Higher class numbers in extensions of number fields

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Abstract Let $F/\mathbb{Q}$ be a complex Galois extension with Galois group V_4 or S_3. This paper proves that certain quotients of higher class numbers corresponding to the intermediate fields take on a determined finite set of values, assuming the motivic formulation of the Lichtenbaum conjecture.

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1 Introduction

Let F be a number field, $\mathcal{O}_F$ the ring of integers of F and $\zeta_F(s)$ the Dedekind zeta function of F. It is known that one has the analytic class number formula

$$\zeta_F^*(0) = -\frac{R_1(F)h_F}{w_1(F)}, \tag{1}$$

where $w_1(F)$ is the number of roots of unity in F, h_F is the class number of F, $R_1(F)$ is the first regulator of F and $\zeta_F^*(0)$ is the first non-vanishing coefficient in the Taylor-expansion of the zeta-function $\zeta_F(s)$ around $s = 0$.

Let E/F be a Galois extension of number fields with Galois group G. When G is a dihedral group of order $2p$, the Brauer-Kuroda formula for the class number can be interpreted in terms of a unit index(See [1, 2, 9]).

There are conjectural analogues of the formula (1) when 0 is replaced by negative integers. One of them says

Motivic formulation of the Lichtenbaum Conjecture. For any number field F and for any integer $n \geq 2$,

$$\zeta_F^*(1-n) = \pm\frac{R_n^M(F)h_n(F)}{w_n(F)}, \tag{2}$$

where $h_n(F)$ is the order of the motivic cohomology group $H_M^2(\mathcal{O}_F, \mathbb{Z}(n))$, $w_n(F)$ is the order of the torsion subgroup of the motivic cohomology group $H_M^1(\mathcal{O}_F, \mathbb{Z}(n))$ and $R_n^M(F)$ is the motivic regulator of $H_M^1(\mathcal{O}_F, \mathbb{Z}(n))$. In this paper we use the definition of motivic cohomology groups for a field F in terms of Bloch's higher Chow groups:

$$H_M^j(F, \mathbb{Z}(n)) := CH^n(Spec(F), 2n-j).$$

Similarly, for a Dedekind domain $\mathcal{O}_F$ we will use the notation $H_M^j(\mathcal{O}_F, \mathbb{Z}(n))$ for the motivic cohomology groups of $Spec(\mathcal{O}_F)$.

The relationship between motivic cohomology, *é*tale cohomology and K-theory is described via Chern characters (cf. [7], Chapter 2 for overview). Here, we want to describe briefly the profound consequences which the Bloch-Kato Conjecture has for the interplay between the 3 functors. The Bloch-Kato Conjecture states that for any field F and any

$n \geq 1$ the Galois symbol

$$K_n^M(F)/p^m \to H^n(F, \mu_{p^m}^{\otimes n})$$

from Milnor K-theory to Galois cohomology is an isomorphism for any p-power p^m with $p \neq char(F)$. It has been proved by Voevodsky [12]. The special case $p = 2$, i.e., The Milnor Conjecture, has been proved by Voevodsky [11]. The first consequence of the Bloch-Kato Conjecture is that the Quillen-Lichtenbaum Conjecture holds, that is, for any odd prime p and any number field F, the *é*tale Chern characters

$$K_{2n-i}(F) \otimes \mathbb{Z}_p \to H^i_{\acute{e}t}(F, \mathbb{Z}_p(n))$$

are isomorphisms for $n \geq 2$ and $i = 1, 2$. Here $H^i_{\acute{e}t}(F, \bullet)$ denotes the i-th *é*tale cohomology group of $Spec(F)$ with values in a sheaf $\bullet$. The second consequence is that the same result is true for the motivic cohomology groups for all primes p:

$$H^i_M(F, \mathbb{Z}(n)) \otimes \mathbb{Z}_p \cong H^i_{\acute{e}t}(F, \mathbb{Z}_p(n)).$$

For the ring of integers $\mathcal{O}_F$, one uses the localization sequences in K-theory, in *é*tale cohomology and in motivic cohomology to obtain the following analogous result:

Lemma 1 *([7]) Let $\mathcal{O}_F$ be the ring of integers in a number field F with r_1 real embeddings, and let $n \geq 2$. Then for $i = 1, 2$,*

(i) The Chern character

$$K_{2n-i}(\mathcal{O}_F) \to H^i_M(\mathcal{O}_F, \mathbb{Z}(n))$$

is an isomorphism if $2n-i \equiv 0, 1, 2, 7 \pmod 8$, *injective with cokernel* $\cong (\mathbb{Z}/2\mathbb{Z})^{r_1}$ *if* $2n-i \equiv 6 \pmod 8$, *surjective with kernel* $\cong (\mathbb{Z}/2\mathbb{Z})^{r_1}$ *if* $2n-i \equiv 3 \pmod 8$. *In the remaining cases* ($n \equiv 3 \pmod 4$) *there is an exact sequence*

$$0 \to K_{2n-2}(\mathcal{O}_F) \to H^2(\mathcal{O}_F, \mathbb{Z}(n)) \to (\mathbb{Z}/2\mathbb{Z})^{r_1} \to K_{2n-1}(\mathcal{O}_F) \to H^1(\mathcal{O}_F, \mathbb{Z}(n)) \to 0.$$

(ii) $H^i_M(\mathcal{O}_F, \mathbb{Z}(n)) \otimes \mathbb{Z}_p \cong H^i_{\acute{e}t}(\mathcal{O}_F[\frac{1}{p}], \mathbb{Z}_p(n))$, *for all primes p.*

We also note that for all $n \geq 2$, the motivic groups $H^2_M(\mathcal{O}_F, \mathbb{Z}(n))$ are finite, $H^1_M(\mathcal{O}_F, \mathbb{Z}(n)) \cong H^1(F, \mathbb{Z}(n))$ are finitely generated $\mathbb{Z}$-modules, $(H^1_M(\mathcal{O}_F, \mathbb{Z}(n)))_{tors} \cong H^0(F, \mathbb{Q}/\mathbb{Z}(n))$ and

$$d_n = \mathrm{rk}_{\mathbb{Z}}(H^1_M(\mathcal{O}_F, \mathbb{Z}(n))) = \begin{cases} r_1 + r_2, & \text{if } n \text{ is odd,} \\ r_2, & \text{if } n \text{ is even,} \end{cases}$$

where r_1 and r_2 are respectively the numbers of real and complex places of F.

Lemma 2 *([7]) Let E/F be a Galois extension of number fields with Galois group G. Then for each $n \geq 2$ there is an isomorphism*

$$H^1_M(F, \mathbb{Z}(n)) \cong H^1_M(E, \mathbb{Z}(n))^G.$$

Let F be a finite Galois extension of a number field k with the Galois group G. R. Brauer [3] and S. Kuroda [8] proved independently some multiplicative relations between the Dedekind zeta functions of some subfields of F. For every cyclic subgroup H of G,

$$c_G(H) := \frac{1}{(G:H)} \sum_{H^* \text{ cyclic } H\subseteq H^* \subseteq G} \mu((H^*:H)),$$

where μ is the Möbius function. Then

$$\zeta_k(s) = \prod_{H \text{ cyclic } H\subseteq G} \zeta_{F^H}(s)^{c_G(H)}, \tag{3}$$

where F^H is the subfield of F fixed by H. In what follows we usually assume $k = \mathbb{Q}$, then $\zeta_k = \zeta$ is the Riemann zeta function.

Let l be a prime number and D the dihedral group of order $2l$. Let $F/\mathbb{Q}$ be a complex Galois extension with Galois group G, where $G = V_4$ or D. In section 2, when n is even, we will give the Brauer-Kuroda formulae for higher class numbers by an index of the first Motivic cohomology groups using the Brauer-Kuroda relations (3) about zeta-functions and the formula (2). For $G = V_4$, we obtain

$$\frac{h_n(F)h_n(\mathbb{Q})^2}{h_n(F_0)h_n(F_1)h_n(F_2)} \in \{1/2, 1, 2\}$$

where F_0, F_1 and F_2 are all quadratic subfields of F. For $D = D_{2l}$ with $l = 3$, we obtain

$$\frac{h_n(F)h_n(\mathbb{Q})^2}{h_n(k)h_n(K)^2} \in \{1/3, 1, 3, 9\}$$

where k is the quadratic subfield of F and K is the real subfield of F.

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2 Main results

Let F be a number field and $X(F) = Hom(F, \mathbb{C})$ the set of complex embeddings of F. Denote $\mathbb{R}(n-1) = (2\pi i)^{n-1}\mathbb{R}$. The Beilinson regualtor map ρ_n is obtained by composing the various embeddings of F into $\mathbb{C}$ with a Chern character

$$ch_n : K_{2n-1}(\mathbb{C}) \to H^1_D(Spec(\mathbb{C}), \mathbb{R}(n)) \cong \mathbb{R}(n-1)$$

into Deligne-cohomology. We obtain

$$\rho_n : \ K_{2n-1}(\mathcal{O}_F) \to K_{2n-1}(\mathcal{O}_F) \otimes \mathbb{Q} \to (\mathbb{R}(n-1)^{X(F)})^+,$$

where complex conjugation acts on the set of embeddings and on the coefficients $\mathbb{R}(n-1)$(See [10, Neukirch's article] for details). If τ is a complex conjugation of the embedding of F into $\mathbb{C}$ and n is even, then for every $a \in K_{2n-1}(\mathcal{O}_F)$, we have $ch_n(\tau(a)) = -ch_n(a)$. By Lemma 1, we can define the n-th motivic regulator map we shall consider is a homomorphism

$$\rho_n^M : \ H^1_M(F, \mathbb{Z}(n)) \to H^1_M(F, \mathbb{Z}(n)) \otimes \mathbb{Q} \cong K_{2n-1}(\mathcal{O}_F) \otimes \mathbb{Q} \to (\mathbb{R}(n-1)^{X(F)})^+.$$

By Borel's results and the fact that the Beilinson regulator map ρ_n is twice the Borel regulator map, the kernel of ρ_n^M is torsion. The image of ρ_n^M therefore is a full lattice in the real vectorspace $(\mathbb{R}(n-1)^{X(F)})^+$ of dimension d_n.We denote by $\Lambda(F)$ this lattice and denote by $R_n^M(F)$ the covolume of $\Lambda(F)$.

In other words, if $a_1, a_2, \cdots, a_{d_n} \in H^1_M(F, \mathbb{Z}(n))$ is a basis of $H^1_M(F, \mathbb{Z}(n))$, then $\rho_n^M(a_1), \rho_n^M(a_2), \cdots \rho_n^M(a_{d_n}) \in (\mathbb{R}(n-1)^{X(F)})^+$ generate this lattice, so

$$R_n^M(F) = |\det(ch_n(\sigma_j(a_i))_{1\leq i,j\leq d_n})|,$$

where σ_i, $i = 1, 2, \cdots, d_n$ are all infinite places of F when n is odd, all complex places of F when n is even. In the rest of this section, we always assume that n is even.

2.1 Biquadratic fields

Let $F/\mathbb{Q}$ be a biquadratic extension with Galois group $G =< \sigma_1, \sigma_2 >$. Then $H_0 =< \sigma_1\sigma_2 >$, $H_1 =< \sigma_1 >$ and $H_2 =< \sigma_2 >$ are all cyclic non-trivial subgroups of G. For $i = 0, 1, 2$ denote $F_i := F^{H_i}$. Hence we have the following Brauer-Kuroda relation:

$$\zeta_F(s)\zeta_{\mathbb{Q}}(s)^2 = \prod_{i=0}^{2} \zeta_{F_i}(s). \tag{4}$$

Assume that $F/\mathbb{Q}$ is a complex biquadratic and F_0 is the real subfield. So $\sigma_1\sigma_2$ is the complex conjugation. Consequently the two complex places of F are represented by 1 and σ_1. The lattices $\Lambda(F_1)$ and $\Lambda(F_2)$ are 1-dimensional. For $i = 1, 2$ let a_i be a generator of $H^1_M(F_i, \mathbb{Z}(n))/\text{tors}$. Hence $R_n^M(F_i) = |ch_n(a_i)|$. Obviously, the lattice Λ' generated by $\rho_n^M(a_1)$ and $\rho_n^M(a_2)$ is a sublattice in $\Lambda(F)$, and has the covolume equal to the absolute value of the determinant of the matrix

$$\begin{pmatrix} ch_n(a_1) & ch_n(\sigma_1(a_1)) \\ ch_n(a_2) & ch_n(\sigma_1(a_2)) \end{pmatrix} = \begin{pmatrix} ch_n(a_1) & ch_n(a_1) \\ ch_n(a_2) & -ch_n(a_2) \end{pmatrix}.$$

Thus

$$\text{covol}(\Lambda') = 2R_n^M(F_1)R_n^M(F_2). \tag{5}$$

Denote by $u(F, n)$ the torsion part of $H^1_M(F, \mathbb{Z}(n))$. We write $U(F, n) = H^1_M(F, \mathbb{Z}(n))$, $V(F, n) = \prod_{i=1}^{2} H^1_M(F_i, \mathbb{Z}(n))$ and $v(F, n) = u(F, n) \cap V(F, n)$.

Proposition 1

$$\frac{R_n^M(F)}{R_n^M(F_1)R_n^M(F_2)} = \frac{2(u(F, n) : v(F, n))}{(U(F, n) : V(F, n))}.$$

Proof It is easy to see that we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 1 \to & v(F,n) & \to V(F,n) \to & \rho_n^M(V(F,n)) \to 0 \\ & \downarrow & \downarrow f & \downarrow \\ 1 \to & u(F,n) & \to U(F,n) \to & \rho_n^M(U(F,n)) \to 0 \end{array},$$

where the map f is induced by inclusions $H^1_M(F_i,\mathbb{Z}(n)) \subseteq H^1_M(F,\mathbb{Z}(n))$, for $i=1,2$. So, by the snake Lemma, we see that

$$(U(F,n):V(F,n)) = (u(F,n):v(F,n))(\rho_n^M(U(F,n)):\rho_n^M(V(F,n)))$$

since $V(F,n)$ has finite index in $U(F,n)$. Since $\rho_n^M(V(F,n)) = \Lambda'$ is a sublattice of $\rho_n^M(U(F,n)) = \Lambda(F)$, we have $\mathrm{covol}(\Lambda') = R_n^M(F)(\rho_n^M(U(F,n)) : \rho_n^M(V(F,n)))$. Thus by (5),

$$\frac{R_n^M(F)}{R_n^M(F_1)R_n^M(F_2)} = \frac{2(u(F,n):v(F,n))}{(U(F,n):V(F,n))}.$$

Proposition 2

$$\frac{R_n^M(F)}{R_n^M(F_1)R_n^M(F_2)} = 1 \text{ or } 2 \text{ or } 1/2.$$

Proof Consider the commutative diagram

$$\begin{array}{ccccccc} 1 \to & v(F,n) & \to V(F,n) \to & V(F,n)/v(F,n) \to 0 \\ & \downarrow & \downarrow f & \downarrow \overline{f} \\ 1 \to & u(F,n) \to & U(F,n) \to & U(F,n)/u(F,n) \to 0 \end{array}.$$

Since $\overline{f}$ is injective, the snake lemma, applied to the above diagram, implies that

$$\frac{(U(F,n):V(F,n))}{(u(F,n):v(F,n))} = |\mathrm{coker}\overline{f}|.$$

For every $x \in U(F,n)$, we have $x^{1+\sigma_i} \in H^1_M(F_i,\mathbb{Z}(n))(i=1,2)$, $x^{\sigma_1+\sigma_2} \in H^1_M(F_0,\mathbb{Z}(n))$ and $x^{1+\sigma_1+\sigma_2+\sigma_2\sigma_2} \in H^1_M(\mathbb{Q},\mathbb{Z}(n))$ by Lemma 2. We know that $H^1_M(F_0,\mathbb{Z}(n)) \subseteq u(F,n)$ and $H^1_M(\mathbb{Q},\mathbb{Z}(n)) \subseteq u(F,n)$ since $\mathbb{Q}$ and F_0 are two totally real number fields. From the following identity

$$x^{1+\sigma_1}x^{1+\sigma_2}x^{1+\sigma_1+\sigma_2+\sigma_2\sigma_2} = x^2x^{\sigma_1+\sigma_2}x^{1+\sigma_1+\sigma_2+\sigma_2\sigma_2},$$

we have $\overline{x^2} = \overline{x^{1+\sigma_1}x^{1+\sigma_2}}$. That is, for every $\overline{x} \in U(F,n)/u(F,n)$ we have $\overline{x}^2 \in \mathrm{im}(\overline{f})$. Since $\mathrm{rk}_{\mathbb{Z}}(H^1_M(F,\mathbb{Z}(n))) = 2$, we have $|\mathrm{coker}(\overline{f})||4$. By Proposition 1,

$$\frac{R_n^M(F)}{R_n^M(F_1)R_n^M(F_2)} = 1 \text{ or } 2 \text{ or } 1/2.$$

Theorem 1 *Let F be a complex biquadratic extension of $\mathbb{Q}$ with quadratic subfields F_0, F_1 and F_2, where F_0 is real. Then for $n > 1$ we have*

$$\frac{h_n(F)h_n(\mathbb{Q})^2}{\prod_{i=0}^{2} h_n(F_i)} = 1 \text{ or } 2 \text{ or } 1/2.$$

Proof Since F_0 and $\mathbb{Q}$ are totally real number fields, we know that $R_n^M(F_0)$ and $R_n^M(\mathbb{Q})$ are trivial. By Proposition 2, the formulae (2) and (4), we have

$$\frac{h_n(F)h_n(\mathbb{Q})^2}{\prod_{i=0}^{2} h_n(F_i)} = \frac{w_n(F)w_n(\mathbb{Q})^2}{\prod_{i=0}^{2} w_n(F_i)} \text{ or } \frac{w_n(F)w_n(\mathbb{Q})^2}{2\prod_{i=0}^{2} w_n(F_i)} \text{ or } \frac{2w_n(F)w_n(\mathbb{Q})^2}{\prod_{i=0}^{2} w_n(F_i)}.$$

Now, it is necessary to prove

$$\frac{w_n(F)w_n(\mathbb{Q})^2}{\prod_{i=0}^{2} w_n(F_i)} = 1.$$

Let E be a number field. For every prime number p,

$$w_n^{(p)}(E) := \max\{p^v | \mathrm{Gal}(E(\zeta_{p^v})/E) \text{ has exponent dividing } n\}.$$

So it is easy to obtain $w_n(E) = \prod_p w_n^{(p)}(E)$ and the following statements:

(1) Let b be the maximal power of 2 dividing n. Then we have $w_n^{(2)}(F_1) = w_n^{(2)}(F_2) = w_n^{(2)}(\mathbb{Q}) = 2^{2+b}$ and

$$w_n^{(2)}(F) = w_n^{(2)}(F_0) = \begin{cases} 2^{3+b}, & \sqrt{2} \in F, \\ 2^{2+b}, & \text{otherwise.} \end{cases}$$

(2) For every odd prime number p, we have

$$w_n^{(p)}(F) = w_n^{(p)}(F_0) = w_n^{(p)}(F_1) = w_n^{(p)}(F_2) = w_n^{(p)}(\mathbb{Q})$$

if $F \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$. If $F \cap \mathbb{Q}(\zeta_p) \neq \mathbb{Q}$, then $F \cap \mathbb{Q}(\zeta_p)$ is a quadratic subfield of F. Assuming $F \cap \mathbb{Q}(\zeta_p) = F_1$, we have $w_n^{(p)}(F) = w_n^{(p)}(F_1)$ and

$$w_n^{(p)}(F_0) = w_n^{(p)}(F_2) = w_n^{(p)}(\mathbb{Q}).$$

Therefore, we obtain $w_n(F)w_n(\mathbb{Q})^2/\prod_{i=0}^{2} w_n(F_i) = 1$.

Colloary 1 $|K_2(\mathcal{O}_F)| = \prod_{i=0}^{2} |K_2(\mathcal{O}_{F_i})|/2$ or $\prod_{i=0}^{2} |K_2(\mathcal{O}_{F_i})|/4$ or $\prod_{i=0}^{2} |K_2(\mathcal{O}_{F_i})|/8$.

Proof By Lemma 1 (i), we have $|K_2(\mathcal{O}_E)| = h_2(E)$ for every number field E. This result follows from $K_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and Theorem 1.

2.2 The case of the dihedral Galois group

Now let l be an odd prime number. Let D denote the dihedral group of order $2l$:

$$D = \{< \tau, \sigma > | \tau^l = \sigma^2 = 1, \sigma\tau\sigma = \tau^{-1}\}.$$

Let F be a Galois extension of $\mathbb{Q}$ with the Galois group D. It has a unique quadratic subfield k fixed by τ. Let K (resp. $K^{'}$) be the subfield of F fixed by $<\sigma>$ (resp. by $<\tau^2\sigma>$).

Assume that the field F is complex and σ is the complex conjugation. Then K is the unique maximal real subfield of F. We have $r_2(F) = l$, $r_2(k) = 1$ and $r_2(K) = r_2(K^{'}) = (l-1)/2$. Obviously, 1 is the complex place of k, and τ^j, $j = 0, 1, \cdots, l-1$ are complex places of F. Since $\sigma\tau^j = \tau^{-j}\sigma$, we get that complex places of K are $\tau, \tau^2, \cdots, \tau^t$ and complex places of $K^{'}$ are $1, \tau, \tau^2, \cdots, \tau^{t-1}$, where $t = (l-1)/2$.

Now we describe lattices of the fields k, K and $K^{'}$.

Let $H^1_M(k, \mathbb{Z}(n))$ be generated by b_0 and $H^1_M(K, \mathbb{Z}(n))$ (resp. $H^1_M(K^{'}, \mathbb{Z}(n))$) be generated by $b_1, b_2, \cdots, b_t$ (resp. by $b_{t+1}, b_{t+2}, \cdots, b_{2t}$). Then $R^M_n(k) = |ch_n(b_0)|$,

$$R^M_n(K) = |\det(\alpha_1, \alpha_2, \cdots, \alpha_t)|, \quad \text{where } \alpha_j = \begin{pmatrix} ch_n(\tau^j(b_1)) \\ \cdots \\ ch_n(\tau^j(b_t) \end{pmatrix}, \ j = 1, 2, \cdots, t,$$

$$R^M_n(K^{'}) = |\det(\beta_1, \beta_2, \cdots, \beta_t)|, \quad \text{where } \beta_j = \begin{pmatrix} ch_n(\tau^{t+j}(b_{t+1})) \\ \cdots \\ ch_n(\tau^{t+j}(b_{2t})) \end{pmatrix}, \ j = 1, 2, \cdots, t.$$

Since the motivic cohomology group $H^1_M(k, \mathbb{Z}(n))$, $H^1_M(K, \mathbb{Z}(n))$, $H^1_M(K^{'}, \mathbb{Z}(n))$ can be mapped canonically into $H^1_M(F, \mathbb{Z}(n))$, the elements $b_0, b_1, \cdots, b_{2t}$ defined above can be considered as elements of $H^1_M(F, \mathbb{Z}(n))$. Therefore the lattice $\Lambda^{'}$ generated $\rho^M_n(b_j)$, $j = 0, 1, \cdots, 2t$ is a sublattice of the lattice $\Lambda(F)$. Consequently,

$$\text{covol}(\Lambda^{'}) = |\det \begin{pmatrix} ch_n(b_0) & ch_n(\tau(b_0)) & \cdots & ch_n(\tau^{2t}(b_0)) \\ ch_n(b_1) & ch_n(\tau(b_1)) & \cdots & ch_n(\tau^{2t}(b_1)) \\ \cdots & \cdots & \cdots & \cdots \\ ch_n(b_{2t}) & ch_n(\tau(b_{2t})) & \cdots & ch^{2t}_n(\tau(b_{2t})) \end{pmatrix}|.$$

The first row of this matrix is simply

$$(ch_n(b_0), ch_n(\tau(b_0)), \cdots, ch_n(\tau^{2t}(b_0))) = ch_n(b_0)(1, 1, \cdots, 1).$$

The (j+1)st row, where $1 \leq j \leq t$, is

$$(ch_n(b_j), ch_n(\tau(b_j)), \cdots, ch_n(\tau^{2t}(b_j)))$$

$$= (0, ch_n(\tau(b_j)), \cdots, ch_n(\tau^t(b_j)), \cdots, -ch_n(\tau^t(b_j)), \cdots, -ch_n(\tau(b_j))),$$

since $\tau^i(b_j)$ and $\tau^{l-i}(b_j)$ are complex conjugate, and b_j is real.

The (j+1)st row, where $t+1 \leq j \leq 2t$, is

$$(ch_n(b_j), ch_n(\tau(b_j)), \cdots, ch_n(\tau^{2t}(b_j)))$$

$$= (ch_n(b_j), ch_n(\tau(b_j)), \cdots, ch_n(\tau^{t-1}(b_j)), \cdots, -ch_n(\tau^{t-1}(b_j)), \cdots, -ch_n(\tau(b_j)), 0)$$

since $\tau^i(b_j)$ and $\tau^{2t-i-1}(b_j)$ are complex conjugate, and $\tau^{l-1}(b_j)$ is real.

Hence, by [5, Lemma 1], we have

$$\begin{aligned}\mathrm{covol}(\Lambda') &= |ch_n(b_0)||\det \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & \alpha_1 & \cdots & \alpha_{t-1} & \alpha_t & -\alpha_t & \cdots & -\alpha_2 & -\alpha_1 \\ \beta_1 & \beta_2 & \cdots & \beta_t & -\beta_t & -\beta_{t-1} & \cdots & -\beta_1 & 0 \end{pmatrix}| \\ &= lR_n^M(k)R_n^M(K)R_n^M(K').\end{aligned}$$

Since K and K' are isomorphic, they have the same regulators. So we have

$$\mathrm{covol}(\Lambda') = lR_n^M(k)R_n^M(K)^2. \tag{6}$$

Proposition 3 *Let F be a complex Galois extension of $\mathbb{Q}$ with the dihedral Galois group D. Let k be the unique quadratic subfield of F and K (resp. K') the subfield of F fixed by $<\sigma>$ (resp. by $<\tau^2\sigma>$). Then we have*

$$\frac{R_n^M(F)}{R_n^M(k)R_n^M(K)^2} = \frac{l(u(F,n):v(F,n))}{(H_M^1(F,\mathbb{Z}(n)):V(F,n))},$$

where $V(F,n) = H_M^1(k,\mathbb{Z}(n))H_M^1(K,\mathbb{Z}(n))H_M^1(K',\mathbb{Z}(n))$, $u(F,n)$ is the torsion part of $H_M^1(F,\mathbb{Z}(n))$ and $v(F,n) = u(F,n) \cap V(F,n)$.

Proof The proof is the same as that of Proposition 1.

Theorem 2 *With notations as in Proposition 3, we have*

$$\frac{h_n(F)h_n(\mathbb{Q})^2}{h_n(k)h_n(K)^2} = \frac{(H_M^1(F,\mathbb{Z}(n)):V(F,n))}{l(u(F,n):v(F,n))}.$$

Proof By [6, Lemma 1.1], we have $w_n(F) = w_n(k)$ and $w_n(K) = w_n(\mathbb{Q})$. This result follows from the Brauer-Kuroda relation $\zeta_F(s)\zeta_{\mathbb{Q}}(s)^2 = \zeta_k(s)\zeta_K(s)^2$, the formula (2) and Proposition 3.

Proposition 4 *With notations as in Proposition 3, if $l = 3$, we have*

$$\frac{h_n(F)h_n(\mathbb{Q})^2}{h_n(k)h_n(K)^2} = 1/3 \text{ or } 1 \text{ or } 3 \text{ or } 9.$$

Proof Consider the commutative diagram

$$\begin{array}{ccccccc} 1 \to & v(F,n) & \to V(F,n) \to & V(F,n)/v(F,n) \to 0 \\ & \downarrow & \downarrow f & \downarrow \overline{f} \\ 1 \to & u(F,n) \to & H_M^1(F,\mathbb{Z}(n)) \to & H_M^1(F,\mathbb{Z}(n))/u(F,n) \to 0 \end{array}.$$

Since $\overline{f}$ is injective, the snake lemma, applied to the above diagram, implies that

$$\frac{(H^1_M(F,\mathbb{Z}(n)) : V(F,n))}{(u(F,n) : v(F,n))} = |\mathrm{coker}\overline{f}|.$$

For every $x \in H^1_M(F,\mathbb{Z}(n))$, we have $x^{1+\sigma} \in H^1_M(K,\mathbb{Z}(n))$,$x^{1+\sigma\tau} \in H^1_M(K^{'},\mathbb{Z}(n))$, $x^{1+\tau+\tau^2} \in H^1_M(k,\mathbb{Z}(n))$ and $x^{1+\sigma+\tau+\tau^2+\sigma\tau+\tau\sigma} \in H^1_M(\mathbb{Q},\mathbb{Z}(n))$ by Lemma 2. Since $\mathbb{Q}$ is a totally real number field, we know that $H^1_M(\mathbb{Q},\mathbb{Z}(n)) \subseteq u(F,n)$. It is easy to verify the following identities

$$1+\tau\sigma = (1+\sigma)(1+\tau\sigma) - \sigma - \tau^2,$$

$$(\sigma\tau)(\sigma+\tau^2) = \sigma + \tau^2.$$

So $x^{\sigma+\tau^2} \in H^1_M(K^{'},\mathbb{Z}(n))$.

Hence

$$\begin{aligned} x^3x^{1+\sigma+\tau+\tau^2+\sigma\tau+\tau\sigma} &= x^{1+\sigma}x^{1+\tau+\tau^2}x^{1+\sigma\tau}x^{1+\tau\sigma} \\ &= x^{(1+\sigma)(2+\tau\sigma)}x^{1+\tau+\tau^2}x^{1+\sigma\tau-\sigma-\tau^2}. \end{aligned}$$

we have $\overline{x^3} = \overline{x^{(1+\sigma)(2+\tau\sigma)}x^{1+\tau+\tau^2}x^{1+\sigma\tau-\sigma-\tau^2}}$. That is, for every $\overline{x} \in H^1_M(F,\mathbb{Z}(n))/u(F,n)$ we have $\overline{x}^3 \in \mathrm{im}(\overline{f})$. Since $\mathrm{rk}_{\mathbb{Z}}(H^1_M(F,\mathbb{Z}(n))) = 3$, we have $|\mathrm{coker}(\overline{f})|\,|27$. By Theorem 2,

$$\frac{h_n(F)h_n(\mathbb{Q})^2}{h_n(k)h_n(K)^2} = 1/3 \text{ or } 1 \text{ or } 3 \text{ or } 9.$$

Colloary 2 *Let $K = \mathbb{Q}(\sqrt[3]{m})$ and $F = K(\zeta_3)$, where m is a cubefree integers not equal* 1 *and* -1, *ζ_3 is a primitive cube root of unity. Then*

$$|K_2(\mathcal{O}_F)| = |K_2(\mathcal{O}_K)|^2/12 \text{ or } |K_2(\mathcal{O}_K)|^2/4 \text{ or } 3|K_2(\mathcal{O}_K)|^2/4 \text{ or } 9|K_2(\mathcal{O}_K)|^2/4.$$

Proof By Lemma 1 (i), we have $|K_2(\mathcal{O}_E)| = h_2(E)$ for every number field E. Since $k = \mathbb{Q}(\zeta_3)$, we know that $K_2(\mathcal{O}_k)$ is trivial by results of Browkin and Gangl in [4]. This result follows from $K_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and Proposition 4.

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