Concomitants of order statistics and record values from Morgenstern type bivariate generalized exponential distribution

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Abstract

In this paper, we introduce the Morgenstern type bivariate generalized exponential distribution. This distribution is an extension of Morgenstern type bivariate exponential distribution and the marginal distributions are generalized exponential distribution. We study some properties of this bivariate distribution. Also, some distributional properties of concomitants of order statistics as well as record values for the MTBED are studied. Recurrence relations between moments of concomitants are also obtained.

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1 Introduction

Let $F_X(x)$ and $F_Y(y)$ are cumulative distribution functions (cdf), and $f_X(x)$ and $f_Y(y)$ are probability density functions (pdf) of random variables X and Y, respectively. Morgenstern (1956) defined a class of bivariate distributions with the bivariate cdf given by

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)[1 + \lambda(1 - F_X(x))(1 - F_Y(y))], \tag{1.1}$$

which the corresponding pdf given by

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)\left[1 + \lambda(2F_X(x) - 1)(2F_Y(y) - 1)\right].$$

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This class is known as Morgenstern-Farlie-Gumbel class distribution. Farlie (1960) extended it to the multivariate case and investigated by Gumbel (1960) for exponential Marginals.

The Morgenstern family is characterized by the specified marginal cdf's of random variables X and Y, and here the parameter λ measures the association between X and Y. The admissible range of association parameter λ is $[-1, 1]$ and the Pearson correlation coefficient ρ between X and Y can never exceed $\frac{1}{3}$. Some authors considered well-known marginal distributions and studied their properties: for example, exponential distribution (Gumbel, 1960), logistic distribution (Gumbel, 1961), gamma distribution (D'este, 1981), uniform distribution (Bairamov and Bekci, 1999).

Gupta and Kundu (1999) defined the generalized exponential (GE) distribution with the following cdf

$$F_X(x) = (1 - e^{-\theta x})^{\alpha}, \quad x > 0,\ \theta > 0,\ \alpha > 0, \tag{1.2}$$

and we denote this cdf by $GE(\theta, \alpha)$. This distribution is a generalization of the exponential distribution, and is very flexible. For example, the hazard function of the exponential distribution is constant but the hazard function of GE distribution can be constant, increasing or decreasing. Gupta and Kundu (1999) by using the binomial series expansion shown that the kth moment of a random variable with $GE(\theta, \alpha)$ is

$$\mu_k = \frac{\alpha\Gamma(k+1)}{\theta^k} \sum_{i=0}^{\infty} \frac{(-1)^i}{(i+1)^{k+1}} A(\alpha - 1, i), \tag{1.3}$$

where $A(\alpha - 1, i) = \binom{\alpha - 1}{i}$. Also, the expectation, variance and moment generating function of the random variable X with $GE(\theta, \alpha)$ can be given as

$$E(X) = \frac{B(\alpha)}{\theta}, \quad Var(X) = \frac{C(\alpha)}{\theta^2}, \quad M_X(t) = \alpha Beta(\alpha, 1 - t/\theta), \tag{1.4}$$

where

$$B(\alpha) = \psi(\alpha + 1) - \psi(1), \quad C(\alpha) = \psi'(1) - \psi'(\alpha + 1), \quad Beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

and $\psi(.)$ is the digamma function and $\psi'(.)$ is its derivative.

In this paper, we consider the Morgenstern type bivariate generalized exponential distribution (MTBGED). In fact, it is a special case of Morgenstern family with the GE distribution as the marginal distribution function. The Morgenstern type bivariate exponential distribution (MTBED) is a spacial case of MTBGED. Balasubramanian and Beg (1997) studied the concomitants of order statistics in MTBED. Recently, some estimators for a parameter of MTBED based on ranked set sampling and concomitants of order statistics were proposed by Chacko and Thomas (2008, 2011). This paper is organized as follows: in Section 2, we introduce the MTBGED and obtain some of its properties. In Section 3, we discuss some distributional properties of concomitants of order statistics for MTBGED. Finally, some distributional properties of concomitants of record values are proposed in Section 4.

2 The MTBGED and some of its properties

The joint cdf of (X, Y) is defined as

$$F_{X,Y}(x,y) = (1-e^{-\theta_1 x})^{\alpha_1}(1-e^{-\theta_2 y})^{\alpha_2}\{1+\lambda[1-(1-e^{-\theta_1 x})^{\alpha_1}][1-(1-e^{-\theta_2 y})^{\alpha_2}]\}, \quad (2.1)$$

$$x, y > 0, \quad \theta_1, \theta_2, \alpha_1, \alpha_2 > 0, \quad -1 \le \lambda \le 1,$$

with the corresponding pdf

$$\begin{aligned} f_{X,Y}(x,y) &= \alpha_1\alpha_2\theta_1\theta_2 e^{-\theta_1 x-\theta_2 y}(1-e^{-\theta_1 x})^{\alpha_1-1}(1-e^{-\theta_2 y})^{\alpha_2-1} \\ &\quad \times\{1+\lambda[2(1-e^{-\theta_1 x})^{\alpha_1}-1][2(1-e^{-\theta_2 y})^{\alpha_2}-1]\}. \end{aligned} \quad (2.2)$$

The moments of the MTBGED are given based on the properties of the GE distribution as

$$E\left(X^n Y^m\right) = E\left(X^n\right) E\left(Y^m\right) + \lambda\left(E\left(U^n\right) - E\left(X^n\right)\right)\left(E\left(V^m\right) - E\left(Y^m\right)\right),$$

where U and V are independent random variables from $GE(\theta_1, 2\alpha_1)$ and $GE(\theta_2, 2\alpha_2)$, respectively. Also, we can conclude that

$$\begin{aligned} E\left(X\right) &= \frac{B(\alpha_1)}{\theta_1}, \qquad E\left(Y\right) = \frac{B(\alpha_2)}{\theta_2}, \\ Var\left(X\right) &= \frac{C(\alpha_1)}{\theta_1^2}, \qquad Var\left(Y\right) = \frac{C(\alpha_2)}{\theta_2^2}, \\ E\left(XY\right) &= \frac{B(\alpha_1)B(\alpha_2) + \lambda D(\alpha_1)D(\alpha_2)}{\theta_1\theta_2}, \end{aligned}$$

where $D(\alpha) = B(2\alpha) - B(\alpha)$. Therefore, the coefficient of correlation between X and Y is

$$\rho_{XY} = \frac{\lambda D(\alpha_1)D(\alpha_2)}{\sqrt{C(\alpha_1)C(\alpha_2)}}.$$

Consider

$$\mathrm{g}\left(\alpha\right) = \frac{D(\alpha)}{\sqrt{C(\alpha)}} = \frac{\psi\left(2\alpha+1\right) - \psi\left(\alpha+1\right)}{\sqrt{\psi'\left(1\right) - \psi'\left(\alpha+1\right)}}. \quad (2.3)$$

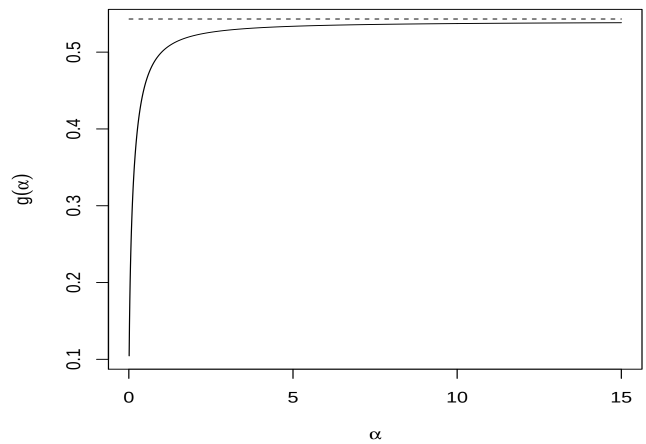

Figure 1: Plot of g(α) and its upper bound.

The plot of g(α) is given in Fig 1. We can see that g(α) is increasing and positive function with respect to α. Therefore, if $\lambda > 0$ then ρ_{XY} increasing and positive function and if $\lambda < 0$ then ρ_{XY} decreasing and negative function with respect to α_1 (and α_2) . Also, we can show that

$$\lim_{\alpha\to\infty} \mathrm{g}(\alpha) = \frac{\sqrt{6}\,\log(2)}{\pi} \approx 0.5404, \qquad \lim_{\alpha\to 0^+} \mathrm{g}(\alpha) = 0.$$

Therefore,

$$-0.2921 \leq \rho_{XY} \leq 0.2921. \tag{2.4}$$

Remark 2.1 *If $\alpha_1 = \alpha_2 = 1$, then the correlation coefficient of X and Y in MTBED is equal to*

$$\rho_{XY} = \lambda \mathrm{g}(1) = \frac{\lambda}{4},$$

that it agrees with the result of Gumbel (1960).

The conditional distribution of X given $Y = y$ has pdf

$$f_{X|Y}(x \mid y) = f_X(x)\left[1 + \lambda\left(2F_X(x) - 1\right)\left(2F_Y(y) - 1\right)\right].$$

Therefore, the regression curve of X given $Y = y$ for MTBGED is

$$\begin{aligned} E\left(X \mid Y = y\right) &= E(X) + \lambda\left(2F_Y(y) - 1\right)\left(E(U) - E(X)\right) \\ &= \frac{1}{\theta_1}\left[B(\alpha_1) + \lambda\, D(\alpha_1)\left(2\left(1 - e^{-\theta_2 y}\right)^{\alpha_2} - 1\right)\right], \end{aligned} \tag{2.5}$$

where U has $GE(\theta_1, 2\alpha_1)$, and the conditional expectation is non-linear with respect to y.

3 Some distributional properties of concomitants of order statistics

Consider a random sample $(X_i, Y_i), i = 1, 2, ..., n$ from a bivariate distribution. If $X_{(r:n)}$ denote the rth order statistic of the marginal X observations, then the Y observations accompanying with $X_{(r:n)}$ is denoted by $Y_{[r:n]}$ and is called the concomitant of rth order statistic. $Y_{[r:n]}$ is called as the rth induced order statistic by Bhattacharya (1974). We refer the reader to David and Nagaraja (1998) for more details. In this section, we consider the concomitants of rth order statistic of MTBGED and obtain the properties of their marginal and joint distributions. Also, we propose an unbiased estimator for $\mu = E(Y)$.

3.1 Marginal distribution of concomitants

From Scaria and Nair (1999), the pdf of $Y_{[r:n]}$ in MTBGED is obtained as

$$f_{Y_{[r:n]}}(y) = \alpha_2\theta_2 e^{-\theta_2 y}(1 - e^{-\theta_2 y})^{\alpha_2 - 1}[1 + \delta_r - 2\delta_r(1 - e^{-\theta_2 y})^{\alpha_2}], \;\; 1 \leq r \leq n, \tag{3.1}$$

where $\delta_r = \lambda(n - 2r + 1)/(n + 1)$. Obviously, we can find that

$$f_{Y_{[r:n]}}(y) = (1 + \delta_r)f_Y(y) - \delta_r f_V(y),$$

where $f_Y(y)$ and $f_V(y)$ are pdf's of random variables Y and V with $GE(\theta_2, \alpha_2)$ and $GE(\theta_2, 2\alpha_2)$, respectively. Therefore, the moment generating function of $Y_{[r:n]}$ is obtained by

$$\begin{aligned} M_{Y_{[r:n]}}(t) &= (1 + \delta_r)M_Y(t) - \delta_r M_V(t) \\ &= \alpha_2[(1 + \delta_r)Beta(\alpha_2, 1 - \frac{t}{\theta_2}) - 2\delta_r Beta(2\alpha_2, 1 - \frac{t}{\theta_2})], \end{aligned} \tag{3.2}$$

where $M_Y(t)$ and $M_V(t)$ are moment generating functions of the random variables Y and V, respectively. With differentiating (3.2) with respect to t and using (1.3), we get the kth moment of $Y_{[r:n]}$ as

$$\begin{aligned} \mu^{(k)}_{[r:n]} &= E(Y^k_{[r:n]}) \\ &= (1 + \delta_r)E(Y^k) - \delta_r E(V^k) \\ &= \frac{\alpha_2\Gamma(k + 1)}{\theta_2^k}\sum_{i=0}^{\infty}\frac{(-1)^i}{(i + 1)^{k+1}}[(1 + \delta_r)A(\alpha_2 - 1, i) - 2\delta_r A(2\alpha_2 - 1, i)]. \end{aligned} \tag{3.3}$$

Since (3.3) is a convergent series for any $k \geq 0$, so all the moments exist for integer values of α_2. With putting $k = 1$, we obtain the mean as

$$\begin{aligned}
\mu_{[r:n]} &= E(Y_{[r:n]}) \\
&= \frac{\alpha_2}{\theta_2} \sum_{i=0}^{\infty} \frac{(-1)^i}{(i+1)^2} [(1+\delta_r) A(\alpha_2 - 1, i) - 2\delta_r A(2\alpha_2 - 1, i)] \\
&= \frac{1}{\theta_2} [B(\alpha_2) - \delta_r D(\alpha_2)]. \qquad (3.4)
\end{aligned}$$

The difference between the means of Y and $Y_{[r:n]}$ is

$$h(r, \lambda, \alpha_2) = E(Y_{[r:n]}) - E(Y) = \frac{-\delta_r D(\alpha_2)}{\theta_2}. \qquad (3.5)$$

which is negative, positive or zero whenever $(-1 \leq \lambda < 0, r > (n+1)/2, \alpha_2 > 1)$, $(0 < \lambda \leq 1, r > (n+1)/2, \alpha_2 > 1)$, or $(r = (n+1)/2$ or $\lambda = 0)$, respectively.

From (3.4), we have the following recurrence relations between the means of concomitants as

$$\mu_{[r+1:n]} = \frac{\mu_{[r+2:n]} + \mu_{[r:n]}}{2} = \mu_{[r:n]} + \frac{2\lambda D(\alpha_2)}{(n+1)\theta_2} \qquad (3.6)$$

$$\mu_{[r:n]} - \mu_{[r:n-1]} = \frac{-\lambda D(\alpha_2)}{(n+1)\theta_2} \qquad (3.7)$$

$$\mu = \frac{1}{n} \sum_{r=1}^{n} \mu_{[n-r+1:n]} = \frac{\alpha_2}{\theta_2} \sum_{i=0}^{\infty} \frac{(-1)^i}{(i+1)^2} A(\alpha_2 - 1, i) = \frac{B(\alpha_2)}{\theta_2}.$$

Relations (3.6) and (3.7) can be extended as

$$\mu_{[r:n]} - \mu_{[r-i:n]} = \frac{2\lambda i D(\alpha_2)}{(n+1)\theta_2}, \quad 1 \leq i \leq r-1,$$

$$\mu_{[r:n]} - \mu_{[r:n-j]} = \frac{-\lambda j D(\alpha_2)}{(n+1)\theta_2}, \quad 1 \leq j \leq n-j.$$

Moreover, for $i = r - 1$ and $j = n - r$ the recurrence relations are obtained as follows:

$$\mu_{[r:n]} = \mu_{[1:n]} + \frac{2\lambda(r-1)D(\alpha_2)}{(n+1)\theta_2}, \quad 1 \leq r \leq n, \qquad (3.8)$$

$$\mu_{[r:n]} = \mu_{[r:r]} - \frac{\lambda(n-r)D(\alpha_2)}{(n+1)\theta_2}, \quad 1 \leq r \leq n. \qquad (3.9)$$

By using (3.8), we get

$$\mu_{[n:n]} - \mu_{[1:n]} = \frac{2\lambda(n-1)D(\alpha_2)}{(n+1)\theta_2}, \qquad (3.10)$$

which is positive, negative or zero whenever $(-1 \leq \lambda < 0, \alpha_2 > 1)$, $(0 < \lambda \leq 1, \alpha_2 > 1)$, or $(n = 1$ or $\lambda = 0)$, respectively.

Finally, we obtain the recurrence relations

$$\mu_{[n-r+1:n]} = \mu_{[r:n]} + \frac{2\delta_r D(\alpha_2)}{\theta_2},$$
$$\mu_{[r:n]} = \mu_{[r\gamma:(n+1)\gamma-1]}, \quad \gamma \geq 1,$$
$$\mu_{[r:n]} = \sum_{s=n-r+1}^{n} (-1)^{s-n+r-1} \binom{s-1}{n-r} \binom{n}{s} \mu_{[1:s]}.$$

Also, an analytical expression of the conditional expectation of $Y_{[r:n]}$ given $X_{(r:n)} = x$ is obtained from (2.5)as

$$E(Y_{[r:n]}|X_{(r:n)} = x) = E(Y|X = x) = \mu[1 + \frac{\lambda\ D(\alpha_2)}{B(\alpha_2)}(2(1 - e^{-\theta_1 x})^{\alpha_1} - 1)].$$

By using (3.2), we obtain the variance of $Y_{[r:n]}$ as

$$Var(Y_{[r:n]}) = \frac{1}{\theta_2^2}[C(\alpha_2) + \delta_r(C(2\alpha_2) - C(\alpha_2))], \tag{3.11}$$

and the difference between the variances $\sigma^2 = Var(Y)$ and $\sigma^2_{[r:n]} = Var(Y_{[r:n]})$ is

$$\sigma^2_{[r:n]} - \sigma^2 = \frac{\delta_r}{\theta_2^2}[C(2\alpha_2) - C(\alpha_2)],$$

which is negative, positive or zero whenever $(-1 \leq \lambda < 0, r > (n+1)/2, \alpha_2 > 1)$, $(0 < \lambda \leq 1, r > (n+1)/2, \alpha_2 > 1)$, or ($r = (n+1)/2$ or $\lambda = 0$), respectively.

3.2 Joint distribution of concomitants

From Scaria and Nair (1999), the joint pdf of $(Y_{[r:n]}, Y_{[s:n]})$, for $r < s$ is given by

$$\begin{aligned} f_{[r,s:n]}(z,w) &= (\alpha_2\theta_2)^2 e^{-\theta_2(z+w)}[(1-e^{-\theta_2 z})(1-e^{-\theta_2 w})]^{\alpha_2-1}\{1+\delta_r - 2\delta_r(1-e^{-\theta_2 z})^{\alpha_2}) \\ &\quad +\delta_s(1-2(1-e^{-\theta_2 w})^{\alpha_2}) \\ &\quad +\delta_{r,s}(1-2(1-e^{-\theta_2 z})^{\alpha_2})(1-2(1-e^{-\theta_2 w})^{\alpha_2})\}, \qquad z < w, \end{aligned} \tag{3.12}$$

where $\delta_{r,s} = \lambda^2[(n-2s+1)/(n+1) - 2r(n-2s)/(n+1)(n+2)]$. Therefore, the joint moment generating function of $Y_{[r:n]}$ and $Y_{[s:n]}$ is obtained as

$$\begin{aligned} M_{Y_{[r:n]},Y_{[s:n]}}(t_1,t_2) &= (\delta_r + \delta_{r,s})\alpha_2^2 Beta(\alpha_2, 1-\frac{t_2}{\theta_2})[Beta(\alpha_2, 1-\frac{t_1}{\theta_2}) - 2Beta(2\alpha_2, 1-\frac{t_1}{\theta_2})] \\ &\quad +(1+\delta_{r,s})\alpha_2^2 Beta(\alpha_2, 1-\frac{t_1}{\theta_2})Beta(\alpha_2, 1-\frac{t_2}{\theta_2}) \\ &\quad -2(\delta_s + \delta_{r,s})\alpha_2^2 Beta(\alpha_2, 1-\frac{t_1}{\theta_2})Beta(2\alpha_2, 1-\frac{t_2}{\theta_2}) \\ &\quad +4\delta_{r,s}\alpha_2^2 Beta(2\alpha_2, 1-\frac{t_1}{\theta_2})Beta(2\alpha_2, 1-\frac{t_2}{\theta_2}). \end{aligned}$$

The product moment $E[Y_{[r:n]}Y_{[s:n]}] = \mu_{[r,s:n]}$ is obtained directly from (3.12) as

$$\mu_{[r,s:n]} = \frac{1}{\theta_2^2}\{(1+\delta_r+\delta_s+\delta_{r,s})B^2(\alpha_2) - (\delta_r+\delta_s+2\delta_{r,s})B(\alpha_2)B(2\alpha_2) + \delta_{r,s}B^2(2\alpha_2)\}.$$

Therefore, the covariance between $Y_{[r:n]}$ and $Y_{[s:n]}$ is given as

$$Cov(Y_{[r:n]}, Y_{[s:n]}) = \frac{D^2(\alpha_2)[\delta_{r,s} - \delta_r\delta_s]}{\theta_2^2}. \tag{3.13}$$

Thus the r-th and s-th concomitants are positively correlated and its value decreases as r and s pull apart. From (3.13) and (3.11), we can obtain the coefficient of correlation between $Y_{[r:n]}$ and $Y_{[s:n]}$ as

$$\rho_{[r,s:n]} = \frac{D^2(\alpha_2)[\delta_{r,s} - \delta_r\delta_s]}{\{[C(\alpha_2) + \delta_r(C(2\alpha_2) - C(\alpha_2))][C(\alpha_2) + \delta_s(C(2\alpha_2) - C(\alpha_2))]\}^{\frac{1}{2}}}. \tag{3.14}$$

We calculate the values of $\rho_{[r,s:n]}$ for $\alpha_2 = 0.2, 2,\ \lambda = 0.2, 0.5, 0.8, 1,$ and $1 < r < s \leq 10$ $(1 < m < n \leq 10)$. We found that the correlation is extremely small, i.e. $\rho_{[r,s:n]} < 0.02$. In fact, $\rho_{[r,s:n]}$ has minimum value when $r = 1$ and $s = 10$, and it has maximum value when $r = 5$ and $s = 6$ for given α_2 and λ.

3.3 Inferences based on concomitants

Here, we consider estimation of the parameter $\mu = B(\alpha_2)/\theta_2$. It is known that in GE distribution there is not a closed form for the maximum likelihood estimation for this parameter. Also, there is not an unbiased estimation (see Gupta and Kundu, 1999). But based on concomitant of order statistics of MTBGED, an unbiased estimator for μ can be given as

$$T_r = \frac{Y_{[r:n]} + Y_{[n-r+1:n]}}{2}, \quad r = 1, 2, ..., n.$$

The variance of T_r is

$$Var(T_r) = \frac{1}{2\theta_2^2}\{C(\alpha_2) + \frac{[2r\lambda D(\alpha_2)]^2}{(n+1)^2(n+2)}\}, \tag{3.15}$$

which is an increasing function of r for $r \geq 1$. Therefore, among these unbiased estimators, T_1 has minimum variance as

$$Var(T_1) = \frac{1}{2\theta_2^2}\{C(\alpha_2) + \frac{[2\lambda D(\alpha_2)]^2}{(n+1)^2(n+2)}\}. \tag{3.16}$$

Recently, Tahmasebi and Behboodian (2012a, 2012b) obtined an explicit expression of Shannon entropy for concomitants of order statistics in Morgenstern family. Applying this expression for $Y_{[r:n]}$ in MTBGED, we have

$$H(Y_{[r:n]}) = W_{\lambda,n}(r) - \log(\alpha_2\theta_2) + B(\alpha_2) - \delta_r D(\alpha_2) + \frac{\alpha_2 - 1}{\alpha_2}[1 + \frac{\delta_r}{2}],$$

where

$$W_{\lambda,n}(r)=\frac{1}{8\delta_r}\left\{(1-\delta_r)^2\left[2\log(1-\delta_r)-1\right]-(1+\delta_r)^2\left[2\log(1+\delta_r)-1\right]\right\}.$$

Finally, when α_2 and λ are known, then an analytical expression of Fisher information in $Y_{[r:n]}$ about θ_2 is derived as

$$\begin{aligned}I_{\theta_2}(Y_{[r:n]}) &= \frac{1}{\theta_2^2}+2\alpha_2(\alpha_2-1)\theta_2\sum_{i=0}^{\infty}\frac{(-1)^i}{(i\lambda+2\theta_2)^3}[A(\alpha_2-3,i)\\&+\delta_r(A(\alpha_2-3,i)-2A(2\alpha_2-3,i))]\\&-4\delta_r\alpha_2^2\theta_2\sum_{i=0}^{\infty}\frac{(-1)^i}{(i\lambda+3\theta_2)^3}[A(2\alpha_2-2,i)-2A(2\alpha_2-3,i)]\\&+E_{f_{[r:n]}}\left[\frac{2\delta_r\alpha_2Ye^{-\theta_2Y}(1-e^{-\theta_2Y})^{\alpha_2-1}}{1+\delta_r(1-2(1-e^{-\theta_2Y})^{\alpha_2})}\right]^2.\end{aligned}$$

4 Some distributional properties of concomitants of record values

Let $(X_1,Y_1),(X_2,Y_2),\cdots$ be a sequence of bivariate random variables from a continuous distribution. If $\{R_n, n\geq 1\}$ is the sequence of record values in the sequence of X's, then the Y which corresponds with the nth record will be called the concomitant of nth record and is denoted by $R_{[n]}$. The concomitants of record values arise in a wide variety of practical experiments such as industrial stress testing, life time experiments, meteorological analysis, sporting matches and some other experimental fields. For other important applications of record values and their concomitants see Arnold et al. (1998), Ahsanullah (1995), Barakat et al. (2013), and Bdair and Raqab (2013). Some properties from concomitants of record values were discussed in Houchens (1984), Nevzorov and Ahsanullah (2000), Ahsanullah (2009), and Ahsanullah and Shakil (2013).

Houchens (1984) has obtained the pdf of concomitant of nth record value for $n\geq 1$ arising in (2.2) as

$$h_{[n]}(y)=\int_{-\infty}^{+\infty}f(y|x)f_n(x)dx=\alpha_2\theta_2e^{-\theta_2y}(1-e^{-\theta_2y})^{\alpha_2-1}[1+\lambda_n(1-2(1-e^{-\theta_2y})^{\alpha_2})],\quad(4.1)$$

where $\lambda_n=\lambda(2^{-n}-1)$ and $f_n(x)$ is pdf of R_n. The moments of $R_{[n]}$ is readily obtained from (4.1) as

$$\mu_{R_{[n]}}^{(k)}=\frac{\alpha_2\Gamma(k+1)}{\theta_2^k}\sum_{i=0}^{\infty}\frac{(-1)^i}{(i+1)^{k+1}}[(1+\lambda_n)A(\alpha_2-1,i)-2\lambda_nA(2\alpha_2-1,i)].$$

Also, the moment generating function of $R_{[n]}$ is obtained by

$$\begin{aligned} M_{R_{[n]}}(t) &= (1+\lambda_n)M_Y(t) - 2\lambda_n M_V(t) \\ &= \alpha_2[(1+\lambda_n)Beta(\alpha_2, 1-\frac{t}{\theta_2}) - 2\lambda_n Beta(2\alpha_2, 1-\frac{t}{\theta_2})], \end{aligned}$$

Therefore, the mean and variance of $R_{[n]}$ are given as

$$\mu_{R_{[n]}} = E(R_{[n]}) = \frac{1}{\theta_2}[B(\alpha_2) - \lambda_n D(\alpha_2)],$$
$$\sigma^2_{R_{[n]}} = Var(R_{[n]}) = \frac{1}{\theta_2^2}[C(\alpha_2) - \lambda_n(C(2\alpha_2) - C(\alpha_2))].$$

The difference between the means of Y and $R_{[n]}$ is

$$\mu_{R_{[n]}} - \mu = \frac{-\lambda_n D(\alpha_2)}{\theta_2}.$$

which is negative, positive or zero whenever $(-1 \le \lambda < 0, \alpha_2 > 1)$, $(0 < \lambda \le 1, \alpha_2 > 1)$, or $(n = 1$ or $\lambda = 0)$, respectively. The asymptotic mean and variance of $R_{[n]}$ are respectively

$$\mu_{R_{[n]}} \to \frac{1}{\theta_2}[B(\alpha_2) + \lambda D(\alpha_2)], \qquad \sigma^2_{R_{[n]}} \to \frac{1}{\theta_2^2}[C(\alpha_2) + \lambda(C(2\alpha_2) - C(\alpha_2))]. \tag{4.2}$$

The joint pdf of $R_{[n]}$ and $R_{[m]}$ for $n < m$ is given by

$$\begin{aligned} f_{R_{[n]},R_{[m]}}(z_1, z_2) &= (\alpha_2\theta_2)^2 e^{-\theta_2(z_1+z_2)}[(1-e^{-\theta_2 z_1})(1-e^{-\theta_2 z_2})]^{\alpha_2-1}[1+\lambda_n+\lambda_m+\lambda_{n,m}] \\ &\quad -2(\alpha_2\theta_2)^2 e^{-\theta_2(z_1+z_2)}(1-e^{-\theta_2 z_1})^{2\alpha_2-1}(1-e^{-\theta_2 z_2})^{\alpha_2-1}[\lambda_n+\lambda_{n,m}] \\ &\quad -2(\alpha_2\theta_2)^2 e^{-\theta_2(z_1+z_2)}(1-e^{-\theta_2 z_1})^{2\alpha_2-1}(1-e^{-\theta_2 z_2})^{\alpha_2-1}[\lambda_m+\lambda_{n,m}] \\ &\quad +4\lambda_{n,m}(\alpha_2\theta_2)^2 e^{-\theta_2(z_1+z_2)}(1-e^{-\theta_2 z_1})^{2\alpha_2-1}(1-e^{-\theta_2 z_2})^{2\alpha_2-1}, \end{aligned} \tag{4.3}$$

where $\lambda_{n,m} = \lambda^2[(2^{n-m+2} - 3^{n+1})/3^{n+1} - (\lambda_n + \lambda_m)/\lambda]$. Using (4.3), the joint moment generating function of $R_{[n]}$ and $R_{[m]}$ is obtained

$$\begin{aligned} M_{R_{[n]},R_{[m]}}(t_1, t_2) &= (1+\lambda_n+\lambda_m+\lambda_{n,m})\alpha_2^2 Beta(\alpha_2, 1-\frac{t_2}{\theta_2})Beta(\alpha_2, 1-\frac{t_1}{\theta_2}) \\ &\quad -2(\lambda_n+\lambda_{n,m})\alpha_2^2 Beta(2\alpha_2, 1-\frac{t_1}{\theta_2})Beta(\alpha_2, 1-\frac{t_2}{\theta_2}) \\ &\quad -2(\lambda_m+\lambda_{n,m})\alpha_2^2 Beta(\alpha_2, 1-\frac{t_1}{\theta_2})Beta(2\alpha_2, 1-\frac{t_2}{\theta_2}) \\ &\quad +4\alpha_2^2\lambda_{n,m} Beta(2\alpha_2, 1-\frac{t_1}{\theta_2})Beta(2\alpha_2, 1-\frac{t_2}{\theta_2}). \end{aligned} \tag{4.4}$$

The expression for the product moment $E[R_{[n]}R_{[m]}]$ directly follows from (4.4) as

$$\begin{aligned} E[R_{[n]}R_{[m]}] &= \frac{1}{\theta_2^2}\{(1+\lambda_n+\lambda_m+\lambda_{n,m})B^2(\alpha_2) \\ &\quad -(\lambda_n+\lambda_m+2\lambda_{n,m})B(\alpha_2)B(2\alpha_2) + \lambda_{n,m}B^2(2\alpha_2)\}. \end{aligned} \tag{4.5}$$

Therefore, covariance between $R_{[n]}$ and $R_{[m]}$ follows as

$$Cov(R_{[n]}, R_{[m]}) = \frac{D^2(\alpha_2)[\lambda_{n,m} - \lambda_n\lambda_m]}{\theta_2^2}. \tag{4.6}$$

Note that $Cov(R_{[n]}, R_{[m]})$ decreases as m and n pull apart and the record concomitants $R_{[n]}$ and $R_{[m]}$ are positively correlated. Further, the correlation coefficient between $R_{[n]}$ and $R_{[m]}$ is

$$\rho(R_{[n]}, R_{[m]}) = \frac{D^2(\alpha_2)[\lambda_{n,m} - \lambda_n\lambda_m]}{\{[C(\alpha_2) + \lambda_n(C(2\alpha_2) - C(\alpha_2))][C(\alpha_2) + \lambda_m(C(2\alpha_2) - C(\alpha_2))]\}^{\frac{1}{2}}}.$$

We calculate the values of $\rho(R_{[n]}, R_{[m]})$ for $\alpha_2 = 0.2, 2$, $\lambda = 0.2, 0.5, 0.8, 1$, and $1 < r < s \leq 10$ $(1 < m < n \leq 10)$. We found that the correlation is extremely small, i.e. $\rho(R_{[n]}, R_{[m]}) < 0.01$.

An explicit expression of Shannon entropy for concomitants of record values in MTBGED is derived as

$$H(R_{[n]}) = C_\lambda(n) - \log(\alpha_2\theta_2) + B(\alpha_2) - \lambda_n D(\alpha_2) + \frac{\alpha_2 - 1}{\alpha_2}[1 + \frac{\lambda_n}{2}],$$

where

$$C_\lambda(n) = \frac{1}{8\lambda_n}\{(1-\lambda_n)^2[2\log(1-\lambda_n) - 1] - (1+\lambda_n)^2[2\log(1+\lambda_n) - 1]\}. \tag{4.7}$$

Finally, when α_2 and λ are known, then an analytical expression of Fisher information in $R_{[n]}$ about θ_2 is derived as

$$\begin{aligned} I_{\theta_2}(R_{[n]}) \quad = \quad & \frac{1}{\theta_2^2} + 2\alpha_2(\alpha_2 - 1)\theta_2 \sum_{i=0}^{\infty} \frac{(-1)^i}{(i\lambda + 2\theta_2)^3}[A(\alpha_2 - 3, i) \\ & +\lambda_n(A(\alpha_2 - 3, i) - 2A(2\alpha_2 - 3, i))] \\ & -4\lambda_n\alpha_2^2\theta_2 \sum_{i=0}^{\infty} \frac{(-1)^i}{(i\lambda + 3\theta_2)^3}[A(2\alpha_2 - 2, i) - 2A(2\alpha_2 - 3, i)] \\ & +E_{h_{[n]}}\left[\frac{2\lambda_n\alpha_2 Y e^{-\theta_2 Y}(1 - e^{-\theta_2 Y})^{\alpha_2 - 1}}{1 + \lambda_n(1 - 2(1 - e^{-\theta_2 Y})^{\alpha_2})}\right]^2. \end{aligned}$$

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