

$$\begin{aligned}
|(\Phi x)(t_1) - (\Phi x)(t_2)| &= \left|\int_0^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \int_0^{t_1} \frac{(t_1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\right| \\
&= \left|\int_0^{t_1} \frac{f(s,x(s))}{\Gamma(q)} \left[(t_2-s)^{q-1} - (t_1-s)^{q-1}\right] ds\right. \\
&\quad \left. - \int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\right| \\
&\leq \frac{f_{\max}}{\Gamma(q+1)} \left|2(t_2-t_1)^q + t_1^q - t_2^q\right|,
\end{aligned}$$

which is independent of x. Thus, Φ is equicontinuous. Since f maps bounded subsets into relatively compact subsets, one can conclude that $\Phi(C_{bs})(t)$ is relatively compact in X for every t, where C_{bs} is bounded subset of C. Therefore, $\Phi(\cdot)$ is relatively compact on B_r, and hence, by the Arzelà-Ascoli theorem, Φ is compact on B_r and the hypotheses (H1) and (H2) are satisfied. Consequently, by Theorem 2.4, we conclude that the nonlinear boundary value problem (1.1) has at least one solution on $[0,1]$. □

Theorem 3.4 Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, there exists a constant c that satisfies such that $0 \leq c < 1/\delta$, where

$$\delta = \frac{\mu}{(1-\mu)\Gamma(q+2)} + \frac{2-\mu}{(1-\mu)\Gamma(q+1)} + \frac{4-3\mu}{4(1-\mu)\Gamma(q)}.$$

Let $M > 0$ such that $|f(t,x)| \leq c\|x\| + M$ and $x(t) \in \mathbb{R}$ for all $t \in [0,1]$. Then the boundary value problem (1.1) has at least one solution.

Proof. Define a fixed point problem by

$$x = Fx, \tag{3.6}$$

where F is defined in Equation (3.1). To prove the existence of at least one solution $x \in C[0,1]$ satisfying Equation (3.6), define a suitable ball $B_{r_0} \subset C[0,1]$ with radius $r_0 > 0$ as

$$B_{r_0} = \left\{ x \in C[0,1] : \max_{t\in[0,1]} |x(t)| < r_0 \right\},$$

where r_0 will be evaluated later.

By Lemma 3.1, we know that $F : C \to C$ is completely continuous, then it is easy to prove that $h_\lambda(x)$ is also completely continuous, where $h_\lambda(x)$ is defined by

$$h_\lambda(x) = x - H(\lambda, x) = x - \lambda Fx, \quad x \in C[0,1], \quad \lambda \in [0,1].$$

Now, it is sufficient to show that mapping $Fx : \overline{B_{r_0}} \to C[0,1]$ satisfies

$$x \neq \lambda Fx, \quad \forall x \in \partial B_{r_0} \text{ and } \forall \lambda \in [0,1]. \tag{3.7}$$

If Equation (3.7) holds, then by the homotopy invariance of topological degree in Leray-Schauder degree theory, it follows that

$$\begin{aligned} \deg(h_\lambda, B_{r_0}, 0) &= \deg(I - \lambda Fx, B_{r_0}, 0) = \deg(h_1, B_{r_0}, 0) = \deg(h_0, B_{r_0}, 0) \\ &= \deg(I, B_{r_0}, 0) = 1 \neq 0, \quad 0 \in B_{r_0}, \end{aligned}$$

where I denotes the unit operator. By the nonzero property of the Leray-Schauder degree, one can conclude that there exists at least one x that belongs to open ball B_{r_0} as $h_1(x) = x - \lambda Fx = 0$.

In what follows, we first prove Equation (3.7). We assume that $x = \lambda Fx$ for some $\lambda \in [0,1]$ and all $t \in [0,1]$, then

$$\begin{aligned} |x(t)| &= |\lambda Fx(t)| \\ &\leq \frac{\mu}{1-\mu} \int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} |f(s,x(s))| ds \\ &\quad + \frac{\mu}{1-\mu} \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds + \frac{\mu}{4(1-\mu)} \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} |f(s,x(s))| ds \\ &\quad + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+\frac{t}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds \\
\leq\ & (c\|x\|+M)\left[\frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^{q}}{\Gamma(q+1)}ds+\frac{\mu}{1-\mu}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}ds\right. \\
&+\frac{\mu}{4(1-\mu)}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}ds+\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}ds+\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}ds \\
&\left.+\frac{1}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}ds+\frac{t}{2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}ds\right] \\
\leq\ & (c\|x\|+M)\left[\frac{\mu}{(1-\mu)\Gamma(q+2)}+\frac{2-\mu}{(1-\mu)\Gamma(q+1)}+\frac{4-3\mu}{4(1-\mu)\Gamma(q)}\right] \\
=\ & (c\|x\|+M)\delta.
\end{aligned}
$$

By taking norm $\sup_{t\in[0,1]}|x(t)|=\|x\|$, simple computation yields

$$
\|x\|\leq\frac{M\delta}{1-c\delta},
$$

when we choose $r_0=1+M\delta/(1-c\delta)$, Equation (3.7) holds. This completes the proof. □

Remark 3.2 By Equation (2.16), if $\mu=0$, then $x(0)\neq 0$ unless $\sigma(t)\equiv 0$ on $t\in[0,1]$. This means that the boundary conditions in problem (1.1) cannot be replaced with the integral boundary conditions in [7, 15] or the anti-periodic boundary conditions considered in [4]. Therefore fractional boundary value problem (1.1) is novel and unique.

4 Examples

Now we provide several examples to demonstrate the applications of the theoretical results in the previous sections.

Example 4.1 Consider the fractional boundary value problem

$$
\begin{cases}
{}^{C}D^{q}x(t)=f(t,x(t)), \quad 0<t<1, \quad 1<q\leq 2,\\
x(1)=\mu\int_0^1 x(s)ds, \ x'(0)+x'(1)=0,
\end{cases}
\tag{4.1}
$$

where $q=1.86$, $0<\mu=0.75<1$ and $f(t,x)=t(x-\sin(x))+\sqrt{1+x^2}-1$. As $\lim_{x\to 0}f(t,x)/x=0$, the hypotheses of Theorem 3.1 are satisfied. Hence, by Theorem 3.1, the problem (4.1) has at least one solution on $t\in[0,1]$.

Example 4.2 Consider the fractional boundary value problem

$$\begin{cases} {}^{C}D^{q}x(t) = f(t, x(t)), \quad 0 < t < 1, \quad 1 < q \leq 2, \\ x(1) = \mu \int_0^1 x(s)ds, \ x'(0) + x'(1) = 0, \end{cases} \tag{4.2}$$

where $q = 1.99$, $0 < \mu = 0.8 < 1$ and $f(t, x) = \|x\| / \left[(t + 10)^2(1 + \|x\|)\right]$. Obviously $L = 1/100$ as $\|f(t, x) - f(t, y)\| \leq \|x - y\|/100$. Moreover,

$$\Delta = 2L\left[\frac{\mu}{(1-\mu)\Gamma(q+2)} + \frac{2-\mu}{(1-\mu)\Gamma(q+1)} + \frac{4-3\mu}{4(1-\mu)\Gamma(q)}\right] = 0.1142 < 1.$$

Therefore, hypotheses of Theorem 3.2 are satisfied. Hence, by Theorem 3.2, the problem (4.2) has a unique solution on $t \in [0, 1]$.

Example 4.3 Consider the fractional boundary value problem

$$\begin{cases} {}^{C}D^{q}x(t) = f(t, x(t)), \quad 0 < t < 1, \quad 1 < q \leq 2, \\ x(1) = \mu \int_0^1 x(s)ds, \ x'(0) + x'(1) = 0, \end{cases} \tag{4.3}$$

where $q = 1.8$, $0 < \mu = 0.9 < 1$ and $f(t, x) = \|x\| / \left[(t + 5)^2(1 + \|x\|)\right]$. Obviously $L = 1/25$ as $\|f(t, x) - f(t, y)\| \leq \|x - y\|/25$. Moreover, $|f(t, x)| \leq \lambda(t) = 1/(t + 5)^2 \in L^1([0, 1], \mathbb{R}^+)$, and

$$L\left[\frac{\mu}{(1-\mu)\Gamma(q+2)} + \frac{1}{(1-\mu)\Gamma(q+1)} + \frac{4-3\mu}{4(1-\mu)\Gamma(q)}\right] = 0.9097 < 1.$$

Therefore, hypotheses of Theorem 3.3 are satisfied. Hence, by Theorem 3.3, the problem (4.3) has at least one solution on $t \in [0, 1]$.

Example 4.4 Consider the fractional boundary value problem

$$\begin{cases} {}^{C}D^{q}x(t) = f(t, x(t)), \quad 0 < t < 1, \quad 1 < q \leq 2, \\ x(1) = \mu \int_0^1 x(s)ds, \ x'(0) + x'(1) = 0, \end{cases} \tag{4.4}$$

where $q = 1.75$, $0 < \mu = 0.85 < 1$ such that

$$\delta = \frac{\mu}{(1-\mu)\Gamma(q+2)} + \frac{2-\mu}{(1-\mu)\Gamma(q+1)} + \frac{4-3\mu}{4(1-\mu)\Gamma(q)} = 8.6774.$$

If $f(t, x) = |x| / \left[(t + 3)^2(1 + |x|)\right]$, then there exists infinitely many positive constant M, such that $|f(t, x)| \leq |x|/9 + M \leq |x|/\delta + M$. Hence, by Theorem 3.4, the problem (4.4) has at least one solution on $t \in [0, 1]$.

5 Conclusions

In this article, we studied a class of fractional boundary value problems with integral and anti-periodic boundary conditions. By using the contraction mapping principle and some fixed point theorems, the existence and uniqueness of the fractional boundary value problems have been obtained. Since the integral and the anti-periodic boundary conditions cannot be replaced with only one of them, the results of this work are different from those given in [1,4,7,9,15,28,31,33].

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