Arithmetic-analytical expression and application of the Hilbert-type space-filling curves*

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Abstract

A new viewpoint is used to understand the generation process of the Hilbert curve. A one-to-one correspondence between the 4-adic expansion of the unit interval and the fractal curve's iterative generating process is established, and an analytical expression of the level-n Hilbert curve is obtained. This expression can take limit and represent the curve with 2-adic series. Though composition of functions this expression can substitute the generator of the Hilbert curve, while it can be proved, by using the expression, that the generation of the Hilbert curve depends on how the subsquares are connected rather than the shape of the generator.

Keywords: arithmetic-analytical expression, Hilbert-type space-filling curve, parent Hilbert curve, analytic transformation.

1 Introduction

In 1890, G. Peano discovered a continuous curve which could fill a square in a geometric way [1]. After that, in 1891, D. Hilbert also constructed another curve with the same property, and it was called the Hilbert curve [2]. For more than one hundred years, space-filling curves have attracted great interest for their wonderful construction and mathematical properties, so it brings a lot of research. Most of the fundamental results can be found in [3]. The value of space-filling curves lies in their capability to establish a corresponding relationship between one dimensional space and high dimensional space, which helps to map the data of multidimensional space into one dimensional space. In 1969 Butz advanced the theories and properties of Hilbert curve [4], pointing out that Hilbert scanning is a kind of two dimensional space scanning method which is continuous, without interleaving and passing through the consecutive points. C. Gostman and M. Lindenbaum have proved it is the scanning curve that best

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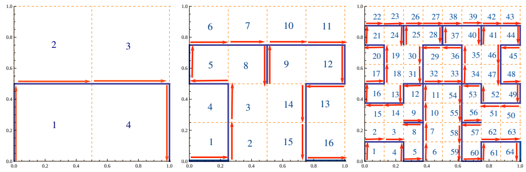
keep the spacial local adjacency [5], and shows an obvious superiority over other scanning methods. Therefore it is regarded as a tool of descending the spacial dimensions and is successfully applied in image processing and indexing of multi-dimensional data. As a facility of ergode and descending the spacial dimensions, Hilbert curve has constantly expanded its application domains recently [6–18]. Because the applications of the Hilbert curve depends on the calculation of its codings, and the classical geometric-based algorithm is a symbolic-system recursive algorithm, when encountering large-scaled computation or high dimensional application problem, it is time-consuming. Thus it is unfavourable to the practical application and popularization of space-filling curves. Many have done a lot of work on researching the fast generation algorithm [19–30]. However, these method of calculating the coding did not take advantage of the analytic expression of Hilbert curve, which reflects the one-to-one correspondence relationship between the data and the image. By applying our unique method of establishing arithmetic-analytical expression of fractals curves, this paper provides an arithmetic-analytic expression of the Hilbert space-filling curve. We also explain the difference between our method and that of H. Sagan's: in section 4 it is shown that our expression is capable of performing composition of functions to create "new" Hilbert curves, which can be proved to be the classic Hilbert curve despite the differences among their generators. Consequently, there is an important conclusion that the Hilbert space-filling curve is only determined by the sequence of how the 4 subsquares connect, and that the type of curves used to connect them are irrelevant. Thus we show that this expression can be use to study a series of analytic properties of the Hilbert curve. In section 5 we clarify further how to use this analytical expression, and briefly introduce our following research based on it.

2 The geometric procedure of filling curves of Hilbert

E. H. Moore [33] was the first to recognize a general geometrical generating procedure that allowed the construction of an entire class of space-filling curves. Following that, after 100 years or so, some others continued to construct geometric space-filling curves, see [3]. Moore gave the principle as following: If the interval I can be mapped continuously onto the square S, then after partitioning I into four congruent subintervals, and S into four congruent subsquares, then each subinterval can be mapped continuously onto one of the subsquare. Next, each subinterval is also partitioned into four congruent subintervals, and each subsquare into four congruent subsquares, and so on. I and S are partitioned into congruent replicas for $n = 1, 2, \cdots$. If a square corresponds to an interval, then its subsquares correspond to the subintervals of that interval. Apparently, determining the order of ubsquares is the key to the generating procedure. We use the bottom edge of a subsquare to represent it (Fig. 1(a)). Notice how the subsquares are replaced by smaller subsquares in Fig. 1, where Fig. 1(a) can be seen as the generator of this fractal curve. For convenience, we call it the parent Hilbert curve, which has the same order of subsquares as the original curve.



(a) (b) (c)

Fig. 1 Generating the parent Hilbert curve

In Fig. 2, replace each of the four subsquares whose bottom edges is represented by arrows with the generator of the Hilbert curve and a "tail". The tails connect the four parts so that we can get a Hilbert curve. Note that in level-2 Hilbert curve we use what is shown in 2(e) while when constructing a level-3 Hilbert curve, the "tail" may be either (c), (d) or (e). The Hilbert curve can then generated by the parent Hilbert curve, this generating procedure is shown in Fig. 2 and Fig. 3.

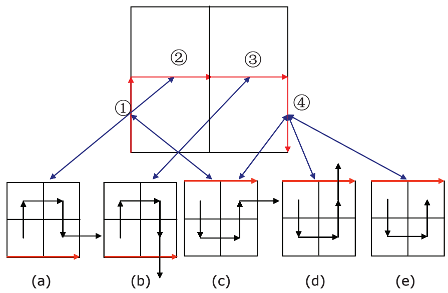


Fig. 2 The five transformation of generating Hilbert curve

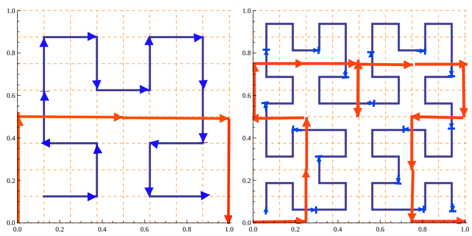


Fig. 3 The generating procedure of the Hilbert curve

3 The series expression of the Hilbert space-filling curve

H. Sagan has indicated in [3] that "Apparently, no attempt at an arithmetic analytic representation of the Hilbert curve has been made during the past 100 years in the belief that such an attempt would be very tedious". It is well known that space-filling curves are classic fractals which can also be generated by iterated function systems (IFS). And in [3], H. Sagan gave an arithmetic process to determine the parent Hilbert curve. In fact, on the complex plane, the IFS of parent Hilbert space-filling curve is as follows

$$M(Z) = \begin{cases} \frac{1}{2}\bar{Z}i \\ \frac{1}{2}Z + \frac{i}{2} \\ \frac{1}{2}Z + \frac{1}{2} + \frac{i}{2} \\ -\frac{1}{2}\bar{Z}i + 1 + \frac{i}{2} \end{cases} \quad Z = x + yi, 0 \leq x, y \leq 1 \tag{1}$$

and let $t \in [0,1]$ to be represented in quaternary form

$$t = 0_4.b_1 b_2 \cdots b_n \cdots = \sum_{n=1}^{\infty} \frac{b_n}{4^n}, \quad b_n \in \{0,1,2,3\} \tag{2}$$

or binary form

$$t = \frac{a_1}{2} + \frac{\tau_1}{2} = \cdots = \sum_{k=1}^{n} \frac{a_k}{2^k} + \frac{\tau_n}{2^n} = \cdots = \sum_{n=1}^{\infty} \frac{a_n}{2^n}, \tag{3}$$

and

$$Z = x(t) + y(t)i,$$

where $b_n = 2a_{n-1} + a_n$, $\{a_1, a_2, \cdots, a_{2n}\}$ are the first $2n$ digits of the binary expansion of t, and τ_{2n} is the remainder. The relation between a_1, a_2, τ_2 and t is shown in Fig. 4.

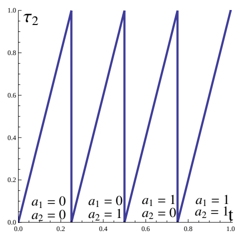


Fig. 4 The Relation between a_1, a_2, τ_2 and t.

To solve the problem, we now only need to find the curve's analytical representation of the following form:

$$\left\{ \begin{array}{l} x = \phi(t) \\ y = \varphi(t) \end{array} \right. , t = \sum_{i=1}^{\infty} \frac{a_i}{2_i}, \; a_i \in \{0, 1\}.$$

because

$$\begin{cases} a_1 = 0, a_2 = 0 & \text{if } t \in [0, \frac{1}{4}); \\ a_1 = 0, a_2 = 1 & \text{if } t \in [\frac{1}{4}, \frac{1}{2}); \\ a_1 = 1, a_2 = 0 & \text{if } t \in [\frac{1}{2}, \frac{3}{4}); \\ a_1 = 1, a_2 = 1 & \text{if } t \in [\frac{3}{4}, 1] \end{cases}$$

So we can combine the equations in (1) in one formula:

$$M(Z) = \frac{1}{2} e^{h_1 \frac{\pi}{2} i} \left((1 - g_1 Z + g_1 \bar{Z} \right) + \frac{u_1}{2} + \frac{v_1}{2} i. \tag{4}$$

where

$$\begin{array}{l} h_1 = h(a_1, a_2) = 1 - a_1 - a_2, \\ g_1 = g(a_1, a_2) = 1 - a_1 - a_2 + 2a_1 a_2, \\ u_1 = (a_1, a_2) = a_1(1 + a_2), \; v_1 = (a_1, a_2) = a_1 + a_2 - a_1 a_2 \end{array}$$

Let Z be the initiator Z_0, then $Z_0 = x(t) = t, t \in [0, 1]$.

$t = a_1/2 + a_2/2^2 + \tau_2/2^2$ stands for the first step in the parent Hilbert curve construction. With (4) we have

$$M(t) = \frac{1}{2} e^{h_1 \frac{\pi}{2} i} \tau_2 + \frac{u_1}{2} + \frac{v_1}{2} i.$$

where $u_1/2 + v_1 i/2$ is to determine the starting points of directed line segments, τ_2 draws the lines as it changes between $\in [0, 1]$.

$t = a_1/2 + a_2/2^2 + a_3/2^3 + a_4/2^4 + \tau_4/2^4$ stands for the second step, that is, representing τ_2 in quaternary form. In Figure 1(b) each of the four directed line segments are replaced by four new line segments (16 line segments in total). So for τ_2, we have

$$M(\tau_2) = \frac{1}{2} e^{h_2 \frac{\pi}{2} i} \tau_4 + \frac{u_2}{2} + \frac{v_2}{2} i.$$

inserted into (4)

$$M_2(t) = M(M(\tau_2)) = \frac{1}{2} e^{(h_1 + (-1)^{g_1} h_2 \frac{\pi}{2} i} \tau_4 + \frac{1}{2} e^{h_1 \frac{\pi}{2} i} \Big(\frac{u_2}{2} + (-1)^{g_1} \frac{v_2}{2} i \Big) + \frac{u_1}{2} + \frac{v_1}{2} i.$$

Carry out this process for nth time, where

$$t = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots + \frac{a_{2n-1}}{2^{2n-1}} + \frac{a_{2n}}{2^{2n}} + \frac{\tau_{2n}}{2^{2n}}$$

the $n-1$ times expansion of τ_2:

$$\begin{aligned} M_n(t) = M(M_{n-1}(\tau_2)) = & \frac{\tau_{2n}}{2^{2n}} e^{\sum\limits_{k=1}^{n} (-1)^{\sum\limits_{s=1}^{k-1} g_s} h_k \frac{\pi}{2} i} \\ & + \sum_{j=1}^{n} \frac{1}{2^j} e^{\sum\limits_{k=1}^{j-1} (-1)^{\sum\limits_{s=1}^{k-1} g_s} h_k \frac{\pi}{2} i} \cdot \left(u_j + (-1)^{\sum\limits_{k=1}^{j-1} g_k} v_j i\right). \end{aligned} \tag{5}$$

Let $n \to \infty$,

$$M^*(t) = \lim_{n \to \infty} M_n(t) = \sum_{j=1}^{\infty} \frac{1}{2^j} e^{\sum\limits_{k=1}^{j-1} (-1)^{\sum\limits_{s=1}^{k-1} g_s} h_k \frac{\pi}{2} i} \cdot \left(u_j + (-1)^{\sum\limits_{k=1}^{j-1} g_k} v_j i\right). \tag{6}$$

where τ_{2n}, $a_i \in \{0, 1\}, i = 1, 2, \cdots, n$ and$t$ satisfy equation:

$$t = \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_{2n}}{2^{2n}} + \frac{\tau_{2n}}{2^{2n}}$$

therefore a_i as the function of t can be determined in the following way:

$$a_j = a_j(t) = \begin{cases} [2^j t] - 2[2^{j-1} t] & : t \neq 1 \\ 1 & : t = 1 \end{cases}, j = 1, 2, 3, \cdots, n.$$

where $[x]$ is the integer part of x. τ_{2n} is also a function of t:

$$\tau_{2n} = 2^{2n} t - 2^{2n-1} a_1 - 2^{2n-2} a_2 - 2^{2n-3} a_3 - \cdots - 2a_{2n-1} - a_{2n}$$

$u_j = u(a_{2j-1}, a_{2j}) = a_{2j-1}(1 + a_{2j})$, $v_j = v(a_{2j-1}, a_{2j}) = a_{2j-1} + a_{2j} - a_{2j} a_{2j-1}(1 + a_{2j})$,
$g_j = g(a_{2j-1}, a_{2j}) = 1 - a_{2j-1} - a_{2j} + 2a_{2j-1} a_{2j}$, $h_j = h(a_{2j-1}, a_{2j}) = 1 - a_{2j-1} - a_{2j}$,

Let

$$\begin{cases} x = \varphi(t) = Re(M^*(t)) \\ y = \phi(t) = Im((M^*(t)) \end{cases}$$

is the analytical expression of parametric equation for the nth iterated parent Hilbert curve, which is show in Fig 1.

(6) is an unusual function representation, where $M * (t)$ as the function of t is determined by the binary expansion digits $\{a_{2n-1}, a_{2n}\}_{n=1}^{\infty}$. So we call it the arithmetic-analytical expression.

In order to explain each part of (5), we denote it as

$$M_n(t) = \frac{1}{2^n} \tau_{2n} e^{A \frac{\pi}{2} i} + C.$$

When t changes from 0 to 1, τ_{2n} changes from 0 to 1 repeatedly for 2^{2n} times. In (5), $\tau_{2n}e^{A\frac{\pi}{2}i}/2^n$ represents 2^{2n} line segments with length $1/2^n$, and $e^{A\frac{\pi}{2}i}$ controls which direction each line segment goes. C determines the starting point of each line segment.

Theorem 1. *The analytical expression of nth parent Hilbert curve $M_n(t)$ is uniform continuous on interval [0,1]*

Proof. $\forall \varepsilon > 0$ and $\forall t'$, $t'' \in [0, 1]$, $\exists$ integer $N > 0$, such that $1/2^{n-1} < 1/2^{N-2}$, $1/2^{N-4} < \varepsilon$, when $|t' - t''| < 1/2^{2N}$,

$$t' = \sum_{k=1}^{2N-1} \frac{a_k}{2^{2k}} + \sum_{k=2N}^{2n} \frac{a'_k}{a_{2k}} + \frac{\tau_{2n}}{2^{2n}}, t'' = \sum_{n=1}^{2N-1} \frac{a_k}{2^{2k}} + \sum_{k=2N}^{2n} \frac{a''_k}{a_{2k}} + \frac{\tau_{2n}}{2^{2n}}.$$

Because when $x, y \in \{0, 1\}$, $|u(x, y)| = |x(1+y)| < 2$, $|v(x, y)| = |x+y-xy| < 1$,

$$\begin{aligned}\|M_n(t') - M_n(t'')\| \leqslant & \frac{1}{2^{n-1}} + \left|\sum_{j=N}^{2n} \frac{u(a'_{2j-1}, a'_{2j})}{2^j}\right| + \left|\sum_{j=N}^{2n} \frac{u(a''_{2j-1}, a''_{2j})}{2^j}\right| \\ & + \left|\sum_{j=N}^{2n} \frac{v(a'_{2j-1}, a'_{2j})}{2^j}\right| + \left|\sum_{j=N}^{2n} \frac{v(a''_{2j-1}, a''_{2j})}{2^j}\right| \\ \leqslant & \frac{1}{2^{n-1}} + 6\sum_{j=N}^{\infty} \frac{1}{2^j} \leqslant \frac{1}{2^{N-4}} < \varepsilon\end{aligned}$$

So $M_n(t)$ is uniform continuous on [0,1]. □

From Fig. 3, we can know that the nth Hilbert curve can be obtained from the $(n-1)$th parent Hilbert curve through 5 types of transformation as is shown in Fig.2, in which the transformation can be attributed to replacing τ_{2n-2} in M_{n-1} with the following formula:

$$\begin{aligned}q(t, n) =& \frac{1 - \prod_{j=1}^{2n} a_j}{2} \tau_{2n} \cdot e^{(-1)^{\sum_{j=1}^{n-1} g_j}((1-2a_{2n-1})(1-a_{2n}) + \sum_{j=1}^{n} a_{2j-1} \prod_{k=j}^{n-1} (-1) a_{2k-1} a_{2k}) \frac{\pi}{2} i} \\ & + \frac{1 + 2a_{2n-1}}{4} + (-1)^{\sum_{j=1}^{n-1} g_j} \frac{1 + 2g_n}{4}\end{aligned} \tag{7}$$

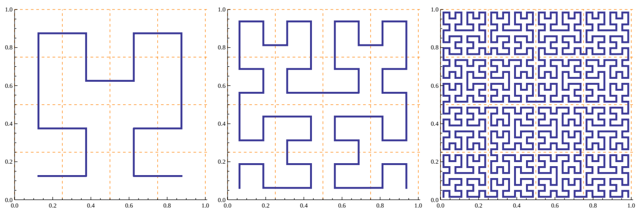
and the obtained nth analytic representations of ordinarily Hilbert space-filling curve is

$$H_n(t) = \frac{1-\prod_{j=1}^{2n} a_j}{2^n}\tau_{2n} e^{\sum_{j=1}^{n-1}(-1)^{\sum_{k=1}^{j-1} g_k} h_j \frac{\pi}{2} i} \cdot e^{(-1)^{\sum_{j=1}^{n-1} g_j}((1-2a_{2n-1})(1-a_{2n})+\sum_{j=1}^{n} a_{2j-1}\prod_{k=j}^{n-1}(-1)a_{2k-1}a_{2k}))\frac{\pi}{2}i}$$
$$+ e^{\sum_{j=1}^{n-1}(-1)^{\sum_{k=1}^{j-1} g_k} h_j \frac{\pi}{2} i} \cdot (\frac{1+2a_{2n-1}}{2^{n+1}} + (-1)^{\sum_{j=1}^{n-1} g_j}\frac{1+2g_n}{2^{n+1}} i)$$
$$+ \sum_{j=1}^{n-1}\frac{1}{2^j} e^{(\sum_{k=1}^{j-1}(-1)^{\sum_{s=1}^{k-1} g_s} h_k \frac{\pi}{2} i} \cdot (u_j + (-1)^{\sum_{k=1}^{j-1} g_k} v_j i). \tag{8}$$

Taking $n \to \infty$ get

$$H(t) = \lim_{n\to\infty} H_n(t) = M^*(t) \tag{9}$$

Where τ_{2n} and $a_i \in \{0, 1\}, i = 1, 2, \cdots, n$ are same before . Taking $n = 2, 3, 5$, we draw Fig. 5 by using equation (8).



$n = 2$ $n = 3$ $n = 5$

Fig. 5 ordinarily Hilbert space-filling curve

In order to explain each part of (8), we denote it as

$$M_n(t) = \frac{B}{2^n}\tau_{2n} e^{A\frac{\pi}{2}i} + C.$$

When t changes from 0 to 1, τ_{2n} changes from 0 to 1 repeatedly for 2^{2n} times. In (5), $\tau_{2n} e^{A\frac{\pi}{2}i}/2^n$ represents 2^{2n} line segments with length 2^{-n}; B controls whether the the line segment has a "tail" attached to it and $e^{A\frac{\pi}{2}i}$ controls which direction each line segment goes. C determines the starting point of each line segment.

Theorem 2. *The analytical expression of nth Hilbert curve $H_n(t)$ is uniform continuous on interval [0,1]*

Proof. $\forall \varepsilon > 0$ and $\forall t'$, $t'' \in [0, 1]$, $\exists$ integer $N > 0$, such that $1/2^{n-1} < 1/2^{N-2}$, $1/2^{N-5} < \varepsilon$, when $|t' - t''| < 1/2^{2N}$,

$$t' = \sum_{k=1}^{2N-1}\frac{a_k}{2^{2k}} + \sum_{k=2N}^{2n}\frac{a'_k}{a_{2k}} + \frac{\tau_{2n}}{2^{2n}}, t'' = \sum_{n=1}^{2N-1}\frac{a_k}{2^{2k}} + \sum_{k=2N}^{2n}\frac{a''_k}{a_{2k}} + \frac{\tau_{2n}}{2^{2n}}.$$

Because when $x, y \in \{0,1\}$, $|g(x,y)| = |1 - x - y + 2xy| < 1$, $|u(x,y)| = |x(1+y)| < 2$,

$$\begin{aligned}\|H_n(t') - H_n(t'')\| \leqslant & \frac{1}{2^{n-1}} + \left|\frac{1+2a_{2n-1}}{2^{n+1}}\right| + \left|\frac{1+2g(a_{2n-1}, a_{2n})}{2^{n+1}}\right| \\ & + \left|\sum_{j=N}^{2n} \frac{u(a'_{2j-1}, a'_{2j})}{2^j}\right| + \left|\sum_{j=N}^{2n} \frac{u(a''_{2j-1}, a''_{2j})}{2^j}\right| \\ & + \left|\sum_{j=N}^{2n} \frac{v(a'_{2j-1}, a'_{2j})}{2^j}\right| + \left|\sum_{j=N}^{2n} \frac{v(a''_{2j-1}, a''_{2j})}{2^j}\right| \\ \leqslant & \frac{1}{2^{n-1}} + \frac{3}{2^n} + 6\sum_{j=N}^{\infty} \frac{1}{2^j} \leqslant \frac{1}{2^{N-4}} = \frac{5}{2^n} + \frac{3}{2^{N-2}} = \frac{1}{2^{N-5}} < \varepsilon\end{aligned}$$

So $H_n(t)$ is uniform continuous on [0,1]. □

Obviously, the nth analytic representations of ordinarily Hilbert space-filling curve given by (8) is also a one-to-one mapping of numbers and graphical elements and suitable to calculate.

4 A class of Hilbert-type space-filling curves and their series expressions

From the deduction of the analytic expression of another Hilbert curve mentioned above, it can be seen that if we use another analytic transformation, the arithmetic expressions of new type Hilbert curves can be similarly obtained through substituting each line segment in parent Hilbert curve with analytical representations of other curves.

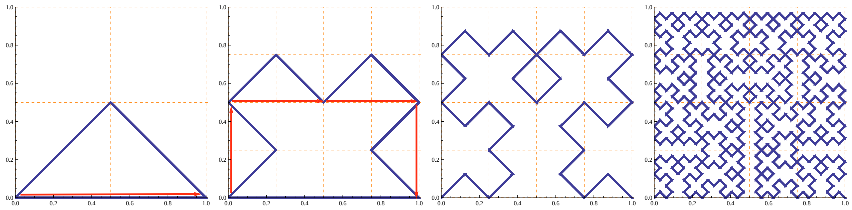
Example 1. *Replacing τ_{2n-2} in M^{n-1} with*

$$q_1(t,n) = (1 + (1 - 2a_{2n-1})(-1)^{\sum_{j=1}^{n-1} g_j} i)\frac{\tau_{2n}}{2} + (1 + (-1)^{\sum_{j=1}^{n-1} g_j} i)\frac{a_{2n-1}}{2} \tag{10}$$

then we obtain

$$\begin{aligned}H_{1n}(t) = & \frac{\tau_{2n}}{2^n}(1 + (1 - 2a_{2n-1})(-1)^{\sum_{j=1}^{n-1} g_j} i) \cdot e^{(\sum_{j=1}^{n-1} (-1)^{\sum_{k=1}^{j-1} g_k} h_j \frac{\pi}{2} i} \\ & + (1 + (-1)^{\sum_{j=1}^{n-1} g_j} i)\frac{a_{2n-1}}{2} \cdot e^{\sum_{j=1}^{n-1} (-1)^{\sum_{k=1}^{j-1} g_k} h_j \frac{\pi}{2} i} \\ & + \sum_{j=1}^{n-1} \frac{1}{2^j} e^{\sum_{k=1}^{j-1} (-1)^{\sum_{s=1}^{k-1} g_s} h_k \frac{\pi}{2} i} \cdot (u_j + (-1)^{\sum_{k=1}^{j-1} g_k} v_j i).\end{aligned} \tag{11}$$

By equation (11) , taking $n = 2, 3, 5,$, we can draw Fig. 6.



$n = 1$ $n = 2$ $n = 3$ $n = 5$

Fig. 6 Hilbert-type space-filling curves in Example 1

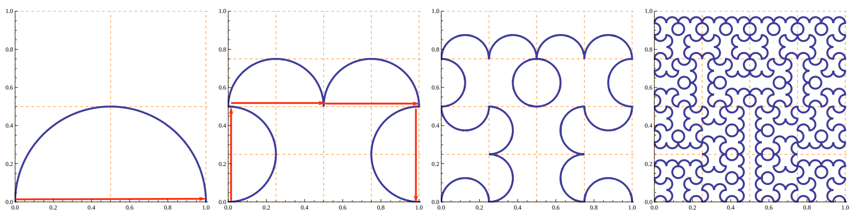
Example 2. *Replacing τ_{2n-2} in M^{n-1} with*

$$q_2(t,n) = \frac{1}{2}(1 + e^{(-1)^{\sum_{j=1}^{n-1} g_j}\pi(1-\tau_{2n-2})i}) \tag{12}$$

then we obtain

$$\begin{aligned} H_{2n}(t) =& \frac{1}{2^n}(1 + e^{(-1)^{\sum_{j=1}^{n-1} g_j}\pi(1-\tau_{2n-2})i}) \cdot e^{\sum_{j=1}^{n-1}(-1)^{\sum_{k=1}^{j-1} g_k} h_j \frac{\pi}{2} i} \\ &+ \sum_{j=1}^{n-1} \frac{1}{2^j} e^{\sum_{k=1}^{j-1}(-1)^{\sum_{s=1}^{k-1} g_s} h_k \frac{\pi}{2} i} \cdot (u_j + (-1)^{\sum_{k=1}^{j-1} g_k} v_j i). \end{aligned} \tag{13}$$

And Fig. 7 is drawn with equation (13), taking $n = 2, 3, 5,$.



$n = 1$ $n = 2$ $n = 3$ $n = 5$

Fig. 7 Hilbert-type space-filling curves in Example 2

With the above method, we can also get other arithmetic expressions of Hilbert-type space-filling curves which are shown in Fig. 8 and Fig. 9.

From the above examples, we can see that many new Hilbert-type space-filling curves can be obtained by replacing the line segments with curve segments in the Hilbert curve, as long as the continuity can be maintained. And their expressions can be obtained by composition of functions. The Hilbert curve

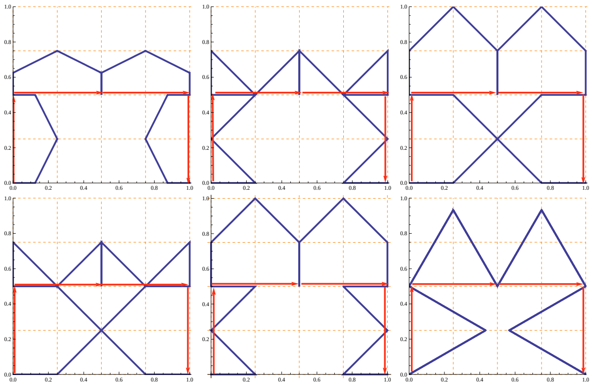


Fig. 8 6 broken line derived from parent Hilbert curve

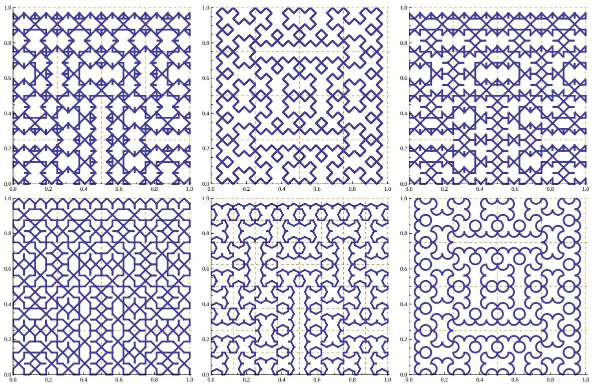


Fig. 9 6 Hilbert-type space-filling curves

is only determined by the way how the four subsquares connect. That is to say, no matter what kind of continuous curve segment is used, the space-filling curve obtained remains the same when the expression takes limit - a property can be readily proved by analytical expression. And this make our analytical expression distinct from Hans Sagan's method.

In fact, our method is applicable to other space-filling curves. So this method also provides an effective way to construct more space-filling curves as well as finding their arithmetic-analytical expressions.

5 Additional explanations on the use of the arithmetic-analytical expression

(5) and (8) provides an analytical expression based on the binary expansion of t. The arithmetic-analytical expression lays the foundation of our study on

the Hilbert curve. Its primary significance is that each generating procedure can be directly and precisely represented by functions. The traditional geometric methods, on the other hand, require previous iteration results before they can calculate the next step of iteration. Because (5) and (8) are two continuous functions that shows how the actual Hilbert-type curves exist, their forms are complicated. On the other hand, they are ideal for generating new variants of Hilbert-type curves by substitute the variable. In (8), let $\tau_{2n} = 0$, we immediately get the formula for the nth Hilbert order and the corresponding coordinates:

$$\begin{aligned} H_{code}(t_k) =& \Big(\frac{1+2a_{2n-1}}{2^{n+1}} + i(-1)^{n-1+\sum\limits_{k=1}^{n-1} g_k} \frac{3-2g_n}{2^{n+1}}\Big) e^{\big(\sum\limits_{k=1}^{n-1}(-1)^{(j-1+\sum\limits_{k=1}^{j-1} g_k)} h_j\big)\frac{\pi}{2} i} \\ &+ \sum_{j=1}^{n-1} \frac{1}{2^j} e^{\big(\sum\limits_{k=1}^{j-1}(-1)^{(k-1+\sum\limits_{s=1}^{k-1} g_s)} h_s\big)\frac{\pi}{2} i} \cdot (u_j + (-1)^{(j-1+\sum\limits_{k=1}^{j-1} g_k)} \cdot v_j i \end{aligned} \tag{14}$$

where $t_k = k/2^{2n}$, $k < 2^{2n}$ is a positive integer, and

$$a_j = a_j(t_k) = \begin{cases} [2^j t_k] - 2[2^{j-1} t_k] & : t \neq 1 \\ 1 & : t = 1 \end{cases}$$

However, for those who often deal with Hilbert encoding/decoding problems, this formula is not very convenient to use. In practical applications, the vertices on the Hilbert curve and their Hilbert order are more important. So we have deducted from our formula another coding formula and its simplified algorithm, whose time complexity for encoding/decoding a single point is no greater than $O(n)$. We will introduce them in another paper entitled "A Formulated Algorithm Based on Binary Expansion Series for Encoding and Decoding the Hilbert Order".

In addition, we have constructed several variants of Moore-Hilbert curve (Fig. 10), as well as the Hilbert curves on rectangular regions (Fig. 11). These results will be discussed in our future papers. We have obtained the arithmetic-analytical expression for 3-dimensional Hilbert curve, but its related problems are yet to be discussed.

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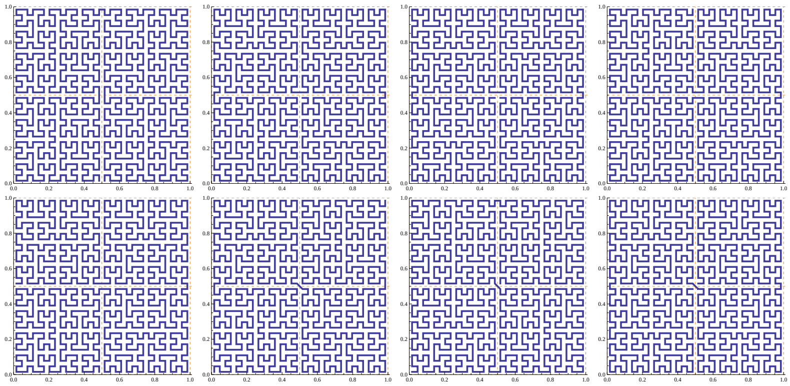

Fig. 10 Some variants of Moore-Hilbert space-filling curves

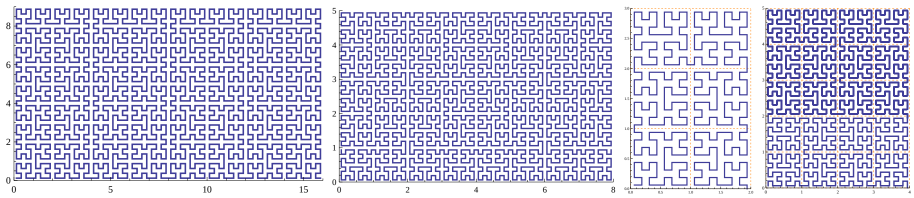

Fig. 11 The Hilbert curves on rectangular regions

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