Certain properties of n-characters and n-homomorphisms on topological algebras

H. Shayanpour, T. G. Honary and M. S. Hashemi

Abstract. We extend the notion of homomorphisms and characters to n-homomorphisms and n-characters on algebras, and then show that some properties of characters are also valid for n-characters on commutative *lmc* topological algebras, and the space of continuous n-characters $M_{(A,n)}$ is relatively compact in A' (the dual space of A), with the weak* topology (Gelfand topology), whenever A is a commutative *lmc* Q-algebra. We also find relations between characters, n-characters and continuous n-characters on commutative Fréchet algebras.

Let B be a topological algebra and $(A_\alpha, \varphi_{\beta\alpha})$ (resp. $(A_\alpha, \varphi_{\alpha\beta})$) be an inductive system (resp. a projective system) of topological algebras. Then we obtain relations between $n-Hom(A_\alpha, B)$ and $n-Hom(A, B)$, or between $\varinjlim M_{(A_\alpha,n)}$, the inductive limit, and $M_{(A,n)}$, where $A = \varinjlim A_\alpha$, is the inductive limit (resp. $A = \varprojlim A_\alpha$, is the projective limit) and $n - Hom(A_\alpha, B)$ (resp. $n - Hom(A, B)$), is the space of all continuous n-homomorphisms from A_α (resp. A) into B.

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1. Introduction

We first bring notation, definitions and known results, which are related to our work. Let A and B be two complex algebras and $n \geq 2$ a fixed integer. A map $\varphi : A \to B$ is called n-multiplicative if $\varphi(a_1a_2\cdots a_n) = \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)$ for all elements $a_1, a_2, ..., a_n \in A$. Moreover, if φ is linear then it is called an n-homomorphism. If $\varphi : A \to \mathbb{C}$ is a non-zero n-homomorphism, then φ is called a complex n-character, or in brief, an n-character of A. Clearly every 2-character is just a character, in the usual sense. The set of all characters of

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A is denoted by S_A and the set of all n-characters of A is denoted by $S_{(A,n)}$. If A is a complex topological algebra, then the set of all continuous n-characters of A is denoted by $M_{(A,n)}$. We call $M_{(A,n)}$ the n-spectrum or the continuous n-character space of A. As usual, the set of all continuous characters of A is denoted by M_A. Obviously, each character is an n-character for every $n \geq 2$, but the converse is not true, in general. For example, if φ is a character and $n \geq 2$ is an odd number, then $\theta = -\varphi$ is an n-character, which is not a character.

The notion of n-homomorphisms between Banach algebras was first introduced, and some of their significant properties were discussed, by Hejazian, Mirzavaziri and Moslehian in 2005 [4]. They also raised the following question in this article: Is every *-preserving n-homomorphism between C*-algebras continuous? This question was answered in the affirmative, by Park and Trout in 2010 [13], but the even and odd n arguments are surprisingly disjoint. They also presented many other interesting results in this article. In 2007, Bračič and Moslehian studied the notion of automatic continuity of 3-homomorphisms on Banach algebras [1]. Later in 2010 and 2011 the problem of automatic continuity on n-homomorphisms between Banach algebras, Fréchet algebras and topological algebras was discussed in [5], [7] and [6], respectively. In 2012 the automatic continuity of higher derivations between Banach algebras was studied by Mirzavaziri and Omidvar Tehrani in [12].

In this paper, we obtain some results on the interesting class of complex-valued n-homomorphisms, which are called n-characters.

The set of all Gelfand transforms of elements of an algebra A is denoted by $\hat{A}$. We endow $S_{(A,n)}$, $M_{(A,n)}$ and S_A with the Gelfand topology, that is, the weakest topology making all elements of $\hat{A}$ continuous on $S_{(A,n)}$, $M_{(A,n)}$ and S_A, respectively. Hence a basis for the neighborhood system of a point $\phi \in S_{(A,n)}$ (resp. $\phi \in M_{(A,n)}$) is given by all sets of the form

$$\{\psi \in S_{(A,n)} : \ |\psi(a_i) - \phi(a_i)| < 1, \ i = 1, ..., r, \ a_i \in A\}$$

$$(\{\psi \in M_{(A,n)} : \ |\psi(a_i) - \phi(a_i)| < 1, \ i = 1, ..., r, \ a_i \in A\}).$$

A locally multiplicatively convex (lmc) algebra is a topological algebra whose topology is defined by a separating family $\mathcal{P} = (p_\alpha)$ of submultiplicative seminorms. Note that here we have not assumed that the multiplication map is jointly continuous. It is, in fact, continuous with respect to each component. A complete metrizable lmc algebra is a Fréchet algebra. A topological algebra A is advertibly complete if every Cauchy net $\{x_\alpha\}$ in A converges in A whenever for some $x \in A$, both $x_\alpha \diamond x = x_\alpha + x - x_\alpha x$ and $x \diamond x_\alpha = x + x_\alpha - x x_\alpha$ converge to zero. An element $a \in A$ is left (right) quasi-invertible if there exists $b \in A$ such that $b \diamond a = 0$ ($a \diamond b = 0$). An element $a \in A$ is quasi-invertible if it is both left and right quasi-invertible in A. A topological algebra A is a Q-algebra if the set of its quasi invertible elements ($q - \text{Inv}A$) is open in A.

An ideal I of an algebra A is a regular (modular) ideal if there exists $a \in A$ such that $ax - x \in I$ and $xa - x \in I$, for every element $x \in A$. Equivalently, a proper ideal I is regular if and only if A/I is a unital algebra. A topological algebra A is called n-functionally continuous if every n-character on A is continuous, in other words, $S_{(A,n)} = M_{(A,n)}$. Clearly every 2-functionally continuous algebra is just functionally continuous, in the usual sense.

Let $(I, \leq)$ be a directed set and let $\{A_\alpha : \alpha \in I\}$ be a family of sets (algebras). Assume that, for every pair of indices (α, β) in I, with $\alpha \leq \beta$, we are given a map (homomorphism) $\varphi_{\beta\alpha} : A_\alpha \to A_\beta$ satisfying the following two conditions:

(i) for all $\alpha \in I$, $\varphi_{\alpha\alpha}$ is the identity map (homomorphism),
(ii) for all $\alpha \leq \beta \leq \gamma$, $\varphi_{\gamma\alpha} = \varphi_{\gamma\beta} \circ \varphi_{\beta\alpha}$.

Then $(A_\alpha, \varphi_{\beta\alpha})$ is called an inductive system of sets (algebras).

Now consider the set $A_0 = \cup_{\alpha \in I}\{(x, \alpha) : x \in A_\alpha\}$. We say that two elements (x, α) and (y, β) of A_0 are equivalent ($x \sim y$) if there exists an index $\gamma \in I$ with $\gamma \geq \alpha$ and $\gamma \geq \beta$ such that $\varphi_{\gamma\alpha}(x) = \varphi_{\gamma\beta}(y)$. Clearly, $\sim$ is an equivalence relation. We now take

$$A = A_0/\sim, \tag{1.1}$$

and let $\psi_\alpha : A_\alpha \to A_0$, defined by $\psi_\alpha(x) = (x, \alpha)$, be the canonical injection map for every $\alpha \in I$, and let $\pi : A_0 \to A$ be the quotient map. Thus $\phi_\alpha = \pi \circ \psi_\alpha : A_\alpha \to A$ is the canonical map for every $\alpha \in I$ and it is easy to see that $A = \cup_{\alpha \in I}\phi_\alpha(A_\alpha)$. The set A is called the inductive limit of the given inductive system of sets (or, algebras) $(A_\alpha, \varphi_{\beta\alpha})$, which is denoted by $A = \varinjlim(A_\alpha, \varphi_{\beta\alpha})$, or simply by $A = \varinjlim A_\alpha$. An inductive system of sets (resp. algebras) $(A_\alpha, \varphi_{\beta\alpha})$, with respect to an index set I, is an inductive system of topological spaces (resp. topological algebras) if for every $\alpha \in I$, A_α is a topological space (resp. topological algebra) and for each $\alpha, \beta \in I$ with $\alpha \leq \beta$, $\varphi_{\beta\alpha} : A_\alpha \to A_\beta$ is a continuous map (resp. continuous homomorphism). If $(A_\alpha, \varphi_{\beta\alpha})$ is an inductive system of topological spaces (resp. topological algebras) and $A = \varinjlim A_\alpha$ then A is called the inductive limit of topological spaces (resp. topological algebras).

Let $A = \varinjlim A_\alpha$, for an inductive system of topological spaces (resp. topological algebras) $(A_\alpha, \varphi_{\beta\alpha})$, with respect to an index set I and let $\phi_\alpha : A_\alpha \to A$ be the canonical map defined as above. The final topology on A, defined by the maps $\phi_\alpha, \alpha \in I$, is the strongest topology on A for which the maps ϕ_α are continuous. If $(A_\alpha, \varphi_{\beta\alpha})$ is an inductive system of topological algebras, then by [10, IV. pp. 111-112] $A = \varinjlim A_\alpha$ is an algebra, ϕ_α is a homomorphism and by [10, IV. Lemma 2.2] A is a topological algebra with the final topology.

Let $(I, \leq)$ be a directed set and let $\{A_\alpha : \alpha \in I\}$ be a family of sets (resp. algebras). Assume that for every pair of indices (α, β) in I, with $\alpha \leq \beta$, we are given a map (resp. homomorphism) $\varphi_{\alpha\beta} : A_\beta \to A_\alpha$ in such a way that the following two conditions are satisfied:

(i) for all $\alpha \in I$, $\varphi_{\alpha\alpha}$ is the identity map (resp. homomorphism).
(ii) for all $\alpha \leq \beta \leq \gamma$, $\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma}$.

Then $(A_\alpha, \varphi_{\alpha\beta})$ is called a projective system of sets (resp. algebras). Also the set

$$A = \{x = \{x_\alpha\} \in \prod_{\alpha \in I} A_\alpha : \ \varphi_{\alpha\beta}(x_\beta) = x_\alpha \text{ for every } \alpha, \beta \in I \text{ with } \alpha \leq \beta\}$$

is the projective limit of the given projective system of sets (resp. algebras) $(A_\alpha, \varphi_{\alpha\beta})$, which is denoted by $A = \varprojlim(A_\alpha, \varphi_{\alpha\beta})$, or simply by $A = \varprojlim A_\alpha$.

Now we define a family $\{\phi_\alpha\}$ of canonical maps (resp. homomorphisms) by the relation $\phi_\alpha = \pi_\alpha|_A : A \to A_\alpha$, where π_α is the canonical projection map of $\prod_{\gamma \in I} A_\gamma$ onto A_α. Thus ϕ_α is not a surjective map, in general, and moreover, $\phi_\alpha = \varphi_{\alpha\beta} \circ \phi_\beta$ for every $\alpha, \beta \in I$ with $\alpha \leq \beta$. A projective system of topological spaces (resp. toppological algebras) is a projective system of sets (resp. algebras) $(A_\alpha, \varphi_{\alpha\beta})$ as above, where A_α is a topological space (resp. topological algebra) for every $\alpha \in I$ and the maps $\varphi_{\alpha\beta}$ are continuous maps (resp. homomorphisms) for all $\alpha, \beta \in I$ with $\alpha \leq \beta$. If $(A_\alpha, \varphi_{\alpha\beta})$ is a projective system of topological spaces (resp. topological algebras) and $A = \varprojlim A_\alpha$, then A is called the projective limit of topological spaces (resp. topological algebras). By [10, III. Lemma 2.1] every projective limit of topological algebras $A = \varprojlim A_\alpha$ is a closed subalgebra of the cartesian topological algebra $\prod_{\gamma \in I} A_\gamma$.

Let B be a topological algebra and let $(A_\alpha, \varphi_{\beta\alpha})$ be an inductive system of topological algebras with respect to a directed index set I. Then $n\text{-}Hom(A_\beta, B)$ is the space of all continuous n-homomorphisms of A_β into B for $\beta \in I$, with the topology of pointwise convergence. We now define the maps $T_{\alpha\beta} : n\text{-}Hom(A_\beta, B) \to n\text{-}Hom(A_\alpha, B)$ by $T_{\alpha\beta}(h) = h \circ \varphi_{\beta\alpha}$ for $\alpha \leq \beta$. It is easy to see that $(n\text{-}Hom(A_\beta, B), T_{\alpha\beta})$ is a projective system of topological spaces.

For further details on the above concepts and properties one can refer, for example, to [2], [3], [9] and [10].

In Section 2 it is shown that some properties of the continuous character space M_A are also valid for the continuous n-character space $M_{(A,n)}$, when A is a commutative *lmc* algebra. We show that the kernel of every n-character is a regular maximal ideal and there exists a surjection between $M_{(A,n)}$ and the set of closed regular maximal ideals of A, whenever A is a commutative *lmc* algebra. In fact, there exists a bijection between the class of all closed regular maximal ideals of A and the equivalence classes of the elements of $M_{(A,n)}$. Moreover, the space $M_{(A,n)}$ is relatively compact in A' (the dual space of A).

In Section 3 we assume that B is a topological algebra and $(A_\alpha, \varphi_{\beta\alpha})$ is an inductive system of topological algebras, or $(A_\alpha, \varphi_{\alpha\beta})$ is a projective system of topological algebras. We obtain some relations between $n - Hom(A_\alpha, B)$ and $n - Hom(A, B)$, or between $\varinjlim M_{(A_\alpha,n)}$ and $M_{(A,n)}$, whenever $A = \varinjlim A_\alpha$ is the inductive limit, or $A = \varprojlim A_\alpha$ is the projective limit, respectively. Here $n - Hom(A_\alpha, B)$ (resp. $n - Hom(A, B)$), is the space of all continuous n-homomorphisms from A_α (resp. A) into B.

Finally, we show that if $(A_\alpha, \varphi_{\beta\alpha})$ is an inductive system of topological algebras and $A = \varinjlim A_\alpha$, is the inductive limit, then

$$M_{(A,n)} \cup \{0\} = M_{(\varinjlim A_\alpha, n)} \cup \{0\} = \varprojlim M_{(A_\alpha,n)} \cup \{0\}$$

within a homeomorphism of respective topological spaces. Moreover, we prove that if $(A, \{p_\alpha\})$ is a complete *lmc* algebra, then $\varinjlim M_{(A_\alpha,n)} = M_{(A,n)}$ within a continuous bijection of respective topological spaces, where A_α is the completion of $A/\ker p_\alpha$.

2. Relations between n-characters and characters

In general, the kernel of an n-homomorphism may not be an ideal [4]. In the following key lemma we show that the kernel of an n-character is a regular maximal ideal.

Lemma 2.1. *Let A be an algebra and φ be an n-character on A. Then* $\ker \varphi$ *is a regular maximal ideal of A.*

Proof. Since $\varphi \neq 0$, there exists $a_\varphi \in A$ such that $\varphi(a_\varphi) = 1$. Now consider the function $\psi_\varphi : A \to \mathbb{C}$, defined by $\psi_\varphi(x) = \varphi(a_\varphi x)$ for every $x \in A$. For every $x, y \in A$

$$\begin{aligned}\psi_\varphi(xy) =& \varphi(a_\varphi xy) = \varphi(a_\varphi xy)\varphi(a_\varphi)^{n-1} = \varphi(a_\varphi x)\varphi(ya_\varphi)\varphi(a_\varphi)^{n-2} \\ =& \varphi(a_\varphi x)\varphi(ya_\varphi),\end{aligned}$$

$$\begin{aligned}\varphi(ya_\varphi) =& \varphi(a_\varphi)^{n-1}\varphi(ya_\varphi) = \varphi(a_\varphi^{n-1} ya_\varphi) = \varphi(a_\varphi)^{n-2}\varphi(a_\varphi y)\varphi(a_\varphi) \\ =& \varphi(a_\varphi y).\end{aligned}$$

Therefore, $\psi_\varphi(xy) = \psi_\varphi(x)\psi_\varphi(y)$ and hence $\psi_\varphi : A \to \mathbb{C}$ is a homomorphism. Moreover, for every $x \in A$ we have

$$\psi_\varphi(x)^{n-1} = \psi_\varphi(x^{n-1}) = \varphi(x^{n-1}a_\varphi) = \varphi(x)^{n-1}\varphi(a_\varphi) = \varphi(x)^{n-1}. \qquad (2.1)$$

It is now clear by (2.1) that $\ker \psi_\varphi = \ker \varphi$. Hence $\ker \varphi$ is a regular maximal ideal of A by [10, II. Lemma 7.1]. □

Corollary 2.2. *Let A be a topological algebra. Then A is n-functionally continuous if and only if it is functionally continuous.*

Proof. Since every character is an n-character it follows that A is functionally continuous whenever A is n-functionally continuous. Conversely, let A be functionally continuous. If $\varphi \in S_{(A,n)}$ then by the above lemma there exists $\psi_\varphi \in S_A = M_A$ such that $\psi_\varphi^{n-1} = \varphi^{n-1}$ and hence $\ker \psi_\varphi = \ker \varphi$. Since ψ_φ is continuous, $\ker \psi_\varphi$ is closed and hence φ is continuous. Consequently, A is n-functionally continuous. □

It is known that Q-algebras are functionally continuous [11, Lemma E4] and hence by the above corollary they are n-functionally continuous. Actually, E. A. Michael posed the question in 1952 as whether each multiplicative linear functional on a commutative Fréchet algebra is automatically continuous [11]. This question, known as the Michael's problem, have been intensively studied but only partial answers have been obtained so far. For example, it is a result of R. Arens that if A is finitely generated then it is functionally continuous.

Theorem 2.3. *Let A be a commutative lmc algebra. Then there exists a surjection between $M_{(A,n)}$ and the set of closed regular maximal ideals of A, which may not be an injection, in general. However, there is a bijection between the set of closed regular maximal ideals of A and the collection of equivalence classes of the elements of $M_{(A,n)}$.*

Proof. By [10, II. Corollary 7.2] and Lemma 2.1, it is clear that $\varphi \mapsto ker\varphi$ is a surjective map between $M_{(A,n)}$ and the set of closed regular maximal ideals of A. But, unlike the characters, the equality $ker\varphi = ker\psi$ for $\varphi, \psi \in M_{(A,n)}$ may not imply the equality $\varphi = \psi$, in general. For example, if $\varphi \in M_{(A,n)}$ then $-\varphi \in M_{(A,n)}$ for odd n and $ker\varphi = ker(-\varphi)$, whereas $\varphi \neq -\varphi$. Hence the map $\varphi \mapsto ker\varphi$ is not an injection, in general.

We now define an equivalence relation $\sim$ on $M_{(A,n)}$. For $\varphi_1, \varphi_2 \in M_{(A,n)}$, we say that $\varphi_1 \sim \varphi_2$ if $\psi_{\varphi_1} = \psi_{\varphi_2}$, where ψ_{φ_1} and ψ_{φ_2} are the elements of M_A, presented in Lemma 2.1, corresponding to the elements φ_1 and φ_2. Since $ker\varphi_1 = ker(\psi_{\varphi_1})$ and $ker\varphi_2 = ker(\psi_{\varphi_2})$, it follows that $\varphi_1 \sim \varphi_2$, or equivalently, $[\varphi_1] = [\varphi_2]$ if and only if $ker(\varphi_1) = ker(\varphi_2)$. Hence the map $[\varphi] \mapsto ker\varphi$ is a bijection between the equivalence classes of elements of $M_{(A,n)}$ and the collection of all closed regular maximal ideals of A. ☐

Proposition 2.4. *Let A be a real topological algebra. If $n > 2$ is even, then $M_{(A,n)} = M_A$.*

Proof. Let $\varphi \in M_{(A,n)}$. By the argument in Lemma 2.1 there exists $\psi_\varphi \in M_A$ such that $\psi_\varphi^{n-1}(x) = \varphi^{n-1}(x)$ for every $x \in A$. Since $n-1$ is odd and φ is real-valued, it follows that $\psi_\varphi = \varphi$ and hence $\varphi \in M_A$. ☐

Proposition 2.5. *Let A be a commutative lmc algebra such that all maximal ideals of A^+, the unitization of A, are closed. Then $a \in A$ is quasi-invertible if $\varphi(a) \neq 1$ for every $\varphi \in M_A$.*

Proof. If a is not quasi-invertible, then $e_{A^+} - a$ is not invertible. Thus $e_{A^+} - a$ is in a closed maximal ideal of A^+. By [10, II. Corollary 7.2], there exists $\psi \in M_{A^+}$ such that $\psi(e_{A^+} - a) = 0$. On the other hand, there exists $\varphi \in M_A$ such that $\psi(a, \lambda) = \varphi(a) + \lambda$, for every $(a, \lambda) \in A^+$. Thus we have $\varphi(a) = 1$, which is a contradiction. ☐

It is interesting to note that in every commutative *lmc* Q-algebra A, or in each commutative *lmc* algebras A, which is advertibly complete and M_A is equicontinuous, all maximal ideals of A^+ are closed [10, III. Theorem 6.3].

If A is a non-unital complex algebra, then the spectrum of a is

$$sp_A(a) = \{0\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} : \frac{1}{\lambda} a \notin q - InvA\},$$

where $q - InvA$ is the set of quasi-invertible (quasi-regular) elements of A. If A^+ is the unitization of A, then $sp_A(a) = sp_{A^+}((a,0))$ and so $\nu_A(a) = \nu_{A^+}((a,0))$ for all $a \in A$. In the following we show that there exists a relationship between $M_{(A,n)}$, $S_{(A,n)}$ and the spectral radius ν_A.

Proposition 2.6. *Let A be a commutative lmc algebra such that all maximal ideals of A^+ (the unitization of A) are closed. Then for every $a \in A$*

$\{|\widehat{a}(\varphi)| : \varphi \in M_{(A,n)}\} \cup \{0\} = \{|\widehat{a}(\varphi)| : \varphi \in S_{(A,n)}\} \cup \{0\} = \{|\lambda| : \lambda \in sp_A(a)\} \cup \{0\}$.

Moreover, $\nu_A(a) = \sup_{\varphi \in M_{(A,n)}} |\varphi(a)| = \sup_{\varphi \in S_{(A,n)}} |\varphi(a)|$.

Proof. Let $\varphi \in M_{(A,n)}$ and $\varphi(a) = \lambda \neq 0$. We define $\psi_\varphi : A \to \mathbb{C}$ by $\psi_\varphi(x) = \varphi(\frac{1}{\lambda} ax)$, which is a character and moreover, $\psi_\varphi^{n-1}(x) = \varphi^{n-1}(x)$ for every $x \in A$, by the proof of Lemma 2.1. If $\psi_\varphi(a) = \lambda'$ then $\lambda' \in sp_A(a)$, by [10, II. Corollary 7.4]. On the other hand, since $|\lambda| = |\lambda'|$, we have

$$|\widehat{a}(\varphi)| \in \{|\lambda| \ : \ \lambda \in sp_A(a)\}$$

and hence

$$\{|\widehat{a}(\varphi)| \ : \ \varphi \in M_{(A,n)}\} \cup \{0\} \subseteq \{|\lambda| \ : \ \lambda \in sp_A(a)\} \cup \{0\}.$$

If $\lambda \in sp_A(a)$ with $\lambda \neq 0$, then $\frac{1}{\lambda} a$ is not quasi-invertible. Hence, by Proposition 2.5, there exists $\varphi \in M_A$ such that $\varphi(\frac{1}{\lambda} a) = 1$ and thus $\lambda \in \widehat{a}(M_{(A,n)})$. Therefore, $\{|\lambda| \ : \ \lambda \in sp_A(a)\} \cup \{0\} = \{|\widehat{a}(\varphi)| \ : \ \varphi \in M_{(A,n)}\} \cup \{0\}$.

With a similar argument we can also show that

$$\{|\widehat{a}(\varphi)| : \varphi \in S_{(A,n)}\} \cup \{0\} = \{|\lambda| : \lambda \in sp_A(a)\} \cup \{0\}.$$

Since, by the definition of spectral radius, we have

$$\nu_A(a) = sup\{|\lambda| \ : \lambda \in sp_A(a)\},$$

the second assertion is clear.

□

A relatively compact subspace (resp. relatively compact subset) Y of a topological space X is a subset whose closure is compact. We now state the following known result, which is used in the sequel.

Theorem 2.7. [8, Theorem 1, p. 201] *Let A be a topological vector space. Then every equicontinuous subset of A' (topological dual space of A) is relatively compact with the weak* topology.*

It is known that the character space of a unital Banach algebra (resp. Q-algebra) is a compact (resp. relatively compact) space, with the weak*-topology (resp. Gelfand topology) [9, Theorem 1.2.8] (resp. [10, II. Proposition 7.1.]). In the following we show that this is also true for $M_{(A,n)}$, when A is a commutative *lmc* Q-algebra.

Theorem 2.8. *Let A be a commutative lmc Q-algebra. Then $M_{(A,n)}$ is relatively compact in A', with the weak*-topology.*

Proof. Since A is a Q-algebra, by [10, I. Lemma 6.4] there exists a balanced neighborhood U of $0 \in A$ such that $U \subseteq q-InvA$. We show that $U \subseteq M^{\circ}_{(A,n)}$, where $M^{\circ}_{(A,n)}$ is the polar set of $M_{(A,n)}$. Suppose on the contrary that there exists $x \in U$ and $|\varphi(x)| > 1$, for some $\varphi \in M_{(A,n)}$. Then $|\psi_\varphi(x)| > 1$, where ψ_φ has been defined in the proof of Lemma 2.1. Thus $\psi_\varphi(x) = \lambda \neq 0$ such that $|\frac{1}{\lambda}| < 1$, which implies that $\frac{1}{\lambda}x \in U$ and $\psi_\varphi(\frac{1}{\lambda}x) = 1$. By [10, II. Lemma 7.4] $\frac{1}{\lambda}x$ is not quasi-invertible, which contradicts $\frac{1}{\lambda}x \in U \subseteq q-InvA$. Therefore, $U \subseteq M^{\circ}_{(A,n)}$ and hence $|\varphi(x)| \leq 1$ for every $x \in U$ and $\varphi \in M_{(A,n)}$. Therefore, $M^{\circ}_{(A,n)}$ is a neighborhood of $0 \in A$ and thus $M_{(A,n)} \subseteq M^{\circ\circ}_{(A,n)}$ is an equicontinuous subset of A'. By Theorem 2.7, $M_{(A,n)}$ is relatively compact. □

Proposition 2.9. *Let A be a topological algebra. Then the following statements are equivalent:*

(*i*) *an element $a \in A$ is quasi-invertible if and only if $|\varphi(a)| \neq 1$ for every $\varphi \in M_{(A,n)}$,*

(*ii*) *an element $a \in A$ is quasi-invertible if and only if $|\varphi(a)| \neq 1$ for every $\varphi \in M_A$.*

Proof. It is clear that $M_A \subseteq M_{(A,n)}$. By the proof of Lemma 2.1, for each $\varphi \in M_{(A,n)}$ there exists $\psi_\varphi \in M_A$ with $|\varphi| = |\psi_\varphi|$, which implies that

$$\{|\varphi(a)| : \varphi \in M_{(A,n)}\} = \{|\varphi(a)| : \varphi \in M_A\}.$$

Hence (i) and (ii) are equivalent.

□

Let A be a topological vector space (TVS). The filter of neighborhoods $\mathcal{F}(x)$ of the point $x \in A$ is the family of sets $x + V$, where V varies over the filter of neighborhoods of the neutral element zero, denoted by $\mathcal{F}(0)$. By [15, Part I, Theorem 3.1] if a filter $\mathcal{F}$ of A is the filter of neighborhoods of the origin in a topology compatible with the linear structure of A, then every $U \in \mathcal{F}$ contains some $V \in \mathcal{F}$ which is balanced. A filter $\mathcal{F}$ on a subset E of the TVS A is said to be a Cauchy filter if for every neighborhood U of 0 in A there is a subset B of E, belonging to $\mathcal{F}$, such that $B - B \subset U$.

In the following we prove a result similar to that of [10, III. Lemma 5.2].

Theorem 2.10. *Let A be an lmc algebra and every $x \in A$ be quasi-invertible if and only if $|\varphi(x)| \neq 1$, for every $\varphi \in M_A$. Then A is advertibly complete.*

Proof. Let $\mathcal{F}$ be a Cauchy filter on A such that for some $x \in A$,

$$\lim(x \diamond \mathcal{F}) = \lim(\mathcal{F} \diamond x) = 0.$$

We claim that x is a quasi-invertible element of A. Otherwise, by the assumption, there exists an element $\varphi \in M_A$ with $|\varphi(x)| = 1$. By the continuity of φ the set $U = \{y \in A : |\varphi(y)| < \frac{1}{8}\}$ is an open neighborhood of 0 in A. Since $\mathcal{F}$ is a Cauchy filter and U is an open neighborhood of 0, by [15, 3, 3.1], there exists a balanced element $B \in \mathcal{F}$ such that $B - B \subset U$ and $x \diamond B = \{x \diamond y : y \in B\} \subseteq U$. Since B is balanced there exists $z \in B$ such that $|\varphi(z)| < \frac{1}{8}$. Hence $|\varphi(y)| < \frac{1}{8} + |\varphi(z)| < \frac{1}{4}$ and $|\varphi(x \diamond y)| < \frac{1}{8}$ for every $y \in B$. Therefore,

$$\frac{1}{8} > |\varphi(x \diamond y)| = |\varphi(x) + \varphi(y) - \varphi(x)\varphi(y)| \geq 1 - 2|\varphi(y)| > \frac{1}{2},$$

for every $y \in B$, which is a contradiction. We omit the proof of the remaining part, for it is similar to the proof of [10, III. Lemma 5.2]. □

Proposition 2.11. *The character space S_A is a subset of $\overline{M_{(A,n)}}$ for each commutative Fréchet algebra A and for each $n \geq 2$, where the closure is taken with respect to the Gelfand topology on $S_{(A,n)}$.*

Proof. Suppose on the contrary that there are $\varphi \in S_A$ and $a_1, ..., a_m \in A$ such that

$$\{\psi \in S_A : \ |\psi(a_i) - \varphi(a_i)| < 1, \ i = 1, ..., m\} \cap M_{(A,n)} = \emptyset. \tag{2.2}$$

Since $\varphi \in S_A$, there exists $c \in A$ such that $\varphi(c) \neq 0$. Set $a_0 = c, \lambda_i = \varphi(a_i)$ and $b_i = a_i - \lambda_i e_{A^+} = (a_i, -\lambda_i)$, for $i = 0, 1, 2, \cdots, m$. Since $M_A \subseteq M_{(A,n)}$, by (2.2), we have

$$\{\psi \in S_A : \ |\psi(a_i) - \varphi(a_i)| < 1, \ i = 1, ..., m\} \cap M_A = \emptyset. \tag{2.3}$$

If $\phi \in M_{A^+}$ and $\widehat{b_i}(\phi) = 0$ for $0 \leq i \leq m$, then $\phi(a_i) = \varphi(a_i)$ and $\phi(a_0) \neq 0$. Hence

$$\phi\,|_A \in \{\psi \in S_A : \ |\psi(a_i) - \varphi(a_i)| < 1, \ i = 1, ..., m\} \cap M_A,$$

which contradicts (2.3). Therefore, $\widehat{b_0}, \widehat{b_1}, ..., \widehat{b_m}$ have no common zero on M_{A^+} and hence, by [9, 6.1.10], there exist $c_0, c_1, ..., c_m \in A^+$ such that

$$\sum_{i=0}^{m} c_i b_i = e_{A^+}.$$

Now define the function $\varphi^+ : A^+ \to \mathbb{C}$ by $\varphi^+(x, \lambda) = \varphi(x) + \lambda$. It is easy to see that $\varphi^+ \in S_{A^+}$ and since $\varphi^+(b_i) = \varphi^+(a_i, -\lambda_i) = \varphi(a_i) - \lambda_i = 0$ for each $i = 0, 1, ..., m$, it follows that $b_0, b_1, ..., b_m \in \ker \varphi^+$, implying that $\varphi^+(e_{A^+}) = 0$, which is a contradiction. □

Corollary 2.12. *The character space $M_A = M_{(A,2)}$ is a dense subset of S_A for each commutative Fréchet algebra A, i.e. $\overline{M_A} = \overline{S_A}$, where the closure is taken with respect to the Gelfand topology on S_A.*

To present more properties of n-characters we first state the following notion: We define $M^m_{(A,n)} = \{\varphi^m : \ \varphi \in M_{(A,n)}\}$, where $\varphi^m(a) = (\varphi(a))^m$ for $\varphi \in M_{(A,n)}$ and $a \in A$.

Proposition 2.13. *If A is a commutative Fréchet algebra then $\overline{M^{n-1}_{(A,n)}} = \overline{S^{n-1}_A}$ for every $n \geq 2$.*

Proof. By Lemma 2.1 $M^{n-1}_{(A,n)} = M^{n-1}_{A} \subseteq S^{n-1}_{A}$), implying that $\overline{M^{n-1}_{(A,n)}} \subseteq \overline{S^{n-1}_{A}}$. By Proposition 2.11 we conclude that $S_A \subseteq \overline{M_{(A,n)}}$, which implies that $S^{n-1}_{A} \subseteq (\overline{M_{(A,n)}})^{n-1} \subseteq \overline{M^{n-1}_{(A,n)}}$. Therefore, $\overline{M^{n-1}_{(A,n)}} = \overline{S^{n-1}_{A}}$. □

Proposition 2.14. *Let A be a commutative Fréchet algebra and $n \geq 2$. For $a_1, a_2, ..., a_m \in A$ let F be defined by*

$$F : S_{(A,n)} \to \mathbb{C}^m, \quad \varphi \mapsto (\varphi^{n-1}(a_1), \varphi^{n-1}(a_2), ..., \varphi^{n-1}(a_m)).$$

Then $F(\overline{M_{(A,n)}} \cap S_{(A,n)}) = F(S_A) = F(M_A)$.

Proof. By Proposition 2.11 we have $F(S_A) \subseteq F(\overline{M_{(A,n)}} \cap S_{(A,n)})$. Hence, by the proof of Lemma 2.1 it is easy to see that $(0, 0, ..., 0) \in F(\overline{M_{(A,n)}} \cap S_{(A,n)})$ if and only if $(0, 0, ..., 0) \in F(S_A)$. We now assume that there exists

$$(\lambda_1, ..., \lambda_m) = \lambda \in F(\overline{M_{(A,n)}} \cap S_{(A,n)}) \setminus F(S_A) \cup \{(0, 0, ..., 0)\}.$$

So there exists $\varphi \in \overline{M_{(A,n)}} \cap S_{(A,n)}$ such that $F(\varphi) = \lambda$ and hence $\varphi^{n-1}(a_i) = \lambda_i$ for $i = 1, 2, ..., m$. Next we define $b_i = a^{n-1}_i - \varphi^{n-1}(a_i)e_{A^+}$. If $\phi \in M(A^+)$ and $\widehat{b_i}(\phi) = 0$, then $\phi^{n-1}(a_i) = \varphi^{n-1}(a_i) = \lambda_i$. Since $\lambda \neq (0, 0, ..., 0)$, it follows that $\phi\mid_A \in M_A \subseteq S_A$ and $\lambda \in F(S_A)$, which contradicts $\lambda \in F(\overline{M_{(A,n)}}) \setminus F(S_A)$. Thus $\widehat{b_1}, ..., \widehat{b_m}$ have no common zero on $M(A^+)$. Hence by [9, 6.1.10] there exist $c_1, ..., c_m \in A^+$ such that

$$\sum_{i=1}^{m} c_i b_i = e_{A^+}.$$

Now consider the function $\varphi^+ : A^+ \to \mathbb{C}$, defined by $\varphi^+(x, \lambda) = \psi_\varphi(x) + \lambda$, where ψ_φ has been defined in Lemma 2.1. It is easy to see that $\varphi^+ \in S_{A^+}$ and $b_1, ..., b_m \in \ker \varphi^+$, which is a contradiction. Finally, the equality $F(S_A) = F(M_A)$ follows by [9, Prop. 10.1.3]. □

3. Relations between n-$Hom(A_\alpha, B)$ and n-$Hom(A, B)$

In this section we first obtain a relationship between n-$Hom(A, B)$ $[M_{(A,n)}]$ and n-$Hom(A_\alpha, B)$ $[M_{(A_\alpha,n)}]$, where B is a topological algebra, $(A_\alpha, \varphi_{\beta\alpha})$ is an inductive system of topological algebras and $A = \varinjlim A_\alpha$.

Theorem 3.1. *Let B be a topological algebra and $(A_\alpha, \varphi_{\beta\alpha})$ be an inductive system of topological algebras with respect to an index set I. If $A = \varinjlim A_\alpha$, then*

$$n\text{-}Hom(A, B) = n\text{-}Hom(\varinjlim A_\alpha, B) = \varprojlim n\text{-}Hom(A_\alpha, B),$$

within a homeomorphism of the respective topological spaces.

Proof. By a similar argument as in [10, V, P. 152] we can get a projective system of topological spaces $(n\text{-}Hom(A_\beta, B), T_{\alpha\beta})$ from the inductive system of topological spaces $(A_\alpha, \varphi_{\beta\alpha})$, where $T_{\alpha\beta}$ is defined as follow:

$$T_{\alpha\beta} : n\text{-}Hom(A_\beta, B) \to n\text{-}Hom(A_\alpha, B)$$
$$T_{\alpha\beta}(h) = h \circ \varphi_{\beta\alpha},$$

for $\alpha \leq \beta$. Suppose that $\phi_\alpha : A_\alpha \to A$ are the canonical continuous maps. We define a map

$$g_\alpha : n\text{-}Hom(A, B) \to n\text{-}Hom(A_\alpha, B)$$
$$g_\alpha(h) = h \circ \phi_\alpha,$$

for $\alpha \in I$. It is easy to see that, $\phi_\beta \circ \varphi_{\beta\alpha} = \phi_\alpha$ for every $\alpha \leq \beta$ in I. Thus

$$T_{\alpha\beta}(g_\beta(h)) = T_{\alpha\beta}(h \circ \phi_\beta) = h \circ \phi_\beta \circ \varphi_{\beta\alpha} = h \circ \phi_\alpha = g_\alpha(h),$$

for every $h \in n\text{-}Hom(A, B)$ and $\alpha \leq \beta$ in I. Hence

$$\{g_\alpha(h)\}_{\alpha \in I} \in \varprojlim n - Hom(A_\alpha, B)$$

for every $h \in n - Hom(A, B)$. Now we define the map

$$G : n\text{-}Hom(A, B) \to \varprojlim n\text{-}Hom(A_\alpha, B)$$
$$G(h) = \{g_\alpha(h)\}_{\alpha \in I},$$

such that $g_\alpha = p_\alpha \circ G$, where $p_\alpha : \varprojlim n\text{-}Hom(A_\alpha, B) \to n\text{-}Hom(A_\alpha, B)$ is the corresponding canonical map for every $\alpha \in I$. Let $f, g \in n\text{-}Hom(A, B)$ such that $f \neq g$. The relation $A = \varinjlim A_\alpha = \cup_{\alpha \in I} \phi_\alpha(A_\alpha)$ implies that $G(f) \neq G(g)$, i.e, G is one-to-one. For each $\{k_\alpha\}_{\alpha \in I} \in \varprojlim n\text{-}Hom(A_\alpha, B)$ we define $h \in n\text{-}Hom(A, B)$ by the relation $k_\alpha = h \circ \phi_\alpha$ for every $\alpha \in I$. Clearly, $G(h) = \{g_\alpha(h)\}_{\alpha \in I} = \{h \circ \phi_\alpha\}_{\alpha \in I} = \{k_\alpha\}_{\alpha \in I}$ and G is onto. Since $\varprojlim n\text{-}Hom(A_\alpha, B)$ is endowed with the initial topology, defined by the maps p_α, for $\alpha \in I$, we conclude that p_α is continuous for all $\alpha \in I$. Clearly g_α and consequently G are continuous for every $\alpha \in I$.

Now to prove the continuity of the inverse map of G, let $\{f_\alpha\}^\delta_{\alpha \in I} = \{f^\delta_\alpha\}_{\alpha \in I}$, $\delta \in K$, be a net in $\varprojlim n\text{-}Hom(A_\alpha, B)$ converging to an element $\{k_\alpha\}_{\alpha \in I}$ in $\varprojlim n\text{-}Hom(A_\alpha, B)$. Then it follows that the net $\{f^\delta_\alpha\}_{\delta \in K}$ converges to k_α in $n\text{-}Hom(A_\alpha, B)$. Let $G(h_\delta) = \{f^\delta_\alpha\}_{\alpha \in I}$ for each $\delta \in K$. If $x \in A = \cup_{\alpha \in I} \phi_\alpha(A_\alpha)$ then $x = \phi_\alpha(x_\alpha)$ and $h_\delta \circ \phi_\alpha = f^\delta_\alpha$ for some $\alpha \in I$. Consequently,

$$\lim_\delta h_\delta(x) = \lim_\delta h_\delta(\phi_\alpha(x_\alpha)) = \lim_\delta f^\delta_\alpha(x_\alpha) = k_\alpha(x_\alpha) = k_\alpha \circ \phi^{-1}_\alpha(x),$$

such that

$$G(k_\alpha \circ \phi^{-1}_\alpha) = \{g_\alpha(k_\alpha \circ \phi^{-1}_\alpha)\}_{\alpha \in I} = \{k_\alpha \circ \phi^{-1}_\alpha \circ \phi_\alpha\}_{\alpha \in I} = \{k_\alpha\}_{\alpha \in I}.$$

Therefore, the net $\{h_\delta\}_{\delta \in K}$ converges to $G^{-1}(\{k_\alpha\}_{\alpha \in I})$ in $n\text{-}Hom(A, B)$ and hence the inverse map of G is continuous. □

Corollary 3.2. *With the same hypothesis as in Theorem* 3.1*, if* $B = \mathbb{C}$ *then*

$$M_{(A,n)} \cup \{0\} = M_{(\varinjlim A_\alpha, n)} \cup \{0\} = \varprojlim M_{(A_\alpha, n)} \cup \{0\},$$

within a homeomorphism of respective topological spaces.

Theorem 3.3. *Let* $(A_\alpha, \varphi_{\beta\alpha})$ *be an inductive system of topological algebras with respect to a directed index set* I*. Let* $A = \varinjlim A_\alpha$ *and* $\phi_\alpha : A_\alpha \to A$ *be the canonical continuous map for every* $\alpha \in I$*. Moreover, assume that* $\varphi_{\beta\alpha}(A_\alpha) \cap (A_\beta \backslash \ker(f)) \neq \emptyset$ *and* $\phi_\alpha(A_\alpha) \cap (A \backslash \ker(g)) \neq \emptyset$ *for every* $f \in M_{(A_\beta,n)}$*,* $g \in M_{(A,n)}$ *and all* $\alpha, \beta \in I$*. Then*

$$M_{(A,n)} = \varprojlim M_{(A_\alpha,n)},$$

within a continuous bijection of respective topological spaces.

Proof. We define the maps $T_{\alpha\beta} : M_{(A_\beta,n)} \to M_{(A_\alpha,n)}$ by $T_{\alpha\beta}(h_\beta) = h_\beta \circ \varphi_{\beta\alpha}$, for every $\alpha \leq \beta$ in I. Then $T_{\alpha\beta}$ is well defined, by the fact that

$$\varphi_{\beta\alpha}(A_\alpha) \cap (A_\beta \backslash \ker(f)) \neq \emptyset.$$

It is easy to see that $(M_{(A_\beta,n)}, T_{\alpha\beta})$ is a projective system of topological spaces. Now we define the map $G : \varprojlim M_{(A_\alpha,n)} \to M_{(A,n)}$ by $G(\{h_\alpha\}) = h$, where $h_\alpha = h \circ \phi_\alpha$, for every $\alpha \in I$. The map G is, in fact, well defined by the assumption $\phi_\alpha(A_\alpha) \cap (A \backslash \ker(g)) \neq \emptyset$ and considering the relation

$$T_{\alpha\beta}(h_\beta) = h_\beta \circ \varphi_{\beta\alpha} = (h \circ \phi_\beta) \circ \varphi_{\beta\alpha} = h \circ (\phi_\beta \circ \varphi_{\beta\alpha}) = h \circ \phi_\alpha = h_\alpha.$$

Finally, by using a similar argument as in the proof of Theorem 3.1 we can show that G is a continuous bijection map. □

We now obtain a relation between $M_{(A,n)}$ and $M_{(A_\alpha,n)}$, where $(A_\alpha, \varphi_{\alpha\beta})$ is a projective system of topological algebras and $A = \varprojlim A_\alpha$. We first bring the following lemmas, which are easy to prove.

Lemma 3.4. *Let* $(A_\alpha, \varphi_{\alpha\beta})$ *be a projective system of topological algebras with respect to a directed index set* I*. If for every* $\alpha \leq \beta$ *in* I*,* $\varphi_{\alpha\beta} : A_\beta \to A_\alpha$ *has dense range, then* $(M_{(A_\alpha,n)}, T_{\beta\alpha})$ *is an inductive system of topological spaces, where the map* $T_{\beta\alpha} : M_{(A_\alpha,n)} \to M_{(A_\beta,n)}$ *is defined by* $T_{\beta\alpha}(h_\alpha) = h_\alpha \circ \varphi_{\alpha\beta}$ *for* $\alpha \leq \beta$ *in* I*.*

Proof. Since h_α is a continuous n-character and $\varphi_{\alpha\beta}$ is a continuous homomorphism, $T_{\beta\alpha}$ is a well defined continuous map for every $\alpha \leq \beta$ in I. It is easy to see that the other properties of an inductive system of topological spaces hold for $(M_{(A_\alpha,n)}, T_{\beta\alpha})$. □

Let $(A_\alpha, \varphi_{\alpha\beta})$ be a projective system of topological algebras with respect to a directed index set I. If for every $\alpha \leq \beta$ in I, $\varphi_{\alpha\beta}$ has dense range and the canonical map $\phi_\alpha : A = \varprojlim A_\alpha \to A_\alpha$ has also dense rage for every $\alpha \in I$, then the system $(A_\alpha, \varphi_{\alpha\beta})$ is a strictly dense projective system of topological algebras and $A = \varprojlim A_\alpha$ is the strictly dense projective limit of topological algebras.

Lemma 3.5. *If* $(A, \{p_\alpha\})$ *is a complete lmc algebra, then* $A = \varprojlim A_\alpha$ *is a strictly dense projective limit of Banach algebras, where* A_α *is the completion of* $A/\ker p_\alpha$*.*

Proof. Let $(A, \{p_\alpha\})$ be a complete *lmc* algebra. By [10, III, Theorem 3.1] A is a projective limit of the net of Banach algebras $\{A_\alpha\}_{\alpha \in I}$, where A_α is the completion of $A/\ker p_\alpha$ for every $\alpha \in I$. Moreover, by [10, p.176], the canonical map $\phi_\alpha : A = \varprojlim A_\alpha \to A_\alpha$ has dense range for every α in I, and hence $A = \varprojlim A_\alpha$ is a strictly dense projective limit of Banach algebras. $\square$

We now bring a known result and then extend it to the n-characters.

Theorem 3.6. [14, p. 139, 4.4] *Let $(A_\alpha, \varphi_{\alpha\beta})$ be a projective system of locally convex spaces and $A = \varprojlim A_\alpha$ such that the canonical map $\phi_\alpha : A = \varprojlim A_\alpha \to A_\alpha$ has dense range for every α in the directed set I. Then the dual A', with the weak* topology* (*c.f.* [14, p. 131, 3.2]), *can be identified with the inductive limit of the family $(A'_\alpha, \varphi'_{\beta\alpha})$, where $\varphi'_{\beta\alpha}$ is the adjoint mapping of $\varphi_{\alpha\beta}$, for every $\alpha \leq \beta$ in I.*

Theorem 3.7. *Let $(A_\alpha, \varphi_{\alpha\beta})$ be a strictly dense projective system of topological algebras with respect to a directed index set I. If $A = \varprojlim A_\alpha$, then there exists a continuous injection of $\varinjlim M_{(A_\alpha,n)}$ to $M_{(A,n)}$. If A_α has continuous multiplication for every $\alpha \in I$, then $\varinjlim M_{(A_\alpha,n)} = M_{(A,n)}$, within a continuous bijection of respective topological spaces.*

Proof. By the hypotheses and Theorem 3.6 we have

$$A' = \varinjlim A'_\alpha = \cup_{\alpha \in I} \phi'_\alpha(A'_\alpha),$$

where the canonical map $\phi'_\alpha : A'_\alpha \to A'$ is defined by $\phi'_\alpha(h_\alpha) = h_\alpha \circ \phi_\alpha$. Note that $\phi_\alpha : A = \varprojlim A_\alpha \to A_\alpha$ is the canonical map for every $\alpha \in I$. Clearly the restriction of ϕ'_α to $M_{(A_\alpha,n)}$ yields a continuous map $\phi'_\alpha|_{M_{(A_\alpha,n)}} := S_\alpha : M_{(A_\alpha,n)} \to M_{(A,n)}$. Now we define a map $G : \varinjlim M_{(A_\alpha,n)} \to M_{(A,n)}$ such that $S_\alpha = G \circ T_\alpha$, where $T_\alpha : M_{(A_\alpha,n)} \to \varinjlim M_{(A_\alpha,n)}$ is the canonical continuous map for every $\alpha \in I$. Then G is a continuous map by the continuity of S_α and T_α for every $\alpha \in I$. Now we show that G is one to one. Suppose that $h_1, h_2 \in \varinjlim M_{(A_\alpha,n)}$ such that $G(h_1) = G(h_2)$. By [10, IV, Lemma 1.1] there exist $\alpha \in I$ and $h_\alpha, h'_\alpha \in M_{(A_\alpha,n)}$ such that $h_1 = T_\alpha(h_\alpha)$ and $h_2 = T_\alpha(h'_\alpha)$. Hence for every $x = \{x_\alpha\} \in A$ we have

$$\begin{aligned} h_\alpha(x_\alpha) = h_\alpha \circ \phi_\alpha(x) = S_\alpha(h_\alpha)(x) = G(h_1)(x) = G(h_2)(x) \\ = S_\alpha(h'_\alpha)(x) = h'_\alpha \circ \phi_\alpha(x) = h'_\alpha(x_\alpha). \end{aligned}$$

The last relation yields $h_\alpha = h'_\alpha$ on $\phi_\alpha(A)$. Since ϕ_α has dense range, by the continuity of ϕ_α one gets $h_\alpha = h'_\alpha$ on $\overline{\phi_\alpha(A)} = A_\alpha$. Therefore, G is one to one. If A_α has continuous multiplication for every α in I, then we show that G is onto. Since $A' = \varinjlim A'_\alpha = \cup_{\alpha \in I} \phi'_\alpha(A'_\alpha)$, for every element $h \in M_{(A,n)} \subseteq A'$ there exists $h' \in A'_\alpha$ for some α such that $h = S_\alpha(h') = h' \circ \phi_\alpha$. Therefore, h' is certainly non-zero n-multiplicative on $\phi_\alpha(A) \subseteq A_\alpha$, and hence $h' \in M_{(\phi_\alpha(A),n)}$. By the continuity of ϕ_α and multiplication of A_α we can extend it uniquely to an element $\overline{h'} \in M_{(A_\alpha,n)}$. If $h'' = T_\alpha(\overline{h'}) \in \varinjlim M_{(A_\alpha,n)}$ then

$$h = S_\alpha(h') = h' \circ \phi_\alpha = \overline{h'} \circ \phi_\alpha = S_\alpha(\overline{h'}) = G \circ T_\alpha(\overline{h'}) = G(h'').$$

Thus G is an onto map and so it is a bijection between $\varinjlim M_{(A_\alpha,n)}$ and $M_{(A,n)}$. □

Corollary 3.8. *If* $(A, \{p_\alpha\})$ *is a complete lmc algebra, then*

$$\varinjlim M_{(A_\alpha,n)} = M_{(A,n)},$$

within a continuous bijection of respective topological spaces, where A_α *is the completion of* $A/\ker p_\alpha$*.*

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H. Shayanpour
Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Shahrekord, P. O. Box 88186-34141, Shahrekord, Iran.
e-mail: h.shayanpour@sci.sku.ac.ir

T. G. Honary
Faculty of Mathematical Sciences and Computer, Kharazmi University (Tarbiat Moallem University), No. 50 Taleghani Avenue, Tehran 1561836314, Iran.
e-mail: honary@tmu.ac.ir

M. S. Hashemi
Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.
e-mail: m.hashemi@srbiau.ac.ir