Asymptotic Behavior of the Time-Dependent Solution of the M/G/1 Queueing Model with Second Optional Service

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Abstract

By studying the spectral properties of the underlying operator corresponding to the $M/G/1$ queueing model with second optional service we obtain that the time-dependent solution of the model strongly converges to its steady-state solution. We also show that the time-dependent queueing size at the departure point converges to the corresponding steady-state queueing size at the departure point.
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1 Introduction

In real life, we may encounter numerous examples of the queuing situations where all arriving customers require the main service and only some may require the subsidiary service provided by the server. For example, all postgraduate students joining a particular department of a university want to complete their MS program, but some of them may join PhD program soon after completing the MS program. In 2000, Madan [11] considered such cases and described them by the M/G/1 queueing system with second optional service. First of all, he established the corresponding mathematical model by using the supplementary variable technique, then he discussed the transient (time-dependent) solution of the model and gave the expression of Laplace-Stieltjes transform of the corresponding probability generating function. Next, he obtained the expression of the steady-state queueing length under the following hypothesis:

$$\lim_{t\to\infty} Q(t) = Q, \quad \lim_{t\to\infty} P_n^{(1)}(x,t) = P_n^{(1)}(x), \quad \lim_{t\to\infty} P_n^{(2)}(t) = P_n^{(1)}, \quad n \geq 0. \tag{H}$$

By reading the paper we find that the above hypothesis (H), in fact, implies the following two hypotheses in view of partial differential equations:
Hypothesis 1. The model has a unique time-dependent solution.
Hypothesis 2. The time-dependent solution converges to its steady-state solution.

Zhao et al. [14] have proved that the model has a unique positive time-dependent solution which satisfies probability condition, that is to say, they proved the hypothesis 1. When the service rate is a constant, Xing [13] has obtained the resolvent set of the operator corresponding to the model and Fang et al [4] have proved that 0 is an eigenvalue of the operator with algebraic multiplicity one. In other words, they have proved the hypothesis 2 for the special case. So far, any other results about hypothesis 2 have not been found in the literature. This paper is just an effort on this subject. First of all, we convert the model into an abstract Cauchy problem by selecting a suitable Banach space as a state space and introducing an operator corresponding to the model and its domain. According to Theorem 14 in Gupur et al. [7] we know that the asymptotic behavior of the time-dependent solution of an abstract Cauchy problem is decided by the spectral properties of the underlying operator. Hence, we study the spectral properties of the operator corresponding to the model. By using probability generating function and the result in Zhao et al. [14] we prove that 0 is an eigenvalue of the operator with geometric multiplicity one. Next, we discuss the resolvent set of the underlying operator. Through investigating the model we find that the main difficult point is the boundary conditions. In 1987, Greiner [5] put forward an idea to perturb the boundary conditions when discussing the population equation which was described by one differential equation and one boundary condition. In 2007, Haji and Radl [8] successfully have used the idea of Greiner [5] to deduce the asymptotic behavior of the time-dependent solution of a reliability model which was described by finite number of partial differential equations with integral boundary conditions. And, Haji and Radl [9] also applied the Greiner's idea to research the asymptotic behavior of the time-dependent solution of the $M/M^B/1$ queueing model which was described by infinitely many partial differential equations with integral boundary conditions and in which the service rate is a constant. The model, which will be discussed in this paper, was described by infinitely many partial differential equations with integral boundary conditions and the service rate is a function. By combining a lemma in Haji and Radl [8, 9] with a result in Nagel [12] we obtain that all points on the imaginary axis except zero belong to the resolvent set of the operator. Lastly, we determine the expression of the adjoint operator and prove that 0 is an eigenvalue of the adjoint operator with geometric multiplicity one. Thus, from these results together with Theorem 14 in Gupur et al. [7] we obtain the desired result in this paper. According to Madan [11], the M/G/1 queueing system with second optional service can be described by the following partial differential equations with integral boundary conditions:

$$\frac{dQ(t)}{dt} = -\lambda Q(t) + \mu_2 P_0^{(2)}(t) + (1-r)\int_0^\infty \mu_1(x) P_0^{(1)}(x,t)dx, \tag{1}$$

$$\frac{dP_0^{(2)}(t)}{dt} = -(\lambda+\mu_2)P_0^{(2)}(t) + r\int_0^\infty \mu_1(x) P_0^{(1)}(x,t)dx, \tag{2}$$

$$\frac{dP_n^{(2)}(t)}{dt} = -(\lambda+\mu_2)P_n^{(2)}(t) + r\int_0^\infty \mu_1(x) P_n^{(1)}(x,t)dx + \lambda P_{n-1}^{(2)}(t), \quad n \geq 1, \tag{3}$$

$$\frac{\partial P_0^{(1)}(x,t)}{\partial t} + \frac{\partial P_0^{(1)}(x,t)}{\partial x} = -(\lambda+\mu_1(x))P_0^{(1)}(x,t), \tag{4}$$

$$\frac{\partial P_n^{(1)}(x,t)}{\partial t} + \frac{\partial P_n^{(1)}(x,t)}{\partial x} = -(\lambda+\mu_1(x))P_n^{(1)}(x,t) + \lambda P_{n-1}^{(1)}(x,t), \quad n \geq 1, \tag{5}$$

$$P_0^{(1)}(0,t) = \mu_2 P_1^{(2)}(t) + (1-r)\int_0^\infty \mu_1(x) P_1^{(1)}(x,t)dx + \lambda Q(t), \tag{6}$$

$$P_n^{(1)}(0,t) = \mu_2 P_{n+1}^{(2)}(t) + (1-r)\int_0^\infty \mu_1(x) P_{n+1}^{(1)}(x,t)dx, \quad n \geq 1, \tag{7}$$

$$Q(0) = 1, \qquad P_n^{(2)}(0) = 0, \quad P_n^{(1)}(x,0) = 0, \quad n \geq 0. \tag{8}$$

Here $(x,t) \in [0,\infty) \times [0,\infty)$, $Q(t)$ represents the probability that at time t, there are no customer in the system and the server is idle. $P_n^{(1)}(x,t)$ represents the probability that at time t, there are n customers in the queue [not system (queue + service)] excluding the one being provided the first essential service and elapsed service time of this customer is x. $P_n^{(2)}(t)$ represents the probability that at time t, there are n customers in the queue excluding one customer being provided the second optional service. λ is the arrival rate of customer. r is the probability that customer may opt the second service. $\mu_1(x)$ is the service completion rate of the first essential service and satisfies

$$\mu_1(x) \geq 0, \quad \int_0^\infty \mu_1(x)dx = \infty.$$

μ_2 is the service completion rate of the second optional service.

In this paper, we use the notations in Zhao et al. [14]. Take a state space

$$X = \left\{ \; g = (P^{(1)}, P^{(2)}) \;\middle|\; \|g\| = \|P^{(1)}\| + \|P^{(2)}\| < \infty \; \right\},$$

here

$$\begin{aligned}
&P^{(1)} = (Q, P_0^{(1)}(x), P_1^{(1)}(x), P_2^{(1)}(x), \cdots) \in \mathbb{R} \times L^1[0,\infty) \times L^1[0,\infty) \times \cdots,\\
&P^{(2)} = (P_0^{(2)}, P_1^{(2)}, P_2^{(2)}, P_3^{(2)}, \cdots\cdots) \in l^1,\\
&\|P^{(1)}\| = |Q| + \sum_{n=0}^{\infty} \|P_n^{(1)}\|_{L^1[0,\infty)}, \qquad \|P^{(2)}\| = \sum_{n=0}^{\infty} |P_n^{(2)}|.
\end{aligned}$$

It is obvious that X is a Banach space. We define an operator as follows.

$$A_m \left(\begin{pmatrix} Q \\ P_0^{(1)}(x) \\ P_1^{(1)}(x) \\ P_2^{(1)}(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} P_0^{(2)} \\ P_1^{(2)} \\ P_2^{(2)} \\ P_3^{(2)} \\ \vdots \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} -\lambda & (1-r)\psi & 0 & 0 & \cdots \\ 0 & \phi & 0 & 0 & \cdots \\ 0 & \lambda & \phi & 0 & \cdots \\ 0 & 0 & \lambda & \phi & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q \\ P_0^{(1)}(x) \\ P_1^{(1)}(x) \\ P_2^{(1)}(x) \\ \vdots \end{pmatrix} + \begin{pmatrix} \mu_2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} P_0^{(2)} \\ P_1^{(2)} \\ P_2^{(2)} \\ P_3^{(2)} \\ \vdots \end{pmatrix}, \right.$$

$$\begin{pmatrix} -(\lambda+\mu_2) & 0 & 0 & 0 & \cdots \\ \lambda & -(\lambda+\mu_2) & 0 & 0 & \cdots \\ 0 & \lambda & -(\lambda+\mu_2) & 0 & \cdots \\ 0 & 0 & \lambda & -(\lambda+\mu_2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} P_0^{(2)} \\ P_1^{(2)} \\ P_2^{(2)} \\ P_3^{(2)} \\ \vdots \end{pmatrix}$$

$$\left. + \begin{pmatrix} 0 & r\psi & 0 & 0 & \cdots \\ 0 & 0 & r\psi & 0 & \cdots \\ 0 & 0 & 0 & r\psi & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q \\ P_0^{(1)}(x) \\ P_1^{(1)}(x) \\ P_2^{(1)}(x) \\ \vdots \end{pmatrix} \right),$$

here

$$\begin{aligned}
&\psi f = \int_0^{\infty} \mu_1(x) f(x) dx, \quad \forall f \in L^1[0,\infty),\\
&\phi g = -\frac{dg(x)}{dx} - (\lambda + \mu_1(x)) g(x), \quad g \in W^{1,1}[0,\infty),
\end{aligned}$$

$$D(A_m) = \left\{ (P^{(1)}, P^{(2)}) \in X \;\middle|\; \begin{array}{l} \frac{dP_n^{(1)}}{dx} \in L^1[0,\infty),\ P_n^{(1)}(x)(n \geq 0) \text{ are} \\ \text{absolutely continuous functions} \\ \text{and } \sum_{n=0}^{\infty} \left\| \frac{dp_n}{dx} \right\|_{L^1[0,\infty)} < \infty \end{array} \right\}.$$

We choose the boundary space of X as

$$\partial X = l^1$$

and define two boundary operators as

$$L: D(A_m) \to \partial X, \quad \Phi: D(A_m) \to \partial X,$$

$$L\left(\begin{pmatrix} Q \\ P_0^{(1)}(x) \\ P_1^{(1)}(x) \\ P_2^{(1)}(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} P_0^{(2)} \\ P_1^{(2)} \\ P_2^{(2)} \\ P_3^{(2)} \\ \vdots \end{pmatrix}\right) = \begin{pmatrix} P_0^{(1)}(0) \\ P_1^{(1)}(0) \\ P_2^{(1)}(0) \\ P_3^{(1)}(0) \\ \vdots \end{pmatrix}$$

$$\Phi\left(\begin{pmatrix} Q \\ P_0^{(1)}(x) \\ P_1^{(1)}(x) \\ P_2^{(1)}(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} P_0^{(2)} \\ P_1^{(2)} \\ P_2^{(2)} \\ P_3^{(2)} \\ \vdots \end{pmatrix}\right)$$

$$= \left(\begin{pmatrix} \lambda & 0 & (1-r)\psi & 0 & 0 & \cdots \\ 0 & 0 & 0 & (1-r)\psi & 0 & \cdots \\ 0 & 0 & 0 & 0 & (1-r)\psi & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q \\ P_0^{(1)}(x) \\ P_1^{(1)}(x) \\ P_2^{(1)}(x) \\ \vdots \end{pmatrix}\right.$$

$$\left. + \begin{pmatrix} 0 & \mu_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \mu_2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \mu_2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \mu_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} P_0^{(2)} \\ P_1^{(2)} \\ P_2^{(2)} \\ P_3^{(2)} \\ \vdots \end{pmatrix}\right).$$

Now we define the underlying operator A and its domain as

$$A(P^{(1)}, P^{(2)}) = A_m(P^{(1)}, P^{(2)}),$$
$$D(A) = \left\{(P^{(1)}, P^{(2)}) \in D(A_m) \,\middle|\, L(P^{(1)}, P^{(2)}) = \Phi(P^{(1)}, P^{(2)})\right\}.$$

Then the above system of equations (1)–(8) can be expressed as an abstract Cauchy problem :

$$\begin{cases} \frac{d(P^{(1)},P^{(2)})(t)}{dt} = A(P^{(1)}, P^{(2)})(t), \qquad \forall t \in \ [0,\infty) \\ (P^{(1)}, P^{(2)})(0) = \left(\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}\right) \end{cases} \tag{9}$$

Zhao et al. [14] have obtained the following results.

Theorem 1. *If $\mu_1(x)$ satisfies $\overline{\mu_1} = \sup\limits_{x\in[0,\infty)} \mu_1(x) < \infty$, then A generates a positive contraction C_0- semigroup $T(t)$. The system (9) has a unique positive time-dependent solution $(P^{(1)}, P^{(2)})(x,t) = T(t)(P^{(1)}, P^{(2)})(0)$ satisfying*

$$Q(t) + \sum_{n=0}^{\infty} \int_0^{\infty} P_n^{(1)}(x,t)dx + \sum_{n=0}^{\infty} P_n^{(2)}(t) = 1, \quad \forall t \in [0,\infty).$$

2 Main Results

Lemma 1. *If* $\lambda\left(\int_0^\infty x\mu_1(x)e^{-\int_0^x \mu_1(\tau)d\tau}dx + r/\mu_2\right) < 1$, *then 0 is an eigenvalue of A with geometric multiplicity one.*

Proof. We consider the equation $A(P^{(1)}, P^{(2)}) = 0$, i.e.,

$$\lambda Q = \mu_2 P_0^{(2)} + (1-r)\int_0^\infty \mu_1(x)P_0^{(1)}(x)dx, \tag{10}$$

$$(\lambda+\mu_2)P_0^{(2)} = r\int_0^\infty \mu_1(x)P_0^{(1)}(x)dx, \tag{11}$$

$$(\lambda+\mu_2)P_n^{(2)} = r\int_0^\infty \mu_1(x)P_n^{(1)}(x)dx + \lambda P_{n-1}^{(2)}, \qquad n\geq 1, \tag{12}$$

$$\frac{dP_0^{(1)}(x)}{dx} = -(\lambda+\mu_1(x))P_0^{(1)}(x), \tag{13}$$

$$\frac{dP_n^{(1)}(x)}{dx} = -(\lambda+\mu_1(x))P_n^{(1)}(x) + \lambda P_{n-1}^{(1)}(x), \qquad n\geq 1, \tag{14}$$

$$P_0^{(1)}(0) = \mu_2 P_1^{(2)} + (1-r)\int_0^\infty \mu_1(x)P_1^{(1)}(x)dx + \lambda Q, \tag{15}$$

$$P_n^{(1)}(0) = \mu_2 P_{n+1}^{(2)} + (1-r)\int_0^\infty \mu_1(x)P_{n+1}^{(1)}(x)dx, \qquad n\geq 1. \tag{16}$$

It is hard to determine the expression of all $P_n^{(1)}(x)$ and $P_n^{(2)}$ for $n\geq 0$ and to verify $(P^{(1)}, P^{(2)}) \in D(A)$. Hence, we use an indirect method. We define the probability generating functions

$$P^{(1)}(x,z) = \sum_{n=0}^\infty P_n^{(1)}(x)z^n, \quad P^{(2)}(z) = \sum_{n=0}^\infty P_n^{(2)}z^n, \quad |z|<1,$$

and Theorem 1 ensures that $P^{(1)}(x,z)$ and $P^{(2)}(z)$ are well-defined. (11) and (12) give

$$(\lambda+\mu_2)\sum_{n=0}^\infty P_n^{(2)}z^n = \sum_{n=0}^\infty r\int_0^\infty \mu_1(x)P_n^{(1)}(x)z^n dx + \sum_{n=1}^\infty \lambda P_{n-1}^{(2)}z^n$$
$$\Longrightarrow$$
$$(\lambda+\mu_2)\sum_{n=0}^\infty P_n^{(2)}z^n = r\int_0^\infty \mu_1(x)\sum_{n=0}^\infty P_n^{(1)}(x)z^n dx + \lambda z\sum_{n=0}^\infty P_n^{(2)}z^n$$
$$\Longrightarrow$$
$$(\lambda+\mu_2)P^{(2)}(z) = r\int_0^\infty P^{(1)}(x,z)\mu_1(x)dx + \lambda z P^{(2)}(z)$$
$$\Longrightarrow$$
$$P^{(2)}(z) = \frac{r}{\lambda-\lambda z+\mu_2}\int_0^\infty P^{(1)}(x,z)\mu_1(x)dx. \tag{17}$$

From (13) and (14) we deduce

$$\frac{\partial\sum_{n=0}^\infty P_n^{(1)}(x)z^n}{\partial x} = -(\lambda+\mu_1(x))\sum_{n=0}^\infty P_n^{(1)}(x)z^n + \lambda\sum_{n=1}^\infty P_{n-1}^{(1)}(x)z^n$$
$$\Longrightarrow$$
$$\frac{\partial P^{(1)}(x,z)}{\partial x} = -(\lambda+\mu_1(x))P^{(1)}(x,z) + \lambda z P^{(1)}(x,z)$$

$$= (\lambda z - \lambda - \mu_1(x))P^{(1)}(x,z)$$

$$\Longrightarrow$$

$$P^{(1)}(x,z) = P^{(1)}(0,z)e^{\int_0^x (\lambda z - \lambda - \mu_1(\tau))d\tau}. \tag{18}$$

From (15) and (16) we have

$$\begin{aligned}
\sum_{n=0}^{\infty} P^{(1)}(0)z^n &= \lambda Q + \mu_2 P_1^{(2)} + (1-r)\int_0^\infty \mu_1(x)P_1^{(1)}(x)dx \\
&\quad + \sum_{n=1}^{\infty} \mu_2 P_{n+1}^{(2)} z^n + \sum_{n=1}^{\infty}(1-r)\int_0^\infty \mu_1(x)P_{n+1}^{(1)}(x)z^n dx \\
&\Longrightarrow \\
P^{(1)}(0,z) &= \lambda Q + \mu_2 \sum_{n=0}^{\infty} P_{n+1}^{(2)} z^n + (1-r)\int_0^\infty \mu_1(x)\sum_{n=0}^{\infty} P_{n+1}^{(1)}(x)z^n dx \\
&= \lambda Q + \frac{\mu_2}{z}\sum_{n=0}^{\infty} P_{n+1}^{(2)} z^{n+1} + \frac{1-r}{z}\int_0^\infty \mu_1(x)\sum_{n=0}^{\infty} P_{n+1}^{(1)}(x)z^{n+1}dx \\
&= \lambda Q + \frac{\mu_2}{z}\left[\sum_{n=0}^{\infty} P_n^{(2)} z^n - P_0^{(2)}\right] \\
&\quad + \frac{1-r}{z}\int_0^\infty \mu_1(x)\left[\sum_{n=0}^{\infty} P_n^{(1)}(x)z^n - P_0^{(1)}(x)\right]dx \\
&= \lambda Q + \frac{\mu_2}{z}P^{(2)}(z) - \frac{\mu_2}{z}P_0^{(2)} \\
&\quad + \frac{1-r}{z}\int_0^\infty \mu_1(x)P^{(1)}(x,z)dx - \frac{1-r}{z}\int_0^\infty \mu_1(x)P_0^{(1)}(x)dx.
\end{aligned} \tag{19}$$

By inserting (10), (17) and (18) into (19) we derive

$$\begin{aligned}
P^{(1)}(0,z) &= \lambda Q - \frac{1}{z}\left[\mu_2 P_0^{(2)} + (1-r)\int_0^\infty \mu_1(x)P_0^{(1)}(x)dx\right] \\
&\quad + \frac{\mu_2}{z}P^{(2)}(z) + \frac{1-r}{z}\int_0^\infty \mu_1(x)P^{(1)}(x,z)dx \\
&= \lambda Q - \frac{1}{z}\lambda Q + \frac{r\mu_2}{z(\lambda - \lambda z + \mu_2)}\int_0^\infty \mu_1(x)P^{(1)}(x,z)dx \\
&\quad + \frac{1-r}{z}\int_0^\infty \mu_1(x)P^{(1)}(x,z)dx \\
&= \frac{z-1}{z}\lambda Q + \frac{r\mu_2}{z(\lambda - \lambda z + \mu_2)}\int_0^\infty \mu_1(x)P^{(1)}(x,z)dx \\
&\quad + \frac{1-r}{z}\int_0^\infty \mu_1(x)P^{(1)}(x,z)dx \\
&= \frac{z-1}{z}\lambda Q + \frac{\mu_2 + (1-r)(\lambda - \lambda z)}{z(\lambda - \lambda z + \mu_2)}\int_0^\infty \mu_1(x)P^{(1)}(x,z)dx \\
&= \frac{z-1}{z}\lambda Q + \frac{\mu_2 + (1-r)(\lambda - \lambda z)}{z(\lambda - \lambda z + \mu_2)}\int_0^\infty P^{(1)}(0,z)\mu_1(x)e^{\int_0^x (\lambda z - \lambda - \mu_1(\tau))d\tau}dx \\
&\Longrightarrow
\end{aligned}$$

$$P^{(1)}(0,z) = (\lambda - \lambda z + \mu_2)(z-1)\lambda Q\left\{z(\lambda - \lambda z + \mu_2)\right.$$

$$- [\mu_2 + (1-r)(\lambda - \lambda z)] \int_0^\infty \mu_1(x) e^{\int_0^x (\lambda z - \lambda - \mu_1(\tau)) d\tau} dx \Big\}^{-1}. \quad (20)$$

By using the L'Hospital rule and $\int_0^\infty \mu_1(x) e^{-\int_0^x \mu_1(\tau) d\tau} dx = 1$, we calculate

$$\begin{aligned}
\lim_{z \to 1} P^{(1)}(0,z) &= \mu_2 \lim_{z \to 1} \lambda Q \Big[\lambda - 2\lambda z + \mu_2 \\
&\quad - [\mu_2 + (1-r)(\lambda - \lambda z)] \int_0^\infty \mu_1(x) \lambda x e^{\int_0^x (\lambda z - \lambda - \mu_1(\tau)) d\tau} dx \\
&\quad + \lambda(1-r) \int_0^\infty \mu_1(x) e^{\int_0^x (\lambda z - \lambda - \mu_1(\tau)) d\tau} dx \Big]^{-1} \\
&= \mu_2 \lambda Q \Big[\mu_2 - \lambda - \mu_2 \int_0^\infty \mu_1(x) \lambda x e^{-\int_0^x \mu_1(\tau) d\tau} dx \\
&\quad + \lambda(1-r) \int_0^\infty \mu_1(x) e^{-\int_0^x \mu_1(\tau) d\tau} dx \Big]^{-1} \\
&= \lambda Q \left[1 - \int_0^\infty \lambda x \mu_1(x) e^{-\int_0^x \mu_1(\tau) d\tau} dx - \frac{r\lambda}{\mu_2} \right]^{-1}. \quad (21)
\end{aligned}$$

By combining (21) with (18) and (17) and using $\int_0^\infty \mu_1(x) e^{-\int_0^x \mu_1(\tau) d\tau} dx = 1$ we estimate

$$\begin{aligned}
\sum_{n=0}^\infty P_n^{(1)}(x) &= \lim_{z \to 1} P^{(1)}(x,z) = \lim_{z \to 1} P^{(1)}(0,z) e^{\int_0^x (\lambda z - \lambda - \mu_1(\tau)) d\tau} \\
&= \lambda Q \left[1 - \int_0^\infty \lambda x \mu_1(x) e^{-\int_0^x \mu_1(\tau) d\tau} dx - \frac{r\lambda}{\mu_2} \right]^{-1} e^{-\int_0^x \mu_1(\tau) d\tau} \\
&\Longrightarrow \\
\sum_{n=0}^\infty \int_0^\infty P_n^{(1)}(x) dx &= \lambda Q \left[1 - \int_0^\infty \lambda x \mu_1(x) e^{-\int_0^x \mu_1(\tau) d\tau} dx - \frac{r\lambda}{\mu_2} \right]^{-1} \\
&\quad \times \int_0^\infty e^{-\int_0^x \mu_1(\tau) d\tau} < \infty. \quad (22) \\
\sum_{n=0}^\infty P_n^{(2)} &= \lim_{z \to 1} P^{(2)}(z) = \lim_{z \to 1} \frac{r}{\lambda - \lambda z + \mu_2} \int_0^\infty P^{(1)}(x,z) \mu_1(x) dx \\
&= \lim_{z \to 1} \frac{r}{\lambda - \lambda z + \mu_2} \int_0^\infty P^{(1)}(0,z) \mu_1(x) e^{\int_0^x (\lambda z - \lambda - \mu_1(\tau)) d\tau} dx \\
&= \frac{r}{\mu_2} \lim_{z \to 1} P^{(1)}(0,z) \int_0^\infty \mu_1(x) e^{-\int_0^x \mu_1(\tau) d\tau} dx \\
&= \frac{r}{\mu_2} \lambda Q \left[1 - \int_0^\infty \lambda x \mu_1(x) e^{-\int_0^x \mu_1(\tau) d\tau} dx - \frac{r\lambda}{\mu_2} \right]^{-1} < \infty. \quad (23)
\end{aligned}$$

(22) and (23) imply

$$\left\| (P^{(1)}, P^{(2)}) \right\| = \left\| P^{(1)} \right\| + \left\| P^{(2)} \right\| < \infty.$$

Which shows that 0 is an eigenvalue of A. Moreover, by solving (10)-(16)

$$\begin{aligned}
P_n^{(1)}(x) &= e^{-\int_0^x (\lambda + \mu_1(\tau)) d\tau} \sum_{k=0}^n \frac{(\lambda x)^k}{k!} a_{n-k}, \quad n \geq 0, \\
P_n^{(2)} &= \sum_{j=0}^n \frac{r\lambda^j}{(\gamma + \lambda + \mu_2)^{j+1}}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^\infty \mu_1(x) e^{-\int_0^x (\lambda+\mu_1(\tau))d\tau} \sum_{k=0}^{n-j} a_{n-k} \frac{(\lambda x)^k}{k!} dx, \quad n \geq 0,\\
\lambda Q =& \frac{\lambda(1-r)}{\lambda+\mu_2} a_0 \int_0^\infty \mu_1(x) e^{-\int_0^x (\lambda+\mu_1(\tau))d\tau} dx,\\
a_0 =& \mu_2 P_1^{(2)} + (1-r)\int_0^\infty \mu_1(x) P_1^{(1)}(x)dx + \lambda Q,\\
a_n =& \mu_2 P_{n+1}^{(2)} + (1-r)\int_0^\infty \mu_1(x) P_{n+1}^{(1)}(x)dx, \quad n \geq 1.
\end{aligned}
$$

From the above, we know that the eigenvectors corresponding to zero span one dimensional linear space, that is to say, geometric multiplicity of 0 is one. □

In the following, by applying a lemma in Haji and Radl [8, 9] we obtain the resolvent set of A on the imaginary axis. To do this, first of all, we need to introduce an operator $(A_0, D(A_0))$, then we determine the expression of $(\gamma I - A_0)^{-1}$, next we consider the explicit expression of $\ker(\gamma I - A_m)$. In addition, we argue the inverse of L in $\ker(\gamma I - A_m)$ and define an operator D_γ which is called Dirichlet operator. Moreover, we give the expressions of D_γ and ΦD_γ. Lastly, by applying the relation between the spectrum of A and the spectrum of ΦD_γ, we deduce the resolvent set of A on the imaginary axis.

We define A_0 and its domain as

$$
\begin{aligned}
A_0(P^{(1)}, P^{(2)}) &= A_m(P^{(1)}, P^{(2)}),\\
D(A_0) &= \left\{ (P^{(1)}, P^{(2)}) \in D(A_m) \,\middle|\, L(P^{(1)}, P^{(2)}) = 0 \right\},
\end{aligned}
$$

and discuss its inverse. For any given $(z^{(1)}, z^{(2)}) \in X$, we consider the equation $(\gamma I - A_0)(P^{(1)}, P^{(2)}) = (z^{(1)}, z^{(2)})$, that is,

$$
(\gamma+\lambda)Q = \mu_2 P_0^{(2)} + (1-r)\int_0^\infty \mu_1(x) P_0^{(1)}(x)dx + z_0, \tag{24}
$$

$$
(\gamma+\lambda+\mu_2)P_0^{(2)} = r\int_0^\infty \mu_1(x) P_0^{(1)}(x)dx + z_0^{(2)}, \tag{25}
$$

$$
(\gamma+\lambda+\mu_2)P_n^{(2)} = r\int_0^\infty \mu_1(x) P_n^{(1)}(x)dx + \lambda P_{n-1}^{(2)} + z_n^{(2)}, \quad n \geq 1, \tag{26}
$$

$$
\frac{dP_0^{(1)}(x)}{dx} = -(\gamma+\lambda+\mu_1(x))P_0^{(1)}(x) + z_0^{(1)}(x), \tag{27}
$$

$$
\frac{dP_n^{(1)}(x)}{dx} = -(\gamma+\lambda+\mu_1(x))P_n^{(1)}(x) + \lambda P_{n-1}^{(1)}(x) + z_n^{(1)}(x), \quad n \geq 1, \tag{28}
$$

$$
P_n^{(1)}(0) = 0, \quad n \geq 0. \tag{29}
$$

By solving (24)-(29), we have

$$
Q = \frac{1}{\gamma+\lambda} z_0 + \frac{\mu_2}{\gamma+\lambda} P_0^{(2)} + \frac{1-r}{\gamma+\lambda}\int_0^\infty \mu_1(x) P_0^{(1)}(x)dx, \tag{30}
$$

$$
P_0^{(2)} = \frac{1}{\gamma+\lambda+\mu_2} z_0^{(2)} + \frac{r}{\gamma+\lambda+\mu_2}\int_0^\infty \mu_1(x) P_0^{(1)}(x)dx, \tag{31}
$$

$$
\begin{aligned}
P_n^{(2)} =& \frac{1}{\gamma+\lambda+\mu_2} z_n^{(2)} + \frac{\lambda}{\gamma+\lambda+\mu_2} P_{n-1}^{(2)}\\
&+ \frac{r}{\gamma+\lambda+\mu_2}\int_0^\infty \mu_1(x) P_n^{(1)}(x)dx, \quad n \geq 1,
\end{aligned} \tag{32}
$$

$$
P_0^{(1)}(x) = e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} \int_0^x z_0^{(1)}(\tau) e^{\int_0^\tau (\gamma+\lambda+\mu_1(\xi))d\xi} d\tau, \tag{33}
$$

$$P_n^{(1)}(x) = e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi}\int_0^x z_n^{(1)}(\tau)e^{\int_0^\tau(\gamma+\lambda+\mu_1(\xi))d\xi}d\tau$$
$$+\lambda e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi}\int_0^x P_{n-1}^{(1)}(\tau)e^{\int_0^\tau(\gamma+\lambda+\mu_1(\xi))d\xi}d\tau, \quad n\geq 1. \tag{34}$$

If we introduce an operator as

$$Ef(x) = e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi}\int_0^x f(\tau)e^{\int_0^\tau(\gamma+\lambda+\mu_1(\xi))d\xi}d\tau, \quad \forall f\in L^1[0,\infty),$$

then (31), (32), (33) and (34) imply

$$P_0^{(1)}(x) = Ez_0^{(1)}(x), \tag{35}$$

$$\begin{aligned}
P_n^{(1)}(x) =& Ez_n^{(1)}(x) + \lambda EP_{n-1}^{(1)}(x)\\
=& Ez_n^{(1)}(x) + \lambda E(Ez_{n-1}^{(1)}(x) + \lambda EP_{n-2}^{(1)}(x))\\
=& Ez_n^{(1)}(x) + \lambda E^2 z_{n-1}^{(1)}(x) + \lambda^2E^2P_{n-2}^{(1)}(x)\\
=& Ez_n^{(1)}(x) + \lambda E^2 z_{n-1}^{(1)}(x) + \lambda^2E^2(Ez_{n-2}^{(1)}(x) + \lambda EP_{n-3}^{(1)}(x))\\
=& Ez_n^{(1)}(x) + \lambda E^2 z_{n-1}^{(1)}(x) + \lambda^2E^3z_{n-2}^{(1)}(x) + \lambda^3E^3P_{n-3}^{(1)}(x)\\
=& \cdots\cdots\\
=& Ez_n^{(1)}(x) + \lambda E^2 z_{n-1}^{(1)}(x) + \lambda^2E^3z_{n-2}^{(1)}(x) + \cdots + \lambda^{n-3}E^{n-2}z_3^{(1)}(x)\\
&+ \lambda^{n-2}E^{n-1}z_2^{(1)}(x) + \lambda^{n-1}E^n z_1^{(1)}(x) + \lambda^n E^n P_0^{(1)}(x)\\
=& Ez_n^{(1)}(x) + \lambda E^2 z_{n-1}^{(1)}(x) + \lambda^2E^3z_{n-2}^{(1)}(x) + \cdots + \lambda^{n-3}E^{n-2}z_3^{(1)}(x)\\
&+ \lambda^{n-2}E^{n-1}z_2^{(1)}(x) + \lambda^{n-1}E^n z_1^{(1)}(x) + \lambda^n E^{n+1} z_0^{(1)}(x)\\
=& \sum_{k=0}^n \lambda^k E^{k+1} z_{n-k}^{(1)}(x), \quad n\geq 1,
\end{aligned} \tag{36}$$

$$P_0^{(2)} = \frac{1}{\gamma+\lambda+\mu_2}z_0^{(2)} + \frac{r}{\gamma+\lambda+\mu_2}\psi Ez_0^{(1)}(x), \tag{37}$$

$$\begin{aligned}
P_n^{(2)} =& \frac{1}{\gamma+\lambda+\mu_2}z_n^{(2)} + \frac{\lambda}{\gamma+\lambda+\mu_2}P_{n-1}^{(2)} + \frac{r}{\gamma+\lambda+\mu_2}\psi P_n^{(1)}(x)\\
=& \frac{1}{\gamma+\lambda+\mu_2}z_n^{(2)} + \frac{\lambda}{\gamma+\lambda+\mu_2}\left(\frac{1}{\gamma+\lambda+\mu_2}z_{n-1}^{(2)} + \frac{\lambda}{\gamma+\lambda+\mu_2}P_{n-2}^{(2)}\right.\\
&\left.+ \frac{r}{\gamma+\lambda+\mu_2}\psi P_{n-1}^{(1)}(x)\right) + \frac{r}{\gamma+\lambda+\mu_2}\sum_{k=0}^n\lambda^k\psi E^{k+1}z_{n-k}^{(1)}(x)\\
=& \frac{1}{\gamma+\lambda+\mu_2}z_n^{(2)} + \frac{\lambda}{(\gamma+\lambda+\mu_2)^2}z_{n-1}^{(2)} + \frac{\lambda^2}{(\gamma+\lambda+\mu_2)^2}P_{n-2}^{(2)}\\
&+ \frac{r\lambda}{(\gamma+\lambda+\mu_2)^2}\sum_{k=0}^{n-1}\lambda^k\psi E^{k+1}z_{n-1-k}^{(1)}(x)\\
&+ \frac{r}{\gamma+\lambda+\mu_2}\sum_{k=0}^n\lambda^k\psi E^{k+1}z_{n-k}^{(1)}(x)\\
=& \frac{1}{\gamma+\lambda+\mu_2}z_n^{(2)} + \frac{\lambda}{(\gamma+\lambda+\mu_2)^2}z_{n-1}^{(2)} + \frac{\lambda^2}{(\gamma+\lambda+\mu_2)^2}\\
&\times\left(\frac{1}{\gamma+\lambda+\mu_2}z_{n-2}^{(2)} + \frac{\lambda}{\gamma+\lambda+\mu_2}P_{n-3}^{(2)} + \frac{r}{\gamma+\lambda+\mu_2}\psi P_{n-2}^{(1)}(x)\right)\\
&+ \frac{r\lambda}{(\gamma+\lambda+\mu_2)^2}\sum_{k=0}^{n-1}\lambda^k\psi E^{k+1}z_{n-1-k}^{(1)}(x)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{r}{\gamma+\lambda+\mu_2}\sum_{k=0}^{n}\lambda^k\psi E^{k+1}z_{n-k}^{(1)}(x)\\
=&\frac{1}{\gamma+\lambda+\mu_2}z_n^{(2)}+\frac{\lambda}{(\gamma+\lambda+\mu_2)^2}z_{n-1}^{(2)}+\frac{\lambda^2}{(\gamma+\lambda+\mu_2)^3}z_{n-2}^{(2)}\\
&+\frac{\lambda^3}{(\gamma+\lambda+\mu_2)^3}P_{n-3}^{(2)}+\frac{r\lambda^2}{(\gamma+\lambda+\mu_2)^3}\sum_{k=0}^{n-2}\lambda^k\psi E^{k+1}z_{n-2-k}^{(1)}(x)\\
&+\frac{r\lambda}{(\gamma+\lambda+\mu_2)^2}\sum_{k=0}^{n-1}\lambda^k\psi E^{k+1}z_{n-1-k}^{(1)}(x)\\
&+\frac{r}{\gamma+\lambda+\mu_2}\sum_{k=0}^{n}\lambda^k\psi E^{k+1}z_{n-k}^{(1)}(x)\\
=&\cdots\cdots\\
=&\frac{1}{\gamma+\lambda+\mu_2}z_n^{(2)}+\frac{\lambda}{(\gamma+\lambda+\mu_2)^2}z_{n-1}^{(2)}+\frac{\lambda^2}{(\gamma+\lambda+\mu_2)^3}z_{n-2}^{(2)}+\cdots\\
&+\frac{\lambda^{n-2}}{(\gamma+\lambda+\mu_2)^{n-1}}z_2^{(2)}+\frac{\lambda^{n-1}}{(\gamma+\lambda+\mu_2)^{n}}z_1^{(2)}+\frac{\lambda^{n}}{(\gamma+\lambda+\mu_2)^{n}}P_0^{(2)}\\
&+\frac{r\lambda^{n-1}}{(\gamma+\lambda+\mu_2)^{n}}\sum_{k=0}^{1}\lambda^k\psi E^{k+1}z_{1-k}^{(1)}(x)\\
&+\frac{r\lambda^{n-2}}{(\gamma+\lambda+\mu_2)^{n-1}}\sum_{k=0}^{2}\lambda^k\psi E^{k+1}z_{2-k}^{(1)}(x)\\
&+\frac{r\lambda^{n-2}}{(\gamma+\lambda+\mu_2)^{n-1}}\sum_{k=0}^{3}\lambda^k\psi E^{k+1}z_{3-k}^{(1)}(x)+\cdots\\
&+\frac{r\lambda^2}{(\gamma+\lambda+\mu_2)^3}\sum_{k=0}^{n-2}\lambda^k\psi E^{k+1}z_{n-2-k}^{(1)}(x)\\
&+\frac{r\lambda}{(\gamma+\lambda+\mu_2)^2}\sum_{k=0}^{n-1}\lambda^k\psi E^{k+1}z_{n-1-k}^{(1)}(x)\\
&+\frac{r}{\gamma+\lambda+\mu_2}\sum_{k=0}^{n}\lambda^k\psi E^{k+1}z_{n-k}^{(1)}(x)\\
=&\sum_{k=0}^{n}\frac{\lambda^k}{(\gamma+\lambda+\mu_2)^{k+1}}z_{n-k}^{(2)}\\
&+\sum_{j=0}^{n}\frac{r\lambda^{n-j}}{(\gamma+\lambda+\mu_2)^{n+1-j}}\sum_{k=0}^{j}\lambda^k\psi E^{k+1}z_{j-k}^{(1)}(x),\quad n\geq 1.
\end{aligned}
\tag{38}
$$

By inserting (35) and (37) into (30), we obtain

$$
\begin{aligned}
Q=&\frac{1}{\gamma+\lambda}z_0+\frac{1-r}{\gamma+\lambda}\psi Ez_0^{(1)}(x)\\
&+\frac{\mu_2}{\gamma+\lambda}\left(\frac{1}{\gamma+\lambda+\mu_2}z_0^{(2)}+\frac{r}{\gamma+\lambda+\mu_2}\psi Ez_0^{(1)}(x)\right)\\
=&\frac{1}{\gamma+\lambda}z_0+\frac{\mu_2}{(\gamma+\lambda)(\gamma+\lambda+\mu_2)}z_0^{(2)}\\
&+\left(\frac{1-r}{\gamma+\lambda}+\frac{r\mu_2}{(\gamma+\lambda)(\gamma+\lambda+\mu_2)}\right)\psi Ez_0^{(1)}(x).
\end{aligned}
\tag{39}
$$

(35)-(39) give the expression of $(\gamma I - A_0)^{-1}$ as follows if $(\gamma I - A_0)^{-1}$ exists.

$$
(\gamma I - A_0)^{-1}\left(\begin{pmatrix} z_0 \\ z_0^{(1)}(x) \\ z_1^{(1)}(x) \\ z_2^{(1)}(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} z_0^{(2)} \\ z_1^{(2)} \\ z_2^{(2)} \\ z_3^{(2)} \\ \vdots \end{pmatrix}\right)
$$

$$
= \left(\begin{pmatrix} \frac{1}{\gamma+\lambda} & \left(\frac{1-r}{\gamma+\lambda} + \frac{\mu_2 r}{(\gamma+\lambda)(\gamma+\lambda+\mu_2)}\right)\psi E & 0 & 0 & \cdots \\ 0 & E & 0 & 0 & \cdots \\ 0 & \lambda E^2 & E & 0 & \cdots \\ 0 & \lambda^2 E^3 & \lambda E^2 & E & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} z_0 \\ z_0^{(1)}(x) \\ z_1^{(1)}(x) \\ z_2^{(1)}(x) \\ \vdots \end{pmatrix}\right.
$$

$$
+ \begin{pmatrix} \frac{\mu_2}{(\gamma+\lambda)(\gamma+\lambda+\mu_2)} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} z_0^{(2)} \\ z_1^{(2)} \\ z_2^{(2)} \\ z_3^{(2)} \\ \vdots \end{pmatrix},
$$

$$
\begin{pmatrix} 0 & \frac{r}{\gamma+\lambda+\mu_2}\psi E & 0 & 0 & \cdots \\ 0 & \frac{\lambda r}{\gamma+\lambda+\mu_2}\psi E^2 + \frac{\lambda r}{(\gamma+\lambda+\mu_2)^2}\psi E & \frac{r}{\gamma+\lambda+\mu_2}\psi E & 0 & \cdots \\ 0 & \frac{\lambda^2 r}{\gamma+\lambda+\mu_2}\psi E^3 + \frac{\lambda^2 r}{(\gamma+\lambda+\mu_2)^2}\psi E^2 + \frac{\lambda^2 r}{(\gamma+\lambda+\mu_2)^3}\psi E & \frac{\lambda r}{\gamma+\lambda+\mu_2}\psi E^2 + \frac{\lambda r}{(\gamma+\lambda+\mu_2)^2}\psi E & \frac{r}{\gamma+\lambda+\mu_2}\psi E & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} z_0 \\ z_0^{(1)}(x) \\ z_1^{(1)}(x) \\ z_2^{(1)}(x) \\ \vdots \end{pmatrix}
$$

$$
\left. + \begin{pmatrix} \frac{1}{\gamma+\lambda+\mu_2} & 0 & 0 & 0 & \cdots \\ \frac{\lambda}{(\gamma+\lambda+\mu_2)^2} & \frac{1}{\gamma+\lambda+\mu_2} & 0 & 0 & \cdots \\ \frac{\lambda^2}{(\gamma+\lambda+\mu_2)^3} & \frac{\lambda}{(\gamma+\lambda+\mu_2)^2} & \frac{1}{\gamma+\lambda+\mu_2} & 0 & \cdots \\ \frac{\lambda^3}{(\gamma+\lambda+\mu_2)^4} & \frac{\lambda^2}{(\gamma+\lambda+\mu_2)^3} & \frac{\lambda}{(\gamma+\lambda+\mu_2)^2} & \frac{1}{\gamma+\lambda+\mu_2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} z_0^{(2)} \\ z_1^{(2)} \\ z_2^{(2)} \\ z_3^{(2)} \\ \vdots \end{pmatrix}\right).
$$

Lemma 2. *If* $0 < \underline{\mu_1} = \inf_{x\in[0,\infty)} \mu_1(x) \le \overline{\mu_1} = \sup_{x\in[0,\infty)} \mu_1(x) < \infty$, *then*

$$
\left\{\gamma \in \mathbb{C} \;\middle|\; \begin{array}{l} Re\gamma + \lambda > 0 \\ Re\gamma + \underline{\mu_1} > 0 \end{array}\right\} \subset \rho(A_0).
$$

Proof. By using integration by parts we have, for any $f \in L^1[0,\infty)$,

$$
\begin{aligned}
\|Ef\|_{L^1[0,\infty)} &= \int_0^\infty |Ef(x)|dx \\
&= \int_0^\infty \left| e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} \int_0^x f(\tau) e^{\int_0^\tau (\gamma+\lambda+\mu_1(\xi))d\xi} d\tau \right| dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^\infty e^{-\int_0^x(\mathrm{Re}\gamma+\lambda+\mu_1(\xi))d\xi}\int_0^x |f(\tau)|e^{\int_0^\tau(\mathrm{Re}\gamma+\lambda+\mu_1(\xi))d\xi}d\tau dx\\
&= \int_0^\infty \frac{-1}{\mathrm{Re}\gamma+\lambda+\mu_1(x)}\int_0^x |f(\tau)|e^{\int_0^\tau(\mathrm{Re}\gamma+\lambda+\mu_1(\xi))d\xi}de^{-\int_0^x(\mathrm{Re}\gamma+\lambda+\mu_1(\xi))d\xi}\\
&\leq \int_0^\infty \frac{-1}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}\int_0^x |f(\tau)|e^{\int_0^\tau(\mathrm{Re}\gamma+\lambda+\mu_1(\xi))d\xi}de^{-\int_0^x(\mathrm{Re}\gamma+\lambda+\mu_1(\xi))d\xi}\\
&= -\frac{1}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}\left\{e^{-\int_0^x(\mathrm{Re}\gamma+\lambda+\mu_1(\xi))d\xi}\int_0^x |f(\tau)|e^{\int_0^\tau(\mathrm{Re}\gamma+\lambda+\mu_1(\xi))d\xi}d\tau\Big|_{x=0}^{x=\infty}\right.\\
&\quad\left.-\int_0^\infty |f(x)|e^{-\int_0^x(\mathrm{Re}\gamma+\lambda+\mu_1(\xi))d\xi}e^{\int_0^x(\mathrm{Re}\gamma+\lambda+\mu_1(\xi))d\xi}dx\right.\\
&\leq \frac{1}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}\|f\|_{L^1[0,\infty)}\\
&\Longrightarrow
\end{aligned}
$$

$$
\|E\| \leq \frac{1}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}. \tag{40}
$$

By using $\|\psi\| \leq \overline{\mu_1}$, (40) and $|\gamma+\lambda+\mu_2| \geq \mu_2$, we estimate, for any $(z^{(1)}, z^{(2)}) \in X$

$$
\begin{aligned}
&\left\|(\gamma I - A_0)^{-1}(z^{(1)}, z^{(2)})\right\|\\
&= \left|\frac{1}{\gamma+\lambda}z_0 + \frac{1-r}{\gamma+\lambda}\psi E z_0^{(1)} + \frac{r\mu_2}{(\gamma+\lambda)(\gamma+\lambda+\mu_2)}\psi E z_0^{(1)}\right.\\
&\quad\left. + \frac{\mu_2}{(\gamma+\lambda)(\gamma+\lambda+\mu_2)}z_0^{(2)}\right| + \|E z_0^{(1)}\|_{L^1[0,\infty)} + \|\lambda E^2 z_0^{(1)} + E z_1^{(1)}\|_{L^1[0,\infty)}\\
&\quad + \|\lambda^2 E^3 z_0^{(1)} + \lambda E^2 z_1^{(1)} + E z_2^{(1)}\|_{L^1[0,\infty)} + \left\|\sum_{k=0}^{3}\lambda^k E^{k+1} z_{3-k}^{(1)}\right\|_{L^1[0,\infty)}\\
&\quad + \cdots\cdots + \left\|\sum_{k=0}^{n}\lambda^k E^{k+1} z_{n-k}^{(1)}\right\|_{L^1[0,\infty)} + \cdots\cdots\\
&\quad + \left|\frac{r}{\gamma+\lambda+\mu_2}\psi E z_0^{(1)}\right| + \left|\frac{\lambda r}{\gamma+\lambda+\mu_2}\psi E^2 z_0^{(1)} + \frac{\lambda r}{(\gamma+\lambda+\mu_2)^2}\psi E z_0^{(1)}\right.\\
&\quad\left. + \frac{r}{\gamma+\lambda+\mu_2}\psi E z_1^{(1)}\right| + \left|\sum_{j=0}^{2}\frac{r\lambda^{2-j}}{(\gamma+\lambda+\mu_2)^{3-j}}\sum_{k=0}^{j}\lambda^k\psi E^{k+1} z_{j-k}^{(1)}\right|\\
&\quad + \left|\sum_{j=0}^{3}\frac{r\lambda^{3-j}}{(\gamma+\lambda+\mu_2)^{4-j}}\sum_{k=0}^{j}\lambda^k\psi E^{k+1} z_{j-k}^{(1)}\right| + \cdots\cdots\\
&\quad + \left|\sum_{j=0}^{n}\frac{r\lambda^{n-j}}{(\gamma+\lambda+\mu_2)^{n+1-j}}\sum_{k=0}^{j}\lambda^k\psi E^{k+1} z_{j-k}^{(1)}\right| + \cdots\cdots\\
&\quad + \left|\frac{1}{\gamma+\lambda+\mu_2}z_0^{(2)}\right| + \left|\frac{\lambda}{(\gamma+\lambda+\mu_2)^2}z_0^{(2)} + \frac{1}{\gamma+\lambda+\mu_2}z_1^{(2)}\right|\\
&\quad + \left|\frac{\lambda^2}{(\gamma+\lambda+\mu_2)^3}z_0^{(2)} + \frac{\lambda}{(\gamma+\lambda+\mu_2)^2}z_1^{(2)} + \frac{1}{\gamma+\lambda+\mu_2}z_2^{(2)}\right|\\
&\quad + \left|\sum_{k=0}^{3}\frac{\lambda^k}{(\gamma+\lambda+\mu_2)^{k+1}}z_{3-k}^{(2)}\right| + \cdots\cdots + \left|\sum_{k=0}^{n}\frac{\lambda^k}{(\gamma+\lambda+\mu_2)^{k+1}}z_{n-k}^{(2)}\right|\\
&\quad + \cdots\cdots
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{|\gamma+\lambda|}|z_0| + \frac{1-r}{|\gamma+\lambda|}\|\psi\|\|E\|\|z_0^{(1)}\|_{L^1[0,\infty)} \\
&\quad + \frac{r\mu_2}{|\gamma+\lambda||\gamma+\lambda+\mu_2|}\|\psi\|\|E\|\|z_0^{(1)}\|_{L^1[0,\infty)} \\
&\quad + \frac{\mu_2}{|\gamma+\lambda||\gamma+\lambda+\mu_2|}|z_0^{(2)}| + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\lambda^k\|E\|^{k+1}\|z_{n-k}^{(1)}\|_{L^1[0,\infty)} \\
&\quad + \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{r\lambda^{n-j}}{|\gamma+\lambda+\mu_2|^{n+1-j}}\sum_{k=0}^{j}\lambda^k\|\psi\|\|E\|^{k+1}\|z_{j-k}^{(1)}\|_{L^1[0,\infty)} \\
&\quad + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\lambda^k}{|\gamma+\lambda+\mu_2|^{k+1}}|z_{n-k}^{(2)}| \\
&\le \frac{1}{|\gamma+\lambda|}|z_0| + \frac{1}{|\gamma+\lambda|}\|\psi\|\|E\|\|z_0^{(1)}\|_{L^1[0,\infty)} + \frac{1}{|\gamma+\lambda|}|z_0^{(2)}| \\
&\quad + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\lambda^k\left(\frac{1}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}\right)^{k+1}\|z_{n-k}^{(1)}\|_{L^1[0,\infty)} \\
&\quad + \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{r\overline{\mu_1}\lambda^{n-j}}{|\gamma+\lambda+\mu_2|^{n+1-j}}\sum_{k=0}^{j}\lambda^k\left(\frac{1}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}\right)^{k+1}\|z_{j-k}^{(1)}\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{|\gamma+\lambda+\mu_2|}\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\left(\frac{\lambda}{|\gamma+\lambda+\mu_2|}\right)^k|z_{n-k}^{(2)}| \\
&= \frac{1}{|\gamma+\lambda|}|z_0| + \frac{1}{|\gamma+\lambda|}\frac{\overline{\mu_1}}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}\|z_0^{(1)}\|_{L^1[0,\infty)} + \frac{1}{|\gamma+\lambda|}|z_0^{(2)}| \\
&\quad + \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{1}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}\left(\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}\right)^k\|z_{n-k}^{(1)}\|_{L^1[0,\infty)} \\
&\quad + \sum_{j=0}^{\infty}\sum_{n=j}^{\infty}\frac{r\overline{\mu_1}}{|\gamma+\lambda+\mu_2|}\left(\frac{\lambda}{|\gamma+\lambda+\mu_2|}\right)^{n-j} \\
&\quad \times \sum_{k=0}^{j}\lambda^k\left(\frac{1}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}\right)^{k+1}\|z_{j-k}^{(1)}\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{|\gamma+\lambda+\mu_2|}\sum_{k=0}^{\infty}\left(\frac{\lambda}{|\gamma+\lambda+\mu_2|}\right)^k\sum_{n=0}^{\infty}|z_n^{(2)}| \\
&\le \frac{1}{|\gamma+\lambda|}|z_0| + \frac{\overline{\mu_1}}{|\gamma+\lambda|(\mathrm{Re}\gamma+\lambda+\underline{\mu_1})}\|z_0^{(1)}\|_{L^1[0,\infty)} + \frac{1}{|\gamma+\lambda|}|z_0^{(2)}| \\
&\quad + \frac{1}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}\sum_{k=0}^{\infty}\left(\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}\right)^k\sum_{n=0}^{\infty}\|z_n^{(1)}\|_{L^1[0,\infty)} \\
&\quad + \sum_{n=0}^{\infty}\frac{r\overline{\mu_1}}{|\gamma+\lambda+\mu_2|}\left(\frac{\lambda}{|\gamma+\lambda+\mu_2|}\right)^n \\
&\quad \times \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{1}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}\left(\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}\right)^k\|z_{j-k}^{(1)}\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{|\gamma+\lambda+\mu_2|-\lambda}\sum_{n=0}^{\infty}|z_n^{(2)}| \\
&\le \frac{1}{|\gamma+\lambda|}|z_0| + \frac{\overline{\mu_1}}{|\gamma+\lambda|(\mathrm{Re}\gamma+\lambda+\underline{\mu_1})}\|z_0^{(1)}\|_{L^1[0,\infty)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{|\gamma+\lambda|}|z_0^{(2)}|+\frac{1}{\mathrm{Re}\gamma+\underline{\mu_1}}\sum_{n=0}^{\infty}\|z_n^{(1)}\|_{L^1[0,\infty)}\\
&+\frac{r\overline{\mu_1}}{|\gamma+\lambda+\mu_2|-\lambda}\frac{1}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}\\
&\times\sum_{k=0}^{\infty}\sum_{j=k}^{\infty}\left(\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}\right)^k\|z_{j-k}^{(1)}\|_{L^1[0,\infty)}\\
&+\frac{1}{|\gamma+\lambda+\mu_2|-\lambda}\sum_{n=0}^{\infty}|z_n^{(2)}|\\
\le&\frac{1}{|\gamma+\lambda|}|z_0|+\frac{\overline{\mu_1}}{|\gamma+\lambda|(\mathrm{Re}\gamma+\lambda+\underline{\mu_1})}\sum_{n=0}^{\infty}\|z_n^{(1)}\|_{L^1[0,\infty)}\\
&+\frac{1}{|\gamma+\lambda|}\sum_{n=0}^{\infty}|z_n^{(2)}|+\frac{1}{\mathrm{Re}\gamma+\underline{\mu_1}}\sum_{n=0}^{\infty}\|z_n^{(1)}\|_{L^1[0,\infty)}\\
&+\frac{r\overline{\mu_1}}{|\gamma+\lambda+\mu_2|-\lambda}\frac{1}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}\\
&\times\sum_{k=0}^{\infty}\left(\frac{\lambda}{\mathrm{Re}\gamma+\lambda+\underline{\mu_1}}\right)^k\sum_{j=0}^{\infty}\|z_j^{(1)}\|_{L^1[0,\infty)}\\
&+\frac{1}{|\gamma+\lambda+\mu_2|-\lambda}\sum_{n=0}^{\infty}|z_n^{(2)}|\\
\le&\frac{1}{|\gamma+\lambda|}|z_0|+\frac{\overline{\mu_1}}{|\gamma+\lambda|(\mathrm{Re}\gamma+\lambda+\underline{\mu_1})}\sum_{n=0}^{\infty}\|z_n^{(1)}\|_{L^1[0,\infty)}\\
&+\frac{1}{|\gamma+\lambda|}\sum_{n=0}^{\infty}|z_n^{(2)}|+\frac{1}{\mathrm{Re}\gamma+\underline{\mu_1}}\sum_{n=0}^{\infty}\|z_n^{(1)}\|_{L^1[0,\infty)}\\
&+\frac{r\overline{\mu_1}}{(|\gamma+\lambda+\mu_2|-\lambda)(\mathrm{Re}\gamma+\underline{\mu_1})}\sum_{n=0}^{\infty}\|z_n^{(1)}\|_{L^1[0,\infty)}\\
&+\frac{1}{|\gamma+\lambda+\mu_2|-\lambda}\sum_{n=0}^{\infty}|z_n^{(2)}|\\
\le&\sup\left\{\frac{1}{|\gamma+\lambda|},\frac{\overline{\mu_1}}{|\gamma+\lambda|(\mathrm{Re}\gamma+\lambda+\underline{\mu_1})}+\frac{1}{\mathrm{Re}\gamma+\underline{\mu_1}}\right.\\
&\left.+\frac{r\overline{\mu_1}}{(|\gamma+\lambda+\mu_2|-\lambda)(\mathrm{Re}\gamma+\underline{\mu_1})}\right\}\left(|z_0|+\sum_{n=0}^{\infty}\|z_n^{(1)}\|_{L^1[0,\infty)}\right)\\
&\left\{\frac{1}{|\gamma+\lambda|}+\frac{1}{|\gamma+\lambda+\mu_2|-\lambda}\right\}\sum_{n=0}^{\infty}|z_n^{(2)}|\\
\le&\sup\left\{\frac{\overline{\mu_1}}{|\gamma+\lambda|(\mathrm{Re}\gamma+\lambda+\underline{\mu_1})}+\frac{1}{\mathrm{Re}\gamma+\underline{\mu_1}}\right.\\
&\left.+\frac{r\overline{\mu_1}}{(|\gamma+\lambda+\mu_2|-\lambda)(\mathrm{Re}\gamma+\underline{\mu_1})},\frac{1}{|\gamma+\lambda|}+\frac{1}{|\gamma+\lambda+\mu_2|-\lambda}\right\}\left\|(z^{(1)},z^{(2)})\right\|\\
<&\infty.
\end{aligned}
$$

This shows that the result of this lemma is right. □

Lemma 3. *Let*

$$0 < \underline{\mu_1} = \inf_{x\in[0,\infty)} \mu_1(x) \le \mu_1 = \sup_{x\in[0,\infty)} \mu_1(x) < \infty.$$

If $\gamma \in \rho(A_0)$, *then*

$$\begin{aligned}
&(P^{(1)}, P^{(2)}) \in \ker(\gamma I - A_m) \Longleftrightarrow \\
&Q = \left(\frac{1-r}{\gamma+\lambda} + \frac{r\mu_2}{(\gamma+\lambda)(\gamma+\lambda+\mu_2)}\right) a_0 \int_0^\infty \mu_1(x) e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} dx \\
&P_0^{(2)} = \frac{r}{\gamma+\lambda+\mu_2} a_0 \int_0^\infty \mu_1(x) e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} dx, \\
&P_n^{(2)} = \sum_{j=0}^{n} \frac{r\lambda^j}{(\gamma+\lambda+\mu_2)^{j+1}} \\
&\qquad \times \int_0^\infty \mu_1(x) e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} \sum_{k=0}^{n-j} \frac{(\lambda x)^k}{k!} a_{n-k} dx, \quad n \ge 1, \\
&P_0^{(1)}(x) = a_0 e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi}, \\
&P_n^{(1)}(x) = e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} \sum_{k=0}^{n} \frac{(\lambda x)^k}{k!} a_{n-k}, \quad n \ge 1, \\
&(a_0, a_1, a_2, \cdots) \in l^1.
\end{aligned}$$

Proof. If $(P^{(1)}, P^{(2)}) \in \ker(\gamma I - A_m)$, then $(\gamma I - A_m)(P^{(1)}, P^{(2)}) = 0$, which is equivalent to

$$(\gamma+\lambda)Q = \mu_2 P_0^{(2)} + (1-r)\int_0^\infty \mu_1(x) P_0^{(1)}(x) dx, \tag{41}$$

$$(\gamma+\lambda+\mu_2) P_0^{(2)} = r\int_0^\infty \mu_1(x) P_0^{(1)}(x) dx, \tag{42}$$

$$(\gamma+\lambda+\mu_2) P_n^{(2)} = r\int_0^\infty \mu_1(x) P_n^{(1)}(x) dx + \lambda P_{n-1}^{(2)}, \quad n \ge 1, \tag{43}$$

$$\frac{dP_0^{(1)}(x)}{dx} = -(\gamma+\lambda+\mu_1(x)) P_0^{(1)}(x), \tag{44}$$

$$\frac{dP_n^{(1)}(x)}{dx} = -(\gamma+\lambda+\mu_1(x)) P_n^{(1)}(x) + \lambda P_{n-1}^{(1)}(x), \quad n \ge 1, \tag{45}$$

By solving (44) and (45) we have

$$P_0^{(1)}(x) = a_0 e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi}, \tag{46}$$

$$\begin{aligned}
P_n^{(1)}(x) =& a_n e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} \\
&+ \lambda e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} \int_0^x P_{n-1}^{(1)}(\tau) e^{\int_0^\tau (\gamma+\lambda+\mu_1(\xi))d\xi} d\tau, \quad n \ge 1.
\end{aligned} \tag{47}$$

By using (46)-(47) repeatedly, we obtain

$$\begin{aligned}
P_1^{(1)}(x) &= a_1 e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} + \lambda e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} \int_0^x a_0 d\tau \\
&= e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} [a_1 + \lambda x a_0],
\end{aligned} \tag{48}$$

$$\begin{aligned}
P_2^{(1)}(x) &= a_2 e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} \\
&\quad + \lambda e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} \int_0^x [a_1 + \lambda\tau a_0] d\tau
\end{aligned}$$

$$= e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} \left[a_2 + \lambda x a_1 + \frac{(\lambda x)^2}{2!} a_0 \right], \tag{49}$$

$$\cdots\cdots$$

$$P_n^{(1)}(x) = e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} \left[a_n + \lambda x a_{n-1} + \frac{(\lambda x)^2}{2!} a_{n-2} + \cdots + \frac{(\lambda x)^n}{n!} a_0 \right]$$

$$= e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} \sum_{k=0}^{n} \frac{(\lambda x)^k}{k!} a_{n-k}, \quad n \geq 1. \tag{50}$$

By inserting (46) and (50) into (42) and (43) respectively and using (42) and (43) repeatedly, we deduce

$$P_0^{(2)} = \frac{r}{\gamma+\lambda+\mu_2} a_0 \int_0^\infty \mu_1(x) e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} dx, \tag{51}$$

$$P_1^{(2)} = \frac{r}{\gamma+\lambda+\mu_2} \int_0^\infty \mu_1(x) e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} [a_1 + \lambda x a_0] dx$$
$$+ \frac{r\lambda}{(\gamma+\lambda+\mu_2)^2} a_0 \int_0^\infty \mu_1(x) e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} dx$$
$$= \sum_{j=0}^{1} \frac{r\lambda^j}{(\gamma+\lambda+\mu_2)^{j+1}}$$
$$\times \int_0^\infty \mu_1(x) e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} \sum_{k=0}^{1-j} \frac{(\lambda x)^k}{k!} a_{1-k} dx, \tag{52}$$

$$P_2^{(2)} = \frac{r}{\gamma+\lambda+\mu_2} \int_0^\infty \mu_1(x) e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} [a_2 + \lambda x a_1 + \frac{(\lambda x)^2}{2!} a_0] dx$$
$$+ \frac{\lambda}{\gamma+\lambda+\mu_2} \left\{ \frac{r}{\gamma+\lambda+\mu_2} \int_0^\infty \mu_1(x) e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} [a_1 + \lambda x a_0] dx \right.$$
$$\left. + \frac{r\lambda}{(\gamma+\lambda+\mu_2)^2} a_0 \int_0^\infty \mu_1(x) e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} dx \right\}$$
$$= \frac{r}{\gamma+\lambda+\mu_2} \int_0^\infty \mu_1(x) e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} \sum_{k=0}^{2} \frac{(\lambda x)^k}{k!} a_{2-k} dx$$
$$+ \frac{r\lambda}{(\gamma+\lambda+\mu_2)^2} \int_0^\infty \mu_1(x) e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} \sum_{k=0}^{1} \frac{(\lambda x)^k}{k!} a_{1-k} dx$$
$$+ \frac{r\lambda^2}{(\gamma+\lambda+\mu_2)^3} a_0 \int_0^\infty \mu_1(x) e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} dx$$
$$= \sum_{j=0}^{2} \frac{r\lambda^j}{(\gamma+\lambda+\mu_2)^{j+1}}$$
$$\times \int_0^\infty \mu_1(x) e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} \sum_{k=0}^{2-j} \frac{(\lambda x)^k}{k!} a_{2-k} dx, \tag{53}$$

$$\cdots\cdots$$

$$P_n^{(2)} = \sum_{j=0}^{n} \frac{r\lambda^j}{(\gamma+\lambda+\mu_2)^{j+1}}$$
$$\times \int_0^\infty \mu_1(x) e^{-\int_0^x (\gamma+\lambda+\mu_1(\xi))d\xi} \sum_{k=0}^{n-j} \frac{(\lambda x)^k}{k!} a_{n-k} dx, \quad n \geq 1. \tag{54}$$

By combining (46) and (51) with (41) we obtain

$$Q = \left(\frac{1-r}{\gamma+\lambda} + \frac{r\mu_2}{(\gamma+\lambda)(\gamma+\lambda+\mu_2)}\right) a_0 \int_0^\infty \mu_1(x) e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi} dx. \tag{55}$$

Since $(P^{(1)}, P^{(2)}) \in \ker(\gamma I - A_m), (P^{(1)}, P^{(2)}) \in D(A_m)$ implies by the imbedding theorem in Adams [1],

$$\begin{aligned}\sum_{n=0}^\infty |a_n| = \sum_{n=0}^\infty \left|P_n^{(1)}(0)\right| &\le \sum_{n=0}^\infty \left\|P_n^{(1)}\right\|_{L^\infty[0,\infty)} \\ &\le \sum_{n=0}^\infty \left\|P_n^{(1)}\right\|_{L^1[0,\infty)} + \sum_{n=0}^\infty \left\|\frac{dP_n^{(1)}}{dx}\right\|_{L^1[0,\infty)} < \infty. \end{aligned} \tag{56}$$

Hence, (56), (55), (54), (50) and (46) are necessary for this lemma.

Conversely, if

$$\begin{aligned}
Q &= \left(\frac{1-r}{\gamma+\lambda} + \frac{r\mu_2}{(\gamma+\lambda)(\gamma+\lambda+\mu_2)}\right) a_0 \int_0^\infty \mu_1(x) e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi} dx \\
P_0^{(2)} &= \frac{r}{\gamma+\lambda+\mu_2} a_0 \int_0^\infty \mu_1(x) e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi} dx, \\
P_n^{(2)} &= \sum_{j=0}^n \frac{r\lambda^j}{(\gamma+\lambda+\mu_2)^{j+1}} \\
&\quad \times \int_0^\infty \mu_1(x) e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi} \sum_{k=0}^{n-j} \frac{(\lambda x)^k}{k!} a_{n-k} dx, \quad n \ge 1, \\
P_0^{(1)}(x) &= a_0 e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi}, \\
P_n^{(1)}(x) &= e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi} \sum_{k=0}^n \frac{(\lambda x)^k}{k!} a_{n-k}, \quad n \ge 1, \\
&(a_0, a_1, a_2, \cdots) \in l^1,
\end{aligned}$$

then by direct calculating we have

$$\frac{dP_0^{(1)}(x)}{dx} = -(\gamma+\lambda+\mu_1(x))P_1^{(1)}(x), \tag{57}$$

$$\frac{dP_n^{(1)}(x)}{dx} = -(\gamma+\lambda+\mu_1(x))P_n^{(1)}(x) + \lambda P_{n-1}^{(1)}(x), \quad n \ge 1. \tag{58}$$

By using $\int_0^\infty x^k e^{-Mx} dx = \frac{k!}{M^{k+1}},\ k \ge 1,\ M > 0$ and the Fubini theorem we estimate

$$\begin{aligned}
\sum_{n=0}^\infty \|P_n^{(1)}\|_{L^1[0,\infty)} &= \sum_{n=0}^\infty \int_0^\infty |P_n^{(1)}(x)| dx \\
&= \sum_{n=0}^\infty \int_0^\infty \left| e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi} \sum_{k=0}^n \frac{(\lambda x)^k}{k!} a_{n-k} \right| dx \\
&\le \sum_{n=0}^\infty \sum_{k=0}^n |a_{n-k}| \frac{\lambda^k}{k!} \int_0^\infty x^k e^{-\int_0^x(\mathrm{Re}\gamma+\lambda+\mu_1(\xi))d\xi} dx \\
&\le \sum_{n=0}^\infty \sum_{k=0}^n |a_{n-k}| \frac{\lambda^k}{k!} \int_0^\infty x^k e^{-\int_0^x(\mathrm{Re}\gamma+\lambda+\underline{\mu_1})d\xi} dx
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}|a_{n-k}|\frac{\lambda^k}{k!}\frac{k!}{(\mathrm{Re}\gamma+\lambda+\underline{\mu_1})^{k+1}} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\lambda^k}{(\mathrm{Re}\gamma+\lambda+\underline{\mu_1})^{k+1}}|a_{n-k}| \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{\lambda^k}{(\mathrm{Re}\gamma+\lambda+\underline{\mu_1})^{k+1}}|a_{n-k}| \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{(\mathrm{Re}\gamma+\lambda+\underline{\mu_1})^{k+1}}\sum_{n=k}^{\infty}|a_{n-k}| \\
&\le \frac{1}{\mathrm{Re}\gamma+\underline{\mu_1}}\sum_{n=0}^{\infty}|a_n| < \infty. \qquad (59)
\end{aligned}
$$

(59) and (57), (58) give

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left\|\frac{dP_n^{(1)}}{dx}\right\|_{L^1[0,\infty)} &\le (|\gamma|+\lambda+\overline{\mu_1})\sum_{n=0}^{\infty}\|P_n^{(1)}\|_{L^1[0,\infty)} \\
&\quad + \lambda\sum_{n=1}^{\infty}\|P_{n-1}^{(1)}\|_{L^1[0,\infty)} \\
&< \infty. \qquad (60)
\end{aligned}
$$

Similarly, it is not difficult to verify $\sum_{n=0}^{\infty}|P_n^{(2)}| < \infty$. This together with (57)-(60) show that $(P^{(1)}, P^{(2)}) \in \ker(\gamma I - A_m)$. □

Observe that the operator L is surjective. So,

$$
L\Big|_{\ker(\gamma I - A_m)} : \ \ker(\gamma I - A_m) \longrightarrow \partial X
$$

is invertible if $\gamma \in \rho(A_0)$. Thus, we introduce an operator D_γ, which is called Dirichlet operator, as follows.

$$
D_\gamma := \left(L\Big|_{\ker(\gamma I - A_m)}\right)^{-1} : \ \partial X \longrightarrow \ker(\gamma I - A_m).
$$

Lemma 3 gives the explicit form of D_γ for $\gamma \in \rho(A_0)$

$$
D_\gamma\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix} = \left(\begin{pmatrix} \left(\frac{(1-r)}{\gamma+\lambda}+\frac{r\mu_2}{(\gamma+\lambda)(\gamma+\lambda+\mu_2)}\right)\psi h_{00} & 0 & 0 & 0 & \cdots \\ h_{11} & 0 & 0 & 0 & \cdots \\ h_{21} & h_{22} & 0 & 0 & \cdots \\ h_{31} & h_{32} & h_{33} & 0 & \cdots \\ h_{41} & h_{42} & h_{43} & h_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{pmatrix}, \right.
$$

$$
\left(\begin{matrix} \frac{r}{\gamma+\lambda+\mu_2}\psi k_{11} \\ \frac{r}{\gamma+\lambda+\mu_2}\psi k_{21} + \frac{r\lambda}{(\gamma+\lambda+\mu_2)^2}\psi k_{11} \\ \frac{r}{\gamma+\lambda+\mu_2}\psi k_{31} + \frac{r\lambda}{(\gamma+\lambda+\mu_2)^2}\psi k_{21} + \frac{r\lambda^2}{(\gamma+\lambda+\mu_2)^3}\psi k_{11} \\ \frac{r}{\gamma+\lambda+\mu_2}\psi k_{41} + \frac{r\lambda}{(\gamma+\lambda+\mu_2)^2}\psi k_{31} + \frac{r\lambda^2}{(\gamma+\lambda+\mu_2)^3}\psi k_{21} + \frac{r\lambda^3}{(\gamma+\lambda+\mu_2)^4}\psi k_{11} \\ \vdots \end{matrix}\right.
$$

$$
\begin{matrix}
0 \\
\frac{r}{\gamma+\lambda+\mu_2}\psi k_{22} \\
\frac{r}{\gamma+\lambda+\mu_2}\psi k_{32}+\frac{r\lambda}{(\gamma+\lambda+\mu_2)^2}\psi k_{22} \\
\frac{r}{\gamma+\lambda+\mu_2}\psi k_{42}+\frac{r\lambda}{(\gamma+\lambda+\mu_2)^2}\psi k_{32}+\frac{r\lambda^2}{(\gamma+\lambda+\mu_2)^3}\psi k_{22} \\
\vdots
\end{matrix}
$$

$$
\left.\left.\begin{matrix}
0 & 0 & \cdots \\
0 & 0 & \cdots \\
\frac{r}{\gamma+\lambda+\mu_2}\psi k_{33} & 0 & \cdots \\
\frac{r}{\gamma+\lambda+\mu_2}\psi k_{43}+\frac{r\lambda}{(\gamma+\lambda+\mu_2)^2}\psi k_{33} & \frac{r}{\gamma+\lambda+\mu_2}\psi k_{44} & \cdots \\
\vdots & \vdots & \ddots
\end{matrix}\right)\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{pmatrix}\right)\right).
$$

Where

$$
\begin{aligned}
& h_{00}=e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi}, \\
& h_{11}=e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi}, \quad h_{21}=\lambda x e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi}, \\
& h_{22}=e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi}, \quad h_{31}=\frac{(\lambda x)^2}{2!}\, e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi}, \\
& h_{32}=\lambda x e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi}, \quad h_{33}=e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi}, \\
& \ldots, h_{ij}=\frac{(\lambda x)^{i-j}}{(i-j)!}\, e^{-(\gamma+\lambda)x-\int_0^x\mu_1(\tau)d\tau},\ i=1,2,\cdots,\ j=1,2,\cdots,i. \\
& k_{11}=e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi}, \quad k_{21}=\lambda x e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi}, \\
& k_{22}=e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi}, \quad k_{31}=\frac{(\lambda x)^2}{2!}\, e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi}, \\
& k_{32}=\lambda x e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi}, \quad k_{33}=e^{-\int_0^x(\gamma+\lambda+\mu_1(\xi))d\xi}, \\
& \cdots, k_{ij}=\frac{(\lambda x)^{i-j}}{(i-j)!}\, e^{-(\gamma+\lambda)x-\int_0^x\mu_1(\tau)d\tau},\ i=1,2,\cdots,\ j=1,2,\cdots,i.
\end{aligned}
$$

From the expression of D_γ and the definition of Φ, it is not difficult to determine the explicit form of ΦD_γ as follows.

$$
\Phi D_\gamma\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix}=\left(\left(\left(\begin{matrix}
\left(\frac{\lambda(1-r)}{\gamma+\lambda}+\frac{r\mu_2\lambda}{(\gamma+\lambda)(\gamma+\lambda+\mu_2)}\right)\psi h_{00}+(1-r)\psi h_{21} \\
(1-r)\psi h_{31} \\
(1-r)\psi h_{41} \\
\vdots
\end{matrix}\right.\right.\right.
$$

$$
\left.\begin{matrix}
(1-r)\psi h_{22} & 0 & 0 & \cdots \\
(1-r)\psi h_{32} & (1-r)\psi h_{33} & 0 & \cdots \\
(1-r)\psi h_{42} & (1-r)\psi h_{43} & (1-r)\psi h_{44} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{matrix}\right)\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix},
$$

$$
\left(\begin{matrix}
\frac{r\mu_2}{\gamma+\lambda+\mu_2}\psi k_{21}+\frac{r\mu_2\lambda}{(\gamma+\lambda+\mu_2)^2}\psi k_{11} \\
\frac{r\mu_2}{\gamma+\lambda+\mu_2}\psi k_{31}+\frac{r\mu_2\lambda}{(\gamma+\lambda+\mu_2)^2}\psi k_{21}+\frac{r\mu_2\lambda^2}{(\gamma+\lambda+\mu_2)^3}\psi k_{11} \\
\frac{r\mu_2}{\gamma+\lambda+\mu_2}\psi k_{41}+\frac{r\mu_2\lambda}{(\gamma+\lambda+\mu_2)^2}\psi k_{31}+\frac{r\mu_2\lambda^2}{(\gamma+\lambda+\mu_2)^3}\psi k_{21}+\frac{r\mu_2\lambda^3}{(\gamma+\lambda+\mu_2)^4}\psi k_{11} \\
\vdots
\end{matrix}\right.
$$

$$
\begin{matrix}
\frac{r\mu_2}{\gamma+\lambda+\mu_2}\psi k_{22} \\
\frac{r\mu_2}{\gamma+\lambda+\mu_2}\psi k_{32}+\frac{r\mu_2\lambda}{(\gamma+\lambda+\mu_2)^2}\psi k_{22} \\
\frac{r\mu_2}{\gamma+\lambda+\mu_2}\psi k_{42}+\frac{r\mu_2\lambda}{(\gamma+\lambda+\mu_2)^2}\psi k_{32}+\frac{r\mu_2\lambda^2}{(\gamma+\lambda+\mu_2)^3}\psi k_{22} \\
\vdots
\end{matrix}
$$

$$
\left.\begin{matrix}
0 & 0 & \cdots \\
\frac{r\mu_2}{\gamma+\lambda+\mu_2}\psi k_{33} & 0 & \cdots \\
\frac{r\mu_2}{\gamma+\lambda+\mu_2}\psi k_{43}+\frac{r\mu_2\lambda}{(\gamma+\lambda+\mu_2)^2}\psi k_{33} & \frac{r\mu_2}{\gamma+\lambda+\mu_2}\psi k_{44} & \cdots \\
\vdots & \vdots & \ddots
\end{matrix}\right)\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix}\Bigg)\Bigg).
$$

Haji and Radl [8, 9] have given the following result.

Lemma 4. *If assume $\gamma \in \rho(A_0)$ and there exists $\gamma_0 \in \mathbb{C}$ such that $1 \notin \sigma(\Phi D_{\gamma_0})$, then*

$$\gamma \in \sigma(A) \Longleftrightarrow 1 \in \sigma(\Phi D_\gamma),$$

From Lemma 4 and Nagel [12], we obtain the resolvent set of A on the imaginary axis.

Lemma 5. *If*

$$0 < \underline{\mu_1} = \inf_{x\in[0,\infty)} \mu_1(x) \le \overline{\mu_1} = \sup_{x\in[0,\infty)} \mu_1(x) < \infty.$$

then all points on the imaginary axis except zero belong to the resolvent set of A.

Proof. Let $\gamma = ia,\ a \in \mathbb{R}\setminus\{0\}$. The Riemann-Lebesgue lemma

$$\lim_{a\to\infty}\int_0^\infty f(x)\cos ax dx = 0, \quad \lim_{a\to\infty}\int_0^\infty f(x)\sin ax dx = 0, \quad f \in L^1[0,\infty)$$

implies that there exists a positive constant $\mathcal{M} > 0$ such that for $\forall |a| > \mathcal{M}$,

$$
\begin{aligned}
\left|\int_0^\infty f(x)e^{-iax}dx\right|^2 &= \left|\int_0^\infty f(x)(\cos ax - i\sin ax)dx\right|^2 \\
&= \left(\int_0^\infty f(x)\cos ax dx\right)^2 + \left(\int_0^\infty f(x)\sin ax dx\right)^2 \\
&< \left(\int_0^\infty f(x)dx\right)^2, \quad 0 < f \in L^1[0,\infty).
\end{aligned}
$$

In this formula, by replacing $f(x)$ with $\mu_1(x)e^{-\int_0^x(\lambda+\mu_1(\tau))d\tau}$, and

$$
\begin{aligned}
&\sum_{j=l}^\infty \left|\int_0^\infty \mu_1(x)\frac{(\lambda x)^{j-l}}{(j-l)!}e^{-\int_0^x(ia+\lambda+\mu_1(\tau))d\tau}\right| dx \\
&\le \sum_{j=l}^\infty \int_0^\infty \mu_1(x)\frac{(\lambda x)^{j-l}}{(j-l)!}|e^{iax}|\left|e^{-\int_0^x(\lambda+\mu_1(\tau))d\tau}\right| dx \\
&= \sum_{j=l}^\infty \int_0^\infty \mu_1(x)\frac{(\lambda x)^{j-l}}{(j-l)!}e^{-\int_0^x(\lambda+\mu_1(\tau))d\tau}dx, \quad l \ge 1.
\end{aligned}
$$

we estimate, for $\vec{a} = (a_0, a_1, a_2, \cdots) \in l^1$,

$$\|\Phi D_\gamma(\vec{a})\| \le \Bigg\{ \left|\frac{\lambda(1-r)}{\gamma+\lambda}\psi h_{00} + \frac{r\mu_2\lambda}{(\gamma+\lambda)(\gamma+\lambda+\mu_2)}\psi h_{00} + (1-r)\psi h_{21}\right|$$

$$
\begin{aligned}
&+\sum_{j=3}^{\infty}|(1-r)\psi h_{j1}|\Bigg\}|a_0|+\sum_{j=2}^{\infty}|(1-r)\psi h_{j2}||a_1|\\
&+\sum_{j=3}^{\infty}|(1-r)\psi h_{j3}||a_2|+\sum_{j=4}^{\infty}|(1-r)\psi h_{j4}||a_3|\\
&+\cdots\cdots+\sum_{j=l}^{\infty}|(1-r)\psi h_{jl}||a_{l-1}|+\cdots\cdots\\
&+\Bigg|\frac{r\mu_2}{\gamma+\lambda+\mu_2}\sum_{j=2}^{\infty}\psi k_{j1}+\frac{r\mu_2\lambda}{(\gamma+\lambda+\mu_2)^2}\sum_{j=1}^{\infty}\psi k_{j1}+\frac{r\mu_2\lambda^2}{(\gamma+\lambda+\mu_2)^3}\sum_{j=1}^{\infty}\psi k_{j1}\\
&+\cdots\cdots+\frac{r\mu_2\lambda^i}{(\gamma+\lambda+\mu_2)^{i+1}}\sum_{j=1}^{\infty}\psi k_{j1}+\cdots\cdots\Bigg||a_0|\\
&+\Bigg|\frac{r\mu_2}{\gamma+\lambda+\mu_2}\sum_{j=2}^{\infty}\psi k_{j2}+\frac{r\mu_2\lambda}{(\gamma+\lambda+\mu_2)^2}\sum_{j=2}^{\infty}\psi k_{j2}+\frac{r\mu_2\lambda^2}{(\gamma+\lambda+\mu_2)^3}\sum_{j=2}^{\infty}\psi k_{j2}\\
&+\cdots\cdots+\frac{r\mu_2\lambda^i}{(\gamma+\lambda+\mu_2)^{i+1}}\sum_{j=2}^{\infty}\psi k_{j2}+\cdots\cdots\Bigg||a_1|\\
&+\Bigg|\frac{r\mu_2}{\gamma+\lambda+\mu_2}\sum_{j=3}^{\infty}\psi k_{j3}+\frac{r\mu_2\lambda}{(\gamma+\lambda+\mu_2)^2}\sum_{j=3}^{\infty}\psi k_{j3}+\frac{r\mu_2\lambda^2}{(\gamma+\lambda+\mu_2)^3}\sum_{j=3}^{\infty}\psi k_{j3}\\
&+\cdots\cdots+\frac{r\mu_2\lambda^i}{(\gamma+\lambda+\mu_2)^{i+1}}\sum_{j=3}^{\infty}\psi k_{j3}+\cdots\cdots\Bigg||a_2|+\cdots\cdots\\
&+\Bigg|\frac{r\mu_2}{\gamma+\lambda+\mu_2}\sum_{j=l}^{\infty}\psi k_{jl}+\frac{r\mu_2\lambda}{(\gamma+\lambda+\mu_2)^2}\sum_{j=l}^{\infty}\psi k_{jl}+\frac{r\mu_2\lambda^2}{(\gamma+\lambda+\mu_2)^3}\sum_{j=l}^{\infty}\psi k_{jl}\\
&+\cdots\cdots+\frac{r\mu_2\lambda^i}{(\gamma+\lambda+\mu_2)^{i+1}}\sum_{j=l}^{\infty}\psi k_{jl}+\cdots\cdots\Bigg||a_{l-1}|+\cdots\cdots\\
\leq&\Bigg\{\frac{\lambda(1-r)}{|\gamma+\lambda|}\psi h_{00}+\frac{\lambda r\mu_2}{|\gamma+\lambda||\gamma+\lambda+\mu_2|}\psi h_{00}\\
&+(1-r)\sum_{j=2}^{\infty}|\psi h_{j1}|\Bigg\}|a_0|+(1-r)\sum_{j=2}^{\infty}|\psi h_{j2}||a_1|\\
&+(1-r)\sum_{j=3}^{\infty}|\psi h_{j3}||a_2|+(1-r)\sum_{j=4}^{\infty}|\psi h_{j4}||a_3|\\
&+\cdots\cdots+\sum_{j=l}^{\infty}|(1-r)\psi h_{jl}||a_{l-1}|+\cdots\cdots\\
&+\frac{r\mu_2}{|\gamma+\lambda+\mu_2|}\sum_{j=2}^{\infty}|\psi k_{j1}||a_0|+\sum_{i=1}^{\infty}\frac{r\mu_2\lambda^i}{|\gamma+\lambda+\mu_2|^{i+1}}\sum_{j=1}^{\infty}|\psi k_{j1}||a_0|\\
&+\sum_{i=0}^{\infty}\frac{r\mu_2\lambda^i}{|\gamma+\lambda+\mu_2|^{i+1}}\sum_{j=2}^{\infty}|\psi k_{j1}||a_1|+\sum_{i=0}^{\infty}\frac{r\mu_2\lambda^i}{|\gamma+\lambda+\mu_2|^{i+1}}\sum_{j=3}^{\infty}|\psi k_{j1}||a_2|
\end{aligned}
$$

$$+\cdots\cdots+\sum_{i=0}^{\infty}\frac{r\mu_2\lambda^i}{|\gamma+\lambda+\mu_2|^{i+1}}\sum_{j=l}^{\infty}|\psi k_{jl}||a_{l-1}|+\cdots\cdots$$

$$=\Bigg\{\frac{\lambda(1-r)}{|\gamma+\lambda|}\int_0^{\infty}\mu_1(x)e^{-\int_0^x(ia+\lambda+\mu_1(\tau))d\tau}dx$$

$$+\frac{\lambda r\mu_2}{|\gamma+\lambda||\gamma+\lambda+\mu_2|}\left|\int_0^{\infty}\mu_1(x)e^{-\int_0^x(ia+\lambda+\mu_1(\tau))d\tau}dx\right|$$

$$+(1-r)\sum_{j=2}^{\infty}\left|\int_0^{\infty}\mu_1(x)\frac{(\lambda x)^{j-1}}{(j-1)!}e^{-\int_0^x(ia+\lambda+\mu_1(\tau))d\tau}dx\right|\Bigg\}|a_0|$$

$$+(1-r)\sum_{j=2}^{\infty}\left|\int_0^{\infty}\mu_1(x)\frac{(\lambda x)^{j-2}}{(j-2)!}e^{-\int_0^x(ia+\lambda+\mu_1(\tau))d\tau}dx\right||a_1|$$

$$+(1-r)\sum_{j=3}^{\infty}\left|\int_0^{\infty}\mu_1(x)\frac{(\lambda x)^{j-3}}{(j-3)!}e^{-\int_0^x(ia+\lambda+\mu_1(\tau))d\tau}dx\right||a_2|$$

$$+(1-r)\sum_{j=4}^{\infty}\left|\int_0^{\infty}\mu_1(x)\frac{(\lambda x)^{j-4}}{(j-4)!}e^{-\int_0^x(ia+\lambda+\mu_1(\tau))d\tau}dx\right||a_3|$$

$$+\cdots\cdots$$

$$+\frac{r\mu_2}{|\gamma+\lambda+\mu_2|}\sum_{j=2}^{\infty}\left|\int_0^{\infty}\mu_1(x)\frac{(\lambda x)^{j-1}}{(j-1)!}e^{-\int_0^x(ia+\lambda+\mu_1(\tau))d\tau}dx\right||a_0|$$

$$+\sum_{i=1}^{\infty}\frac{r\mu_2\lambda^i}{|\gamma+\lambda+\mu_2|^{i+1}}\sum_{j=1}^{\infty}\left|\int_0^{\infty}\mu_1(x)\frac{(\lambda x)^{j-1}}{(j-1)!}e^{-\int_0^x(ia+\lambda+\mu_1(\tau))d\tau}dx\right||a_0|$$

$$+\sum_{i=0}^{\infty}\frac{r\mu_2\lambda^i}{|\gamma+\lambda+\mu_2|^{i+1}}\sum_{j=2}^{\infty}\left|\int_0^{\infty}\mu_1(x)\frac{(\lambda x)^{j-2}}{(j-2)!}e^{-\int_0^x(ia+\lambda+\mu_1(\tau))d\tau}dx\right||a_1|$$

$$+\sum_{i=0}^{\infty}\frac{r\mu_2\lambda^i}{|\gamma+\lambda+\mu_2|^{i+1}}\sum_{j=3}^{\infty}\left|\int_0^{\infty}\mu_1(x)\frac{(\lambda x)^{j-3}}{(j-3)!}e^{-\int_0^x(ia+\lambda+\mu_1(\tau))d\tau}dx\right||a_2|$$

$$+\sum_{i=0}^{\infty}\frac{r\mu_2\lambda^i}{|\gamma+\lambda+\mu_2|^{i+1}}\sum_{j=4}^{\infty}\left|\int_0^{\infty}\mu_1(x)\frac{(\lambda x)^{j-4}}{(j-4)!}e^{-\int_0^x(ia+\lambda+\mu_1(\tau))d\tau}dx\right||a_3|$$

$$+\cdots\cdots$$

$$<(1-r)\sum_{j=1}^{\infty}\left|\int_0^{\infty}\mu_1(x)\frac{(\lambda x)^{j-1}}{(j-1)!}e^{-\int_0^x(ia+\lambda+\mu_1(\tau))d\tau}dx\right||a_0|$$

$$+(1-r)\int_0^{\infty}\mu_1(x)\sum_{j=2}^{\infty}\frac{(\lambda x)^{j-2}}{(j-2)!}e^{-\int_0^x(\lambda+\mu_1(\tau))d\tau}dx|a_1|$$

$$+(1-r)\int_0^{\infty}\mu_1(x)\sum_{j=3}^{\infty}\frac{(\lambda x)^{j-3}}{(j-3)!}e^{-\int_0^x(\lambda+\mu_1(\tau))d\tau}dx|a_2|$$

$$+(1-r)\int_0^{\infty}\mu_1(x)\sum_{j=4}^{\infty}\frac{(\lambda x)^{j-4}}{(j-4)!}e^{-\int_0^x(\lambda+\mu_1(\tau))d\tau}dx|a_3|$$

$$+\cdots\cdots$$

$$+\frac{r\mu_2}{|\gamma+\lambda+\mu_2|}\left|\int_0^{\infty}\mu_1(x)e^{-\int_0^x(ia+\lambda+\mu_1(\tau))d\tau}dx\right||a_0|$$

$$
\begin{aligned}
&+\frac{r\mu_2}{|\gamma+\lambda+\mu_2|}\sum_{j=2}^{\infty}\left|\int_0^{\infty}\mu_1(x)\frac{(\lambda x)^{j-1}}{(j-1)!}e^{-\int_0^x(ia+\lambda+\mu_1(\tau))d\tau}dx\right||a_0|\\
&+\sum_{i=1}^{\infty}\frac{r\mu_2\lambda^i}{|\gamma+\lambda+\mu_2|^{i+1}}\sum_{j=1}^{\infty}\left|\int_0^{\infty}\mu_1(x)\frac{(\lambda x)^{j-1}}{(j-1)!}e^{-\int_0^x(ia+\lambda+\mu_1(\tau))d\tau}dx\right||a_0|\\
&+\sum_{i=0}^{\infty}\frac{r\mu_2\lambda^i}{|\gamma+\lambda+\mu_2|^{i+1}}\int_0^{\infty}\mu_1(x)\sum_{j=2}^{\infty}\frac{(\lambda x)^{j-2}}{(j-2)!}e^{-\int_0^x(\lambda+\mu_1(\tau))d\tau}dx|a_1|\\
&+\sum_{i=0}^{\infty}\frac{r\mu_2\lambda^i}{|\gamma+\lambda+\mu_2|^{i+1}}\int_0^{\infty}\mu_1(x)\sum_{j=3}^{\infty}\frac{(\lambda x)^{j-3}}{(j-3)!}e^{-\int_0^x(\lambda+\mu_1(\tau))d\tau}dx|a_2|\\
&+\sum_{i=0}^{\infty}\frac{r\mu_2\lambda^i}{|\gamma+\lambda+\mu_2|^{i+1}}\int_0^{\infty}\mu_1(x)\sum_{j=4}^{\infty}\frac{(\lambda x)^{j-4}}{(j-4)!}e^{-\int_0^x(\lambda+\mu_1(\tau))d\tau}dx|a_3|\\
&+\cdots\cdots\\
<&(1-r)\int_0^{\infty}\mu_1(x)\sum_{j=1}^{\infty}\frac{(\lambda x)^{j-1}}{(j-1)!}e^{-\int_0^x(\lambda+\mu_1(\tau))d\tau}dx|a_0|\\
&+(1-r)\int_0^{\infty}\mu_1(x)e^{\lambda x}e^{-\int_0^x(\lambda+\mu_1(\tau))d\tau}dx|a_1|\\
&+(1-r)\int_0^{\infty}\mu_1(x)e^{\lambda x}e^{-\int_0^x(\lambda+\mu_1(\tau))d\tau}dx|a_2|\\
&+(1-r)\int_0^{\infty}\mu_1(x)e^{\lambda x}e^{-\int_0^x(\lambda+\mu_1(\tau))d\tau}dx|a_3|\\
&+\cdots\cdots\\
&+\sum_{i=0}^{\infty}\frac{r\mu_2\lambda^i}{|\gamma+\lambda+\mu_2|^{i+1}}\int_0^{\infty}\mu_1(x)\sum_{j=1}^{\infty}\frac{(\lambda x)^{j-1}}{(j-1)!}e^{-\int_0^x(\lambda+\mu_1(\tau))d\tau}dx|a_0|\\
&+\sum_{i=0}^{\infty}\frac{r\mu_2\lambda^i}{|\gamma+\lambda+\mu_2|^{i+1}}\int_0^{\infty}\mu_1(x)e^{\lambda x}e^{-\int_0^x(\lambda+\mu_1(\tau))d\tau}dx|a_1|\\
&+\sum_{i=0}^{\infty}\frac{r\mu_2\lambda^i}{|\gamma+\lambda+\mu_2|^{i+1}}\int_0^{\infty}\mu_1(x)e^{\lambda x}e^{-\int_0^x(\lambda+\mu_1(\tau))d\tau}dx|a_2|\\
&+\sum_{i=0}^{\infty}\frac{r\mu_2\lambda^i}{|\gamma+\lambda+\mu_2|^{i+1}}\int_0^{\infty}\mu_1(x)e^{\lambda x}e^{-\int_0^x(\lambda+\mu_1(\tau))d\tau}dx|a_3|\\
&+\cdots\cdots\\
\leq&(1-r)\int_0^{\infty}\mu_1(x)e^{-\int_0^x\mu_1(\tau)d\tau}dx|a_0|\\
&+(1-r)\int_0^{\infty}\mu_1(x)e^{-\int_0^x\mu_1(\tau)d\tau}dx|a_1|\\
&+(1-r)\int_0^{\infty}\mu_1(x)e^{-\int_0^x\mu_1(\tau)d\tau}dx|a_2|\\
&+(1-r)\int_0^{\infty}\mu_1(x)e^{-\int_0^x\mu_1(\tau)d\tau}dx|a_3|\\
&+\cdots\cdots\\
&+\frac{\mu_2}{|\gamma+\lambda+\mu_2|}\sum_{i=0}^{\infty}\left(\frac{\lambda}{|\gamma+\lambda+\mu_2|}\right)^i
\end{aligned}
$$

$$
\begin{aligned}
&\times r\Bigg\{ \int_0^\infty \mu_1(x) e^{\lambda x} e^{-\int_0^x (\lambda+\mu_1(\tau))d\tau} dx |a_0| \\
&+ \int_0^\infty \mu_1(x) e^{-\int_0^x \mu_1(\tau)d\tau} dx |a_1| \\
&+ \int_0^\infty \mu_1(x) e^{-\int_0^x \mu_1(\tau)d\tau} dx |a_2| \\
&+ \int_0^\infty \mu_1(x) e^{-\int_0^x \mu_1(\tau)d\tau} dx |a_3| + \cdots\cdots \Bigg\} \\
=&(1-r)\sum_{n=0}^\infty |a_n| + \frac{\mu_2}{|\gamma+\lambda+\mu_2|-\lambda} \\
&\times r\Bigg\{ \int_0^\infty \mu_1(x) e^{-\int_0^x \mu_1(\tau)d\tau} dx |a_0| \\
&+ \int_0^\infty \mu_1(x) e^{-\int_0^x \mu_1(\tau)d\tau} dx |a_1| \\
&+ \int_0^\infty \mu_1(x) e^{-\int_0^x \mu_1(\tau)d\tau} dx |a_2| \\
&+ \int_0^\infty \mu_1(x) e^{-\int_0^x \mu_1(\tau)d\tau} dx |a_3| + \cdots\cdots \Bigg\} \\
<&(1-r)\sum_{n=0}^\infty |a_n| + r\sum_{n=0}^\infty |a_n| = \|\vec{a}\|
\end{aligned}
$$

which imply

$$\|\Phi D_\gamma\| < 1. \tag{61}$$

(61) means that $1 \notin \sigma(\Phi D_\gamma)$ when $|a| > \mathcal{M}$. This together with Lemma 4 give

$$\{ia \mid |a| > \mathcal{M}\} \subset \rho(A), \quad \{ia \mid |a| \le \mathcal{M}\} \subset \sigma(A). \tag{62}$$

Since $T(t)$ is a positive contraction C_0- semigroup with spectral bound zero by Theorem 1 and Lemma 1, by Nagal [12] we know that $\sigma(A)$ is imaginary additively cyclic which states that

$$ia \in \sigma(A) \Rightarrow iah \in \sigma(A) \quad \text{for all integer } h.$$

From which together with (62) and Lemma 1 we conclude that $i\mathbb{R} \cap \sigma(A) = \{0\}$. □

It is not difficult to verify that X^*, the dual space of X, is as follows.

$$X^* = \left\{ (P^{(1)^*}, P^{(2)^*}) \;\middle|\; |||\, (P^{(1)^*}, P^{(2)^*}) ||| = \sup\{|||P^{(1)^*}|||, |||P^{(2)^*}|||\} \right\},$$

here

$$
\begin{aligned}
&P^{(1)*} = (Q^*, P_0^{(1)*}(x), P_1^{(1)*}(x), P_2^{(1)*}(x), \cdots) \in \mathbb{R} \times L^\infty[0,\infty) \times L^\infty[0,\infty) \times \cdots, \\
&P^{(2)*} = (P_0^{(2)*}, P_1^{(2)*}, P_2^{(2)*}, P_3^{(2)*}, \cdots\cdots) \in l^1, \\
&|||\, P^{(1)*}||| = \sup\left\{ |Q^*|, \sup_{n\ge 0} \|P_n^{(1)*}\|_{L^\infty[0,\infty)} \right\}, \\
&|||\, P^{(2)*}||| = \sup_{n\ge 0} |P_n^{(2)*}|.
\end{aligned}
$$

It is clear that X^* is a Banach space.

Lemma 6. *A^*, the adjoint operator of A, is as follows.*

$$
\begin{aligned}
&A^*(P^{(1)*}, P^{(2)*})\\
&= \left(\begin{pmatrix} -\lambda & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{d}{dx} - (\lambda + \mu_1(x)) & \lambda & 0 & 0 & \cdots \\ 0 & 0 & \frac{d}{dx} - (\lambda + \mu_1(x)) & \lambda & 0 & \cdots \\ 0 & 0 & 0 & \frac{d}{dx} - (\lambda + \mu_1(x)) & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q^* \\ P_0^{(1)*}(x) \\ P_1^{(1)*}(x) \\ P_2^{(1)*}(x) \\ \vdots \end{pmatrix} \right.\\
&\quad + \begin{pmatrix} 0 & \lambda & 0 & \cdots \\ (1-r)\mu_1(x) & 0 & 0 & \cdots \\ 0 & (1-r)\mu_1(x) & 0 & \cdots \\ 0 & 0 & (1-r)\mu_1(x) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q^* \\ P_0^{(1)*}(0) \\ P_1^{(1)*}(0) \\ P_2^{(1)*}(0) \\ \vdots \end{pmatrix}\\
&\quad + \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ r\mu_1(x) & 0 & 0 & 0 & \cdots \\ 0 & r\mu_1(x) & 0 & 0 & \cdots \\ 0 & 0 & r\mu_1(x) & 0 & \cdots \\ 0 & 0 & 0 & r\mu_1(x) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} P_0^{(2)*} \\ P_1^{(2)*} \\ P_2^{(2)*} \\ P_3^{(2)*} \\ \vdots \end{pmatrix},\\
&\quad \begin{pmatrix} -(\lambda + \mu_2(x)) & \lambda & 0 & 0 & \cdots \\ 0 & -(\lambda + \mu_2(x)) & \lambda & 0 & \cdots \\ 0 & 0 & -(\lambda + \mu_2(x)) & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} P_0^{(2)*} \\ P_1^{(2)*} \\ P_2^{(2)*} \\ \vdots \end{pmatrix}\\
&\quad \left. + \begin{pmatrix} \mu_2 & 0 & 0 & 0 & \cdots \\ 0 & \mu_2 & 0 & 0 & \cdots \\ 0 & 0 & \mu_2 & 0 & \cdots \\ 0 & 0 & 0 & \mu_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q^* \\ P_0^{(1)*}(0) \\ P_1^{(1)*}(0) \\ P_2^{(1)*}(0) \\ \vdots \end{pmatrix} \right)
\end{aligned}
$$

$$
D(A^*) = \left\{ (P^{(1)*}, P^{(2)*}) \in X^* \;\middle|\; \begin{array}{l} \frac{dP_n^{(1)*}(x)}{dx} \text{ are exist and} \\ P_n^{(1)*}(\infty) = \alpha, \quad n \geq 0, \end{array} \right\},
$$

here α in $D(A^*)$ is a constant which is irrelevant to n.

Proof. By using integration by parts and the boundary conditions on $(P^{(1)}, P^{(2)}) \in D(A)$,we have, for $(P^{(1)^*}.P^{(2)^*})$

$$
\begin{aligned}
&\langle A(P^{(1)}, P^{(2)}), (P^{(1)*}, P^{(2)*}) \rangle \\
&= \left\{ -\lambda Q + \mu_2 P_0^{(2)} + (1-r)\int_0^\infty \mu_1(x) P_0^{(1)}(x)dx \right\} Q^* \\
&\quad + \int_0^\infty \left\{ -\frac{dP_0^{(1)}(x)}{dx} - (\lambda + \mu_1(x))P_0^{(1)}(x) \right\} P_0^{(1)*}(x)dx \\
&\quad + \sum_{n=1}^\infty \int_0^\infty \left\{ -\frac{dP_n^{(1)}(x)}{dx} - (\lambda + \mu_1(x))P_n^{(1)}(x) + \lambda P_{n-1}^{(1)}(x) \right\} P_n^{(1)*}(x)dx \\
&\quad + \left\{ -(\lambda + \mu_2)P_0^{(2)} + r\int_0^\infty \mu_1(x)P_0^{(1)}(x)dx \right\} P_0^{(2)*}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}\left\{-(\lambda+\mu_2)P_n^{(2)}+\lambda P_{n-1}^{(2)}+r\int_0^{\infty}\mu_1(x)P_n^{(1)}(x)dx\right\}P_n^{(2)*}\\
&=(-\lambda Q)Q^*+\mu_2P_0^{(2)}Q^*+(1-r)\int_0^{\infty}P_0^{(1)}(x)\mu_1(x)Q^*dx\\
&+\sum_{n=0}^{\infty}\int_0^{\infty}-\frac{dP_n^{(1)}(x)}{dx}P_n^{(1)*}(x)dx-\sum_{n=0}^{\infty}\int_0^{\infty}(\lambda+\mu_1(x))P_n^{(1)}(x)P_n^{(1)*}(x)dx\\
&+\sum_{n=1}^{\infty}\int_0^{\infty}P_{n-1}^{(1)}(x)\lambda P_n^{(1)*}(x)dx-\sum_{n=0}^{\infty}(\lambda+\mu_2)P_n^{(2)}P_n^{(2)*}\\
&+\sum_{n=0}^{\infty}r\int_0^{\infty}P_n^{(1)}(x)\mu_1(x)P_n^{(2)*}dx+\sum_{n=1}^{\infty}\lambda P_{n-1}^{(2)}P_n^{(2)*}\\
&=Q(-\lambda Q^*)+P_0^{(2)}(\mu_2Q^*)+(1-r)\int_0^{\infty}P_0^{(1)}(x)\mu_1(x)Q^*dx\\
&+\sum_{n=0}^{\infty}\left[-P_n^{(1)}(x)P_n^{(1)*}(x)\Big|_{x=0}^{x=\infty}+\int_0^{\infty}P_n^{(1)}(x)\frac{dP_n^{(1)*}(x)}{dx}dx\right]\\
&+\sum_{n=0}^{\infty}\int_0^{\infty}P_n^{(1)}(x)[-(\lambda+\mu_1(x))P_n^{(1)*}(x)]dx\\
&+\sum_{n=1}^{\infty}\int_0^{\infty}P_{n-1}^{(1)}(x)[\lambda P_n^{(1)*}(x)]dx+\sum_{n=0}^{\infty}P_n^{(2)}[-(\lambda+\mu_2)P_n^{(2)*}]\\
&+r\sum_{n=0}^{\infty}\int_0^{\infty}P_n^{(1)}(x)[\mu_1(x)P_n^{(2)*}]dx+\sum_{n=1}^{\infty}P_{n-1}^{(2)}[\lambda P_n^{(2)*}]\\
&=Q(-\lambda Q^*)+P_0^{(2)}(\mu_2Q^*)+(1-r)\int_0^{\infty}P_0^{(1)}(x)\mu_1(x)Q^*dx\\
&+\sum_{n=0}^{\infty}P_n^{(1)}(0)P_n^{(1)*}(0)+\sum_{n=0}^{\infty}\int_0^{\infty}P_n^{(1)}(x)\frac{dP_n^{(1)*}(x)}{dx}dx\\
&+\sum_{n=0}^{\infty}\int_0^{\infty}P_n^{(1)}(x)[-(\lambda+\mu_1(x))P_n^{(1)*}(x)]dx\\
&+\sum_{n=0}^{\infty}\int_0^{\infty}P_n^{(1)}(x)[\lambda P_{n+1}^{(1)*}(x)]dx+\sum_{n=0}^{\infty}P_n^{(2)}[-(\lambda+\mu_2)P_n^{(2)*}]\\
&+\sum_{n=0}^{\infty}\int_0^{\infty}P_n^{(1)}(x)[r\mu_1(x)P_n^{(2)*}]dx+\sum_{n=0}^{\infty}P_n^{(2)}[\lambda P_{n+1}^{(2)*}]\\
&=Q(-\lambda Q^*)+P_0^{(2)}(\mu_2Q^*)+(1-r)\int_0^{\infty}P_0^{(1)}(x)\mu_1(x)Q^*dx\\
&+\left[(1-r)\int_0^{\infty}\mu_1(x)P_1^{(1)}(x)dx+\mu_2P_1^{(2)}+\lambda Q\right]P_0^{(1)*}(0)\\
&+\sum_{n=1}^{\infty}\left[(1-r)\int_0^{\infty}\mu_1(x)P_{n+1}^{(1)}(x)dx+\mu_2P_{n+1}^{(2)}\right]P_n^{(1)*}(0)\\
&+\sum_{n=0}^{\infty}\int_0^{\infty}P_n^{(1)}(x)\left[\frac{dP_n^{(1)*}(x)}{dx}-(\lambda+\mu_1(x))P_n^{(1)*}(x)+\lambda P_{n+1}^{(1)*}(x)+r\mu_1(x)P_n^{(2)*}\right]dx\\
&+\sum_{n=0}^{\infty}P_n^{(2)}\left[-(\lambda+\mu_2)P_n^{(2)*}+\lambda P_{n+1}^{(2)*}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= Q[-\lambda Q^* + \lambda P_0^{(1)*}(0)] + P_0^{(2)}(\mu_2 Q^*) + (1-r)\int_0^\infty P_0^{(1)}(x)\mu_1(x) Q^* dx \\
&\quad + \sum_{n=0}^\infty (1-r)\int_0^\infty P_{n+1}^{(1)}(x)\mu_1(x)P_n^{(1)*}(0)dx + \sum_{n=0}^\infty \mu_2 P_{n+1}^{(2)} P_n^{(1)*}(0) \\
&\quad + \sum_{n=0}^\infty \int_0^\infty P_n^{(1)}(x)\left[\frac{dP_n^{(1)*}(x)}{dx} - (\lambda+\mu_1(x))P_n^{(1)*}(x) + \lambda P_{n+1}^{(1)*}(x) + r\mu_1(x)P_n^{(2)*}\right]dx \\
&\quad + \sum_{n=0}^\infty P_n^{(2)}\left[-(\lambda+\mu_2)P_n^{(2)*} + \lambda P_{n+1}^{(2)*}\right] \\
&= Q[-\lambda Q^* + \lambda P_0^{(1)*}(0)] \\
&\quad + \int_0^\infty P_0^{(1)}(x)\left[\frac{dP_0^{(1)*}(x)}{dx} - (\lambda+\mu_1(x))P_0^{(1)*}(x) + \lambda P_1^{(1)*}(x)\right. \\
&\quad \left. + (1-r)\mu_1(x)Q^* + r\mu_1(x)P_0^{(2)*}\right]dx \\
&\quad + \sum_{n=1}^\infty \int_0^\infty P_n^{(1)}(x)\left[\frac{dP_n^{(1)*}(x)}{dx} - (\lambda+\mu_1(x))P_n^{(1)*}(x) + \lambda P_{n+1}^{(1)*}(x)\right. \\
&\quad \left. + (1-r)\mu_1(x)P_{n-1}^{(1)*}(0) + r\mu_1(x)P_n^{(2)*}\right]dx \\
&\quad + P_0^{(2)}\left[-(\lambda+\mu_2)P_0^{(2)*} + \lambda P_1^{(2)*} + \mu_2 Q^*\right] \\
&\quad + \sum_{n=1}^\infty P_n^{(2)}\left[-(\lambda+\mu_2)P_n^{(2)*} + \lambda P_{n+1}^{(2)*} + \mu_2 P_{n-1}^{(1)*}(0)\right] \\
&= \langle (P^{(1)}, P^{(2)}), A^*(P^{(1)*}, P^{(2)*})\rangle.
\end{aligned}
$$

From the definition of adjoint operator we know that the assertion of this lemma is right. □

From Theorem 1, Lemma 1 and Arendt et al. [2], we know that 0 is an eigenvalue of A^*. Furthermore, we deduce the following result.

Lemma 7. *If* $\lambda\left(\int_0^\infty x\mu_1(x)e^{-\int_0^x \mu_1(\tau)d\tau}dx + r/\mu_2\right) < 1$, *then* 0 *is an eigenvalue of* A^* *with geometric multiplicity one.*

Proof. We consider the equation $A^*(P^{(1)*}, P^{(2)*}) = 0$, which is equivalent to

$$
\begin{aligned}
& -\lambda Q^* + \lambda P_0^{(1)*}(0) = 0, && (63)\\
& \frac{dP_0^{(1)*}(x)}{dx} - (\lambda+\mu_1(x))P_0^{(1)*}(x) + \lambda P_1^{(1)*}(x) + (1-r)\mu_1(x)Q^* \\
& + r\mu_1(x)P_0^{(2)*} = 0, && (64)\\
& \frac{dP_n^{(1)*}(x)}{dx} - (\lambda+\mu_1(x))P_n^{(1)*}(x) + \lambda P_{n+1}^{(1)*}(x) + (1-r)\mu_1(x)P_{n-1}^{(1)*}(0) \\
& + r\mu_1(x)P_n^{(2)*} = 0, \quad n \geq 1, && (65)\\
& -(\lambda+\mu_2)P_0^{(2)*} + \lambda P_1^{(2)*} + \mu_2 Q^* = 0, && (66)\\
& -(\lambda+\mu_2)P_n^{(2)*} + \lambda P_{n+1}^{(2)*} + \mu_2 P_{n-1}^{(1)*}(0) = 0, \quad n \geq 1, && (67)\\
& P_n^{(1)*}(\infty) = \alpha, \quad n \geq 0. && (68)
\end{aligned}
$$

It is easy to see that

$$(P^{(1)*}, P^{(2)*}) = \left(\begin{pmatrix} \alpha \\ \alpha \\ \vdots \end{pmatrix}, \begin{pmatrix} \alpha \\ \alpha \\ \vdots \end{pmatrix} \right) \in D(A^*)$$

is a solution of (63)-(68). In addition, (63)-(68) are equivalent to

$$Q^* = P_0^{(1)*}(0), \tag{69}$$

$$P_1^{(1)*}(x) = \frac{1}{\lambda}\Big\{ -\frac{dP_0^{(1)*}(x)}{dx} + (\lambda + \mu_1(x))P_0^{(1)*}(x) - (1-r)\mu_1(x)Q^* - r\mu_1(x)P_0^{(2)*} \Big\}, \tag{70}$$

$$P_{n+1}^{(1)*}(x) = \frac{1}{\lambda}\Big\{ -\frac{dP_n^{(1)*}(x)}{dx} + (\lambda + \mu_1(x))P_n^{(1)*}(x) - (1-r)\mu_1(x)P_{n-1}^{(1)*}(0) - r\mu_1(x)P_n^{(2)*} = 0 \Big\}, \quad n \geq 1, \tag{71}$$

$$P_1^{(2)*} = \frac{1}{\lambda}\Big\{ (\lambda + \mu_2)P_0^{(2)*} - \mu_2 Q^* \Big\}, \tag{72}$$

$$P_{n+1}^{(2)*} = \frac{1}{\lambda}\Big\{ (\lambda + \mu_2)P_n^{(2)*} - \mu_2 P_{n-1}^{(1)*}(0) \Big\}, \quad n \geq 1, \tag{73}$$

(69)-(73) shows that we can determine each $P_n^{(1)*}(x)$ and $P_n^{(2)*}$ for all $n \geq 0$ if $P_0^{(1)*}(x)$ and $P_0^{(2)*}$ are given . That is to say, geometric multiplicity of zero is one. □

Since Theorem 1, Lemma 1, Lemma 5, Lemma 7 are just conditions of Theorem 14 in Gupur et al [7], we conclude the desired result.

Theorem 2. *If* $0 < \underline{\mu_1} = \inf_{x\in[0,\infty)} \mu_1(x) \leq \mu_1 = \sup_{x\in[0,\infty)} \mu_1(x) < \infty$*, then the time-depended solution of the system* (9) *strongly converges its steady-state solution, that is,*

$$\lim_{t\to\infty} \|(P^{(1)}, P^{(2)})(\cdot,t) - \beta(P^{(1)}, P^{(2)})(\cdot)\| = 0.$$

here $(P^{(1)}, P^{(2)})(x)$ *is eigenvector in Lemma 1 and* β *is decided by the eigenvector in Lemma 7 and the initial value* $(P^{(1)}, P^{(2)})(0)$*.*

Theorem 2 implies that the time-dependent queueing size at the departure point converges a positive number, i.e.,

$$\begin{aligned}\lim_{t\to\infty} \pi_j(t) &= \lim_{t\to\infty} \left\{ K(1-r)\int_0^\infty \mu_1(x)P_j^{(1)}(x,t)dx + K\mu_2 P_j^{(2)}(t) \right\} \\ &= K(1-r)\int_0^\infty \mu_1(x)P_j^{(1)}(x)dx + K\mu_2 P_j^{(2)} = \pi_j, \quad j \geq 0.\end{aligned}$$

According to our experience [3, 10], the result of Theorem 2 is best, that is to say, it is impossible that the time-dependent solution of the system (9) exponentially converges to its steady-state solution. Of course, it needs to verify. That is our next research work.

Remark 1. *By using the idea in Gupur et al. [6, 7], we can prove that the time-dependent queueing length* $L(t)$ *converges to the steady-state queueing length* L*, that is,*

$$\lim_{t\to\infty} L(t) = \lim_{t\to\infty} \left\{ \sum_{n=0}^{\infty} n \int_0^\infty P_n^{(1)}(x,t)dx + \sum_{n=0}^{\infty} nP_n^{(2)}(t) \right\}$$

$$= \sum_{n=0}^{\infty} n \int_0^{\infty} P_n^{(1)}(x)dx + \sum_{n=0}^{\infty} nP_n^{(2)} \overset{set}{=\!=} L.$$

Similarly, we can obtain that the time-dependent waiting time $W_q(t)$ also converges to the corresponding steady-state waiting time W_q.

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