Stronger version of Denjoy-type inequality [1]

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Abstract

Let f be an orientation-preserving circle diffeomorphism with irrational rotation number and $\log f'$ is absolutely continuous, $f''/f' \in L_p$, $p > 1$. By using the ratio distortion approach, we prove a sharper result for the key estimate of the Y. Katznelson and D. Ornstein theorem on the absolute continuity of the conjugacy for such circle diffeomorphism f.

1 Introduction

For an orientation-preserving homeomorphism f of the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$, its *rotation number* $\rho(f)$ is the value of the limit $\rho(f) = \lim_{i\to\infty} L_f^i(x)/i \mod 1$ for a lift L_f of f from S^1 onto $\mathbb{R}$. Here and below, for a given map L, L^i denotes its i-th iteration. It is known since Poincaré that the rotation number does not depend on the starting point $x \in \mathbb{R}$ and is irrational if and only if f has no periodic points (see [1]). We shall assume the rotation number $\rho = \rho(f)$ to be irrational through out this paper. We use the continued fraction representation $\rho = 1/(k_1+1/(k_2+...)) := [k_1, k_2, ..., k_n, ...]$ of the rotation number, which is understood as the limit of the sequence of convergents $p_n/q_n = [k_1, k_2, ..., k_n]$. The sequence of positive integer k_n with $n \geq 1$, which is called *partial quotients*, is uniquely determined for each ρ. The coprimes p_n and q_n satisfy the recursive relations $p_n = k_n p_{n-1} + p_{n-2}$ and $q_n = k_n q_{n-1} + q_{n-2}$ for $n \geq 1$, where we set for convenience, $p_{-1} = 0$, $q_{-1} = 1$, and $p_0 = 1$, $q_0 = k_1$. Let us denote $\mathbf{K}_n = \max_\xi |\log(f^{q_n}(\xi))'| = \|\log(f^{q_n})'\|_0$, where $f'(\xi)$ is the derivative of f at the point ξ. If f is an orientation preserving C^1-diffeomorphism of the circle with irrational rotation number ρ and $\log f'$ has bounded variation then, Denjoy in [6] showed that

$$\mathbf{K}_n \leq v,$$

where $v = Var_{S^1} \log f'$. It is known that this Denjoy's inequality has important applications on many problems of circle homeomorphisms. If Denjoy's inequality hold for f with irrational rotation number ρ, then this fact ensure the existence of a conjugation map h between f and linear rotation $R_\rho : \xi \to \xi + \rho \mod 1$ (see [1]). In this case, the conjugation h satisfying $f = h^{-1} \circ R_\rho \circ h$ is an essentially unique homeomorphism of the circle. In this direction, one can also obtain various degrees of smoothness of the conjugation h when the sequence $\mathbf{K}_n$ satisfies other properties. For example, in the works [3], [5, 6] and [7] it was shown that if diffeomorphism $f \in C^{2+\epsilon}(S^1)$, $\epsilon > 0$ and the rotation number ρ is irrational, then $\mathbf{K}_n$ tends to zero exponentially fast. This fact together with the condition of rotation number of *Diophantine type* ensure the conjugation h to be C^1-smooth. Convergence of the sum of squares of the sequence $\mathbf{K}_n$ on a wider class of circle homeomorphisms, which

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implies the absolute continuity of the conjugating function, was shown by Y. Katznelson, D. Ornstein [4]. More precisely, they showed:

Theorem 1.1. *If* $\log f'$ *is absolutely continuous,* $f''/f' \in L_p$ *for some* $p > 1$ *and the rotation number* ρ *is irrational, then*

$$\sum_{n=1}^{\infty} \boldsymbol{K}_n^2 < \infty.$$

Moreover, if the rotation number is of bounded type, then the conjugating function is absolutely continuous.

Our aim in this present paper is to get sharp estimate for the sequence $\mathbf{K}_n$ under the conditions of Theorem 1.1 - the so-called Katznelson-Ornstein (K.O) conditions. Denote by $d_n = d_n(f) = \|f^{q_n} - Id\|_0$. Our main result is the following:

Theorem 1.2. *Let diffeomorphism* f *satisfies (K.O) conditions and the rotation number* ρ *is irrational. Then there exists a constant* $C = C(f) > 0$ *and a sequence* $\tau_n = \tau_n(f)$ *whose sum of squares converges, such that*

$$\boldsymbol{K}_n \leq C \sum_{k=1}^{n} \frac{d_n}{d_k} \cdot \tau_k.$$

In the proof, it is shown that the sequence d_n/d_k tends to zero exponentially fast and this will ensure the convergence of the sum of squares of the sequence $\mathbf{K}_n$. Prove of our main result is based on the method of ratio distortion estimates. This simple and elementary method allows us to derive this stronger estimate in an easy manner. Moreover we also believe that this method will prove useful in other problems involving circle diffeomorphisms. Note that the first time the method of cross-ratio distortion (ratio distortion) estimates were used in the investigation of smoothness of the conjugation was by K. Khanin and A. Teplinsky in [7]. In fact the proof of our main result in this paper follows closely that of K. Khanin and A. Teplinsky in [7].

2 Necessary Facts and Statements

For $\xi \in S^1$ we define the n-th *fundamental segment* $I_0^n = I_0^n(\xi)$ as the circle arc $[\xi, f^{q_n}(\xi)]$ if n is even and $[f^{q_n}(\xi), \xi]$ if n is odd. We denote two sets of closed intervals of order n : q_n *"lengthy"* intervals: $\left\{I_i^{n-1} = f^i(I_0^{n-1}),\ 0 \leq i < q_n\right\}$ and q_{n-1} *"short"* intervals: $\left\{I_j^n = f^j(I_0^n),\ 0 \leq j < q_{n-1}\right\}$. The lengthy and short intervals are mutually disjoint except for the endpoints and cover the whole circle. The partition obtained by the above construction will be denoted by $\mathbf{P}_n = \mathbf{P}_n(\xi, f)$ and called the n-th *dynamical partition* of the point ξ. Obviously the partition $\mathbf{P}_{n+1}$ is a refinement of the partition $\mathbf{P}_n$. Indeed the short intervals are members of $\mathbf{P}_{n+1}$ and each lengthy interval $I_i^{n-1} \in \mathbf{P}_n,\ 0 \leq i < q_n$, is partitioned into $k_{n+1}+1$ intervals belonging to $\mathbf{P}_{n+1}$ such that

$$I_i^{n-1} = I_i^{n+1} \cup \bigcup_{s=0}^{k_{n+1}-1} I_{i+q_{n-1}+sq_n}^n. \tag{1}$$

The next lemmas are valid for any orientation-preserving diffeomorphisms $f \in C^{1+BV}(S^1)$ (i.e. f' has bounded variation) with irrational rotation number ρ, constitute classical Denjoy's theory.

Lemma 2.1. *Let $\xi \in S^1$, $\eta \in I_0^{n-1}(\xi)$. Then for any $0 \leq k < q_n$ the following inequality holds:*

$$|\log(f^k(\xi))' - \log(f^k(\eta))'| \leq v.$$

An elementary proof of the above lemma can be found in [3]. Now we equip S^1 with the usual metric $|x-y| = \inf\{|\widetilde{x} - \widetilde{y}|,$ where $\widetilde{x}, \widetilde{y}$ range over the lifts of $x, y \in S^1$ respectively$\}$. We need the following lemma to prove our main result.

Lemma 2.2. *For any short interval I_j^{n+m} of $\boldsymbol{P}_{n+m}$, $m > 1$ such that $I_j^{n+m} \subset I_0^n$ the following inequality holds:*

$$\frac{|I_j^{n+m}|}{|I_0^n|} \leq e^v(1+e^v)\frac{d_{n+m}}{d_n}.$$

Proof. Pick out the point $\xi^* \in S^1$ such that $d_n = |I_0^n(\xi^*)|$. Due to the combinatorics of trajectories, there exist $0 \leq i < q_{n+1}$ such that either $I_0^n(\xi^*) \subset f^i(I_0^n) \cup f^{i+q_n}(I_0^n)$ or $I_0^n(\xi^*) \subset f^i(I_0^n) \cup f^{i-q_n}(I_0^n)$, so apply Denjoy's inequality to the last relations we get $|I_0^n(\xi^*)| \leq (1+e^v)|f^i(I_0^n)|$. Using Lemma 2.1, for $0 \leq i < q_{n+1}$ we have

$$\frac{|I_j^{n+m}|}{|I_0^n|} \leq e^v \frac{|f^i(I_j^{n+m})|}{|f^i(I_0^n)|} \leq e^v(1+e^v)\frac{d_{n+m}}{d_n}. \tag{2}$$

□

Note, that the same assertions hold for intervals of any length of dynamical partition $\mathbf{P}_{n+m}$ with the same constants but with respect to d_{n+m-1}/d_n. Using Lemma 2.2 and applying Lemma 2.1 twice, we get the following corollary:

Corollary 2.3. *For any short intervals of $I^{n+m} \in \boldsymbol{P}_{n+m}$, $m > 1$ and $I^n \in \boldsymbol{P}_n$ such that $I^{n+m} \subset I^n$ the following estimate*

$$\frac{|I^{n+m}|}{|I^n|} \leq e^{2v}(1+e^v)\frac{d_{n+m}}{d_n} \tag{3}$$

holds.

Now we estimate the ratio of fundamental segments.

Lemma 2.4. *For any $n \geq 1$ and $m \geq 0$ holds the following estimate*

$$\frac{|I_0^{n+m}|}{|I_0^n|} \leq e^{2v}(1+e^v)\frac{d_{n+m}}{d_n}. \tag{4}$$

Proof. If m is even then the interval I_0^{n+m} is a short interval of P_{n+m} and $I_0^{n+m} \subset I_0^n$. Applying Lemma 2.2 we get

$$\frac{|I_0^{n+m}|}{|I_0^n|} \leq e^v(1+e^v)\frac{d_{n+m}}{d_n}.$$

If m is odd then the interval I_0^{n+m} is a short interval of P_{n+m} but the interval $I_{q_{n+m-1}}^{n+m}$ is a lengthly interval of P_{n+m+1}. Using the above note and Denjoy's inequality we get

$$\frac{|I_0^{n+m}|}{|I_0^n|} = \frac{|I_{q_{n+m-1}}^{n+m}|}{|I_0^n|} \cdot \frac{|I_0^{n+m}|}{|I_{q_{n+m-1}}^{n+m}|} \leq e^{2v}(1+e^v)\frac{d_{n+m}}{d_n}.$$

□

In the next lemma it will be shown that the ratio d_{n+m}/d_n tends to zero exponentially fast.

Lemma 2.5. *There exists a universal constant $C_1 = C_1(f) > 0$ such that $d_{n+m} \leq C_1\lambda^m d_n$, where $\lambda = (1+e^{-v})^{-1/2}$.*

Proof. By the definition of dynamical partitions, it is easy to see that for any $t \in S^1$ we get $|I_0^{n-1}(t)| \geq |I_0^{n+1}(t)| + |I_0^n(t_{q_{n+1}-q_n})|$ and $I_0^{n+1}(t) \subset I_0^n(t_{q_{n+1}})$. Using the last two relations and Denjoy's inequality implies

$$\frac{|I_0^{n-1}(t)|}{|I_0^{n+1}(t)|} \geq 1 + \frac{|I_0^n(t_{q_{n+1}-q_n})|}{|I_0^n(t_{q_{n+1}})|} \geq 1 + e^{-v}.$$

By induction, we get $|I_0^{n+m}(t)| \leq (1+e^{-v})^{-[\frac{m}{2}]}|I_0^n(t)|$. If we pick out the point $t = \xi^* \in S^1$ such that $d_{n+m} = |I_0^{n+m}(\xi^*)|$ then the proof of the lemma follows. □

Suppose f satisfy (K.O) conditions. Using the dynamical partition $\mathbf{P}_n$ we define the sequence of step functions on the circle by the next formula: for any $n \geq 1$ we set

$$\mathcal{M}_n(x) = \frac{1}{|I^n|}\int_{I^n} \frac{f''(x)}{f'(x)}dx, \text{ if } x \in I^n, \ I^n \in \mathbf{P}_n, \tag{5}$$

$$\mathcal{D}n(x) = \mathcal{M}_n(x) - \mathcal{M}_{n-1}(x), \ n \geq 1 \text{ where } \mathcal{M}_0(x) \equiv 0, \ x \in S^1. \tag{6}$$

Denoted by $(\mathbf{P}_n)$ (by abuse of notation) the sequence of algebras generated by dynamical partitions. It is easy to show that the sequence of $(\mathcal{M}_n)$ is a martingale with respect to $(\mathbf{P}_n)$.

Statement 2.6. *Let the diffeomorphism f satisfies the (K.O) conditions. Then the following equalities are true:*

$$(a) \int\limits_{I^{n-1}} \mathcal{D}_n(x)dx = 0 \quad \text{for any} \ \ I^{n-1} \in \boldsymbol{P}_{n-1}; \ (b) \ \frac{f''}{f'} = \sum_{n=1}^{\infty} \mathcal{D}_n \quad (\text{in } L_1 - \text{norm}).$$

Both assertions of the Statement 2.6 are easily verified. The following theorem of Katznelson-Ornstein [4] plays an important role in the proof of our main theorem.

Theorem 2.7. *Suppose f satisfies the (K.O) conditions. Let $(\mathcal{M}_n, \boldsymbol{P}_n)$ be L_p- bounded martingales $0 < p \leq 2$. Then*

$$\sum_{n=1}^{\infty} \|\mathcal{D}_n\|_p^2 < \infty.$$

3 Technical tools

Cross ratio estimates were used in dynamical systems for the first time by J.C. Yoccoz in [10] and later by W. de Melo, S. van Strien in [2] and G. Świątek in [8]. The asymptotic estimates for the cross-ratio distortion with respect to smooth monotone function was very well investigated by A. Teplinsky in [9]. Let $f \in C^1(I)$, and the derivative of f does not have zeros on the interval $I=[a,b]$. The *ratio distortion* of point $c\in I$ and interval I with respect to the function f is

$$\mathcal{R}(c,I;f)=\frac{|f(I)|}{|I|}\frac{1}{f'(c)}.$$

Notice that the ratio distortion is multiplicative with respect to composition: for any two functions f and g we have

$$\mathcal{R}(c,I;f\circ g)=\mathcal{R}(c,I;g)\cdot\mathcal{R}(g(c),g(I);f). \tag{7}$$

Denote by

$$\varepsilon_n=\|\mathcal{D}_{n+1}\|_p+\sum_{k=n+2}^{\infty}\frac{d_{k-1}}{d_{n-1}}\|\mathcal{D}_k\|_p. \tag{8}$$

Due to Theorem 2.7 $\|\mathcal{D}_n\|_p\in\ell_2$ (i.e. convergence of the sum of squares of the sequence $\|\mathcal{D}_n\|_p$). Using this it is easy to see that $\varepsilon_n\in\ell_2$. Fix $\xi_0\in S^1$ and consider dynamical partition $\mathbf{P}_n(\xi_0,f)$. Let $I_0^{n-1}=I_0^{n-1}(\xi_0)$ and $I_0^n=I_0^n(\xi_0)$ be $(n-1)$-th and n-th fundamental segments. We need following lemma.

Lemma 3.1. *Suppose f satisfies the conditions of (K.O). Then there exists $C_2=C_2(f)>0$ such that*

$$\Big|\log\mathcal{R}(\xi_0,I_0^{n-1};f^{q_n})-\frac{(-1)^{n-1}}{2}\sum_{s=0}^{q_n-1}\int\limits_{I_s^{n-1}}\frac{f''(x)}{f'(x)}dx\Big|\le C_2\varepsilon_n, \tag{9}$$

$$\Big|\log\mathcal{R}(\xi_0,I_0^{n};f^{q_{n-1}})-\frac{(-1)^{n}}{2}\sum_{s=0}^{q_{n-1}-1}\int\limits_{I_s^{n}}\frac{f''(x)}{f'(x)}dx\Big|\le C_2\varepsilon_{n+1}. \tag{10}$$

Proof. We prove the first inequality. By the multiplicative nature of $\mathcal{R}(\xi_0,I_0^{n-1};f^{q_n})$ with respect to composition, we have

$$\log\mathcal{R}(\xi_0,I_0^{n-1};f^{q_n})=\sum_{s=0}^{q_n-1}\log\Big[\frac{f(\eta_s)-f(\xi_s)}{\eta_s-\xi_s}\frac{1}{f'(\xi_s)}\Big], \tag{11}$$

where $\eta_s=f^s(\xi_{q_{n-1}})$ and $\xi_s=f^s(\xi_0)$ are end-points of interval I_s^{n-1}. It is clear that if f satisfies the conditions (K.O) then for every $0\le s<q_n$ holds the following equality

$$f(\eta_s)=f(\xi_s)+f'(\xi_s)(\eta_s-\xi_s)+\int_{\xi_s}^{\eta_s}f''(x)(\eta_s-x)dx. \tag{12}$$

Applying this equality to (11) we get

$$\log\mathcal{R}(\xi_0,I_0^{n-1};f^{q_n})=\sum_{s=0}^{q_n-1}\log\Big[1+\int_{\xi_s}^{\eta_s}\frac{f''(x)}{f'(\xi_s)}\cdot\frac{\eta_s-x}{\eta_s-\xi_s}dx\Big]. \tag{13}$$

It is clear

$$\Big|\int_{\xi_s}^{\eta_s}\frac{f''(x)}{f'(\xi_s)}\cdot\frac{\eta_s-x}{\eta_s-\xi_s}dx-\int_{\xi_s}^{\eta_s}\frac{f''(x)}{f'(x)}\cdot\frac{\eta_s-x}{\eta_s-\xi_s}dx\Big|\le$$

$$\leq \frac{1}{(\min_{x\in S^1} f'(x))^2}\int_{\xi_s}^{\eta_s}|f''(x)|dx\cdot\int_{\xi_s}^{\eta_s}|f''(x)|\cdot\frac{\eta_s-x}{\eta_s-\xi_s}dx. \tag{14}$$

Applying Hölder's inequality to the second integral we get

$$\int_{\xi_s}^{\eta_s}|f''(x)|\cdot\frac{\eta_s-x}{\eta_s-\xi_s}dx\leq\|f''\|_p\cdot\Big[\frac{|I_s^{n-1}|}{q+1}\Big]^{\frac{1}{q}}, \tag{15}$$

where $q=p/(p-1)$. According to (11)-(15) we have

$$\log\mathcal{R}(\xi_0,I_0^{n-1};f^{q_n})=\sum_{s=0}^{q_n-1}\Big[\int_{\xi_s}^{\eta_s}\frac{f''(x)}{f'(x)}\cdot\frac{\eta_s-x}{\eta_s-\xi_s}dx+\mathcal{O}\Big(|I_s^{n-1}|^{\frac{1}{q}}\cdot\int_{\xi_s}^{\eta_s}|f''(x)|dx\Big)\Big]. \tag{16}$$

Here and below the signs $\mathcal{O}(\cdot)$ and $\sim$ stand for an estimate, which are universal constants in such estimate depend only on the diffeomorphism f. Note, that the interval $[\xi_s,\eta_s]$ is a $(n-1)$-th fundamental interval, from this implies that the integral $\int_{\xi_s}^{\eta_s}$ changes sign depending on n. More precisely,

$$\int_{\xi_s}^{\eta_s}\frac{f''(x)}{f'(x)}\cdot\frac{\eta_s-x}{\eta_s-\xi_s}dx=(-1)^{n-1}\int_{I_s^{n-1}}\frac{f''(x)}{f'(x)}\cdot\frac{\eta_s-x}{\eta_s-\xi_s}dx.$$

Using this we can rewrite estimate (16) in the form

$$\log\mathcal{R}(\xi_0,I_0^{n-1};f^{q_n})=(-1)^{n-1}\sum_{s=0}^{q_n-1}\int_{I_s^{n-1}}\frac{f''(x)}{2f'(x)}dx+$$

$$+\sum_{s=0}^{q_n-1}(-1)^{n-1}\int_{I_s^{n-1}}\frac{f''(x)}{f'(x)}\cdot\Big(\frac{\eta_s-x}{\eta_s-\xi_s}-\frac{1}{2}\Big)dx+\mathcal{O}(d_{n-1})^{1/q}. \tag{17}$$

Now, we rewrite second sum of the right-hand of equation (17) as follows:

$$\sum_{s=0}^{q_n-1}\int_{I_s^{n-1}}\frac{f''(x)}{f'(x)}\cdot\Big(\frac{\eta_s-x}{\eta_s-\xi_s}-\frac{1}{2}\Big)dx=\sum_{k=1}^{N}\sum_{s=0}^{q_n-1}\int_{I_s^{n-1}}\Big(\frac{\eta_s-x}{\eta_s-\xi_s}-\frac{1}{2}\Big)\mathcal{D}_k(x)dx+$$

$$+\sum_{s=0}^{q_n-1}\int_{I_s^{n-1}}\Big(\frac{\eta_s-x}{\eta_s-\xi_s}-\frac{1}{2}\Big)\cdot\Big(\frac{f''(x)}{f'(x)}-\mathcal{M}_N(x)\Big)dx\equiv S_n^1+S_n^2. \tag{18}$$

To obtain estimate S_n^2 we consider sufficiently large N such that $|S_n^2|\leq\|\mathcal{N}f-\mathcal{M}_N\|_p\leq d_{n-1}$. To the estimate S_n^1 we should divide the first sum into three parts $1\leq k\leq n$, $k=n+1$ and $n+2\leq k\leq N$. If $1\leq k\leq n$ then the function $\mathcal{D}_k(x)$ is a constant on every interval $I_s^{n-1}\in\mathbf{P}_n$. Therefore

$$\int_{I_s^{n-1}}\Big(\frac{\eta_s-x}{\eta_s-\xi_s}-\frac{1}{2}\Big)\mathcal{D}_k(x)dx=0. \tag{19}$$

If $k = n+1$, then we have

$$\Big|\sum_{s=0}^{q_n-1}\int\limits_{I_s^{n-1}}\Big(\frac{\eta_s-x}{\eta_s-\xi_s}-\frac{1}{2}\Big)\mathcal{D}_{n+1}(x)dx\Big| \leq \|\mathcal{D}_{n+1}\|_p. \tag{20}$$

Let $n+2 \leq k \leq N$. In this case we define a new sequence of step functions as the following way. Consider dynamical partition $\mathbf{P}_{k-1}$, $k \geq n+2$ and on the every interval $I \in \mathbf{P}_{k-1}$ such that $I \subset [\xi_s, \eta_s]$, $0 \leq s < q_n$ we define

$$\mathcal{L}_{k,s}(x) = \frac{\eta_s - r(I)}{\eta_s-\xi_s} - \frac{1}{2}, \qquad x \in I,$$

where $r(I)$ is a right end-point of interval I. By construction, the function $\mathcal{L}_{k,s}$ is constant on the every interval $I \in \mathbf{P}_{k-1}$ such that $I \subset I_s^{n-1}$, $0 \leq s < q_n$. Using the first assertion of Statement 2.6 and Corollary 2.3 we have

$$\int\limits_{I_s^{n-1}}\mathcal{D}_k(x)\mathcal{L}_{k,s}(x)dx = 0 \quad \text{and} \quad \Big|\frac{\eta_s-x}{\eta_s-\xi_s}-\frac{1}{2}-\mathcal{L}_{k,s}(x)\Big| \leq e^{2v}(1+e^v)\frac{d_{k-1}}{d_{n-1}} \tag{21}$$

for any $n+2 \leq k \leq N$ and $0 \leq s < q_n$. Finally, combining (19)-(21) we get

$$|S_n^1| = \Big|\sum_{k=n+1}^{N}\sum_{s=0}^{q_n-1}\Big[\int\limits_{I_s^{n-1}}\Big(\frac{\eta_s-x}{\eta_s-\xi_s}-\frac{1}{2}-\mathcal{L}_{k,s}(x)\Big)\mathcal{D}_k(x)dx + \int\limits_{I_s^{n-1}}\mathcal{D}_k(x)\mathcal{L}_{k,s}(x)dx\Big]\Big| \leq C_2\varepsilon_n.$$

The inequality (9) is proved. The inequality (10) also will be proved in the same way as above, but there will be some changes for the relation (18). Since the short intervals of $\mathbf{P}_n$ are members $\mathbf{P}_{n+1}$ we should divide the first sum of (18) into three parts: $1 \leq k \leq n+1$, $k = n+2$ and $n+3 \leq k \leq N$. In the case $1 \leq k \leq n+1$ the function $\mathcal{D}_k(x)$ is also a constant on every interval $I_s^n \in \mathbf{P}_{n+1}$ and we get

$$\int\limits_{I_s^n}\Big(\frac{\widehat{\eta}_s-x}{\widehat{\eta}_s-\xi_s}-\frac{1}{2}\Big)\mathcal{D}_k(x)dx = 0 \tag{22}$$

where $\widehat{\eta}_s = f^s(\xi_{q_n})$. In the case $k = n+2$, we have

$$\Big|\sum_{s=0}^{q_{n-1}-1}\int\limits_{I_s^n}\Big(\frac{\widehat{\eta}_s-x}{\widehat{\eta}_s-\xi_s}-\frac{1}{2}\Big)\mathcal{D}_{n+2}(x)dx\Big| \leq \|\mathcal{D}_{n+2}\|_p. \tag{23}$$

Consider the case $n+3 \leq k \leq N$. In this case the same arguments as above we define a new sequence of step functions $\widehat{\mathcal{L}}_{k,s}(x)$ on the every interval $I \in \mathbf{P}_{k-1}$, $k \geq n+3$ such that $I \subset I_s^n$, $0 \leq s < q_{n-1}$ and using the first assertion of Statement 2.6 and Corollary 2.3 we get

$$\int\limits_{I_s^n}\mathcal{D}_k(x)\widehat{\mathcal{L}}_{k,s}(x)dx = 0 \quad \text{and} \quad \Big|\frac{\widehat{\eta}_s-x}{\widehat{\eta}_s-\xi_s}-\frac{1}{2}-\widehat{\mathcal{L}}_{k,s}(x)\Big| \leq e^{2v}(1+e^v)\frac{d_k}{d_n} \tag{24}$$

for any $n+3 \leq k \leq N$ and $0 \leq s < q_{n-1}$. Finally, considering (22)-(24) it is easy to see that the estimate in (10) will be ε_{n+1}. □

We need the following lemma which is to be used in the proof of Theorem 1.2.

Lemma 3.2. *Suppose f satisfies the conditions of (K.O). Then there exists $C_3 = C_3(f) > 0$ such that*

$$\Big|\log\Big[\frac{|I_{q_n}^{n-1}|}{|I_0^{n-1}|}\cdot\frac{|I_0^{n-1}|-|I_0^{n+1}|}{|I_{q_n}^{n-1}|-|I_{q_n}^{n+1}|}\Big]-\frac{(-1)^n}{2}\sum_{s=0}^{q_n-1}\int\limits_{I_s^{n+1}}\frac{f''(x)}{f'(x)}dx\Big|\leq C_3\varepsilon_n,$$

$$\Big|\log\Big[\frac{|I_{q_n}^{n+1}|}{|I_0^{n+1}|}\cdot\frac{|I_0^{n-1}|-|I_0^{n+1}|}{|I_{q_n}^{n-1}|-|I_{q_n}^{n+1}|}\Big]-\frac{(-1)^n}{2}\sum_{s=0}^{q_n-1}\int\limits_{I_s^{n-1}}\frac{f''(x)}{f'(x)}dx\Big|\leq C_3\varepsilon_n.$$

Proof. We prove only the first inequality, the second inequality can be handled similarly. The following three exact relations are crucial for our proof:

$$\log\Big[\frac{|I_{q_n}^{n-1}|}{|I_0^{n-1}|}\cdot\frac{|I_0^{n-1}|-|I_0^{n+1}|}{|I_{q_n}^{n-1}|-|I_{q_n}^{n+1}|}\Big]=\sum_{s=0}^{q_n-1}\log\Big[\frac{|f(I_s^{n-1})|}{|I_s^{n-1}|}\cdot\frac{|I_s^{n-1}|-|I_s^{n+1}|}{|f(I_s^{n-1})|-|f(I_s^{n+1})|}\Big], \tag{25}$$

$$\frac{|f(I_s^{n-1})|}{|I_s^{n-1}|\cdot f'(\xi_s)}=1+\int_{\xi_s}^{\xi_{s+q_{n-1}}}\frac{f''(y)}{f'(\xi_s)}\cdot\frac{\xi_{s+q_{n-1}}-y}{\xi_{s+q_{n-1}}-\xi_s}dy, \tag{26}$$

$$\frac{|f(I_s^{n-1})|-|f(I_s^{n+1})|}{\Big[|I_s^{n-1}|-|I_s^{n+1}|\Big]\cdot f'(\xi_{s+q_{n+1}})}=1+\int_{\xi_{s+q_{n+1}}}^{\xi_{s+q_{n-1}}}\frac{f''(y)}{f'(\xi_{s+q_{n+1}})}\cdot\frac{\xi_{s+q_{n-1}}-y}{\xi_{s+q_{n-1}}-\xi_{s+q_{n+1}}}dy. \tag{27}$$

Equality (25) comes from the multiplicative nature of ratio distortion with respect to composition and using (12) easily we can get equalities (26) and (27). Since $|I_s^{n-1}|\sim|I_s^{n-1}|-|I_s^{n+1}|$ and using (25)-(27) and similar arguments as in the proof of Lemma 3.1 we get

$$\log\Big[\frac{|I_{q_n}^{n-1}|}{|I_0^{n-1}|}\cdot\frac{|I_0^{n-1}|-|I_0^{n+1}|}{|I_{q_n}^{n-1}|-|I_{q_n}^{n+1}|}\Big]=\sum_{s=0}^{q_n-1}\int_{\xi_{s+q_{n+1}}}^{\xi_s}\frac{f''(x)}{f'(x)}dx+\sum_{s=0}^{q_n-1}\int_{\xi_s}^{\xi_{s+q_{n-1}}}\frac{f''(x)}{2f'(x)}dx-$$

$$-\sum_{s=0}^{q_n-1}\int_{\xi_{s+q_{n+1}}}^{\xi_{s+q_{n-1}}}\frac{f''(x)}{2f'(x)}dx+\mathcal{O}(\varepsilon_n)=\frac{(-1)^n}{2}\sum_{s=0}^{q_n-1}\int\limits_{I_s^{n+1}}\frac{f''(x)}{f'(x)}dx+\mathcal{O}(\varepsilon_n).$$

□

4 Prove of Theorem 1.2

Proof. To prove Theorem 1.2, we will use the following two relations:

$$\frac{|I_{q_n}^{n-1}|}{|I_0^{n-1}|}\cdot\frac{|I_0^{n-1}|-|I_0^{n+1}|}{|I_{q_n}^{n-1}|-|I_{q_n}^{n+1}|}-1=\frac{|I_0^{n+1}|}{|I_0^{n-1}|}\cdot\Big[\frac{|I_{q_n}^{n+1}|}{|I_0^{n+1}|}\cdot\frac{|I_0^{n-1}|-|I_0^{n+1}|}{|I_{q_n}^{n-1}|-|I_{q_n}^{n+1}|}-1\Big], \tag{28}$$

$$(f^{q_n}(\xi_0))'\mathcal{R}(\xi_0,I_0^{n-1};f^{q_n})-1=\frac{|I_0^n|}{|I_0^{n-1}|}\cdot\Big[1-(f^{q_{n-1}}(\xi_0))'\mathcal{R}(\xi_0,I_0^n;f^{q_{n-1}})\Big]. \tag{29}$$

The equality (28) is easily verified and (29) comes from the definition of ratio distortions of $\mathcal{R}(\xi_0, I_0^{n-1}; f^{q_n})$ and $\mathcal{R}(\xi_0, I_0^n; f^{q_{n-1}})$. Denote by

$$m_n = \exp\Big(\frac{(-1)^n}{2}\sum_{s=0}^{q_n-1}\int_{I_s^{n-1}}\frac{f''(x)}{f'(x)}dx\Big).$$

It is clear that

$$\sum_{s=0}^{q_n-1}\int_{I_s^{n-1}}\frac{f''(x)}{f'(x)}dx + \sum_{s=0}^{q_{n-1}-1}\int_{I_s^{n}}\frac{f''(x)}{f'(x)}dx = \int_{S^1}\frac{f''(x)}{f'(x)}dx = 0.$$

Thus we have

$$\exp\Big(\frac{(-1)^{n+1}}{2}\sum_{s=0}^{q_{n-1}-1}\int_{I_s^{n}}\frac{f''(x)}{f'(x)}dx\Big) = m_n.$$

Due to (28) and Lemma 3.2 we get

$$m_{n+1} - 1 = \frac{|I_0^{n+1}|}{|I_0^{n-1}|}(m_n - 1) + \mathcal{O}(\varepsilon_n), \tag{30}$$

which is iterated into

$$m_{n+1} - 1 = \mathcal{O}(\eta_{n+1}), \tag{31}$$

where

$$\eta_{n+1} = |I_0^n||I_0^{n+1}|\sum_{k=1}^{n}\frac{\varepsilon_k}{|I_0^k||I_0^{k+1}|}.$$

It is easy to see that $\eta_{n+1} \in \ell_2$. Due to (29), (31) and Lemma 3.1 we have

$$(f^{q_n}(\xi_0))' - 1 = \frac{|I_0^n|}{|I_0^{n-1}|}(1 - (f^{q_{n-1}}(\xi_0))') + \mathcal{O}(\tau_n), \tag{32}$$

which is iterated into

$$(f^{q_n}(\xi_0))' - 1 = \mathcal{O}\Big(|I_0^n|\sum_{k=1}^{n}\frac{\tau_k}{|I_0^k|}\Big),$$

where $\tau_n = \eta_n + \varepsilon_n$. The proof of Theorem 1.2 now follows from Lemma 2.4. □

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