# UNCERTAINTY PRINCIPLE IN TERMS OF ENTROPY FOR THE RIEMANN-LIOUVILLE OPERATOR

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Abstract. We prove Hausdorff-Young inequality for the Fourier transform connected with Riemann-Liouville operator. We use this inequality to establish the uncertainty principle in terms of entropy. Next, we show that we can derive the Heisenberg-Pauli-Weyl inequality for the precedent Fourier transform.

## 1. Introduction

Uncertainty principles play an important role in harmonic analysis, they state that a function $f$ and its Fourier transform $\widehat{f}$ can not be simultaneously sharply localized in the sense that it is impossible for a nonzero function and its Fourier transform to be simultaneously small.
Many mathematical formulations of this fact can be found in [6, 9, 10, 16, 17, 25]. For a probability density function $f$ on $\mathbb{R}^n$, the entropy of $f$ is defined according to [29] by

$$E(f) = -\int_{\mathbb{R}^n} f(x) \ \ln\big(f(x)\big)dx.$$

The entropy $E(f)$ is closely related to quantum mechanics [7] and constitutes one of the important way to measure the concentration of $f$.
The uncertainty principle in terms of entropy consists to compare the entropy of $|f|^2$ with that of $|\ \widehat{f}\ |^2$. A first result has been given in [21], where the author has established a weak version of this uncertainty principle by showing that for every square integrable function $f$ on $\mathbb{R}^n$ with respect to the Lebesgue measure, such that $||f||_2 = 1$, we have

$$E\big(|f|^2\big) + E\big(|\ \widehat{f}\ |^2\big) \ \geqslant\ 0. \tag{1.1}$$

Later in [5], the author has proved the following stronger inequality, that is for every square integrable function $f$ on $\mathbb{R}^n$; $||f||_2 = 1$,

$$E\big(|f|^2\big) + E\big(|\ \widehat{f}\ |^2\big) \ \geqslant\ n\big(1 - \ln(2)\big). \tag{1.2}$$

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The last inequality constitutes a very powerful result which implies in particular the well known Heisenberg-Pauli-Weyl uncertainty principle.
In [1], the authors have defined the Riemann-Liouville operator $\mathscr{R}_\alpha;\ \alpha \geqslant 0$, by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-1/2} \\ \times(1-s^2)^{\alpha-1}\,dt\,ds, \text{ if } \alpha > 0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f(r\sqrt{1-t^2}, x+rt)\dfrac{dt}{\sqrt{1-t^2}}, \text{ if } \alpha = 0; \end{cases} \tag{1.3}$$

where $f$ is any continuous function on $\mathbb{R}^2$, even with respect to the first variable.
The dual ${}^t\mathscr{R}_\alpha$ is defined by

$$ {}^t\mathscr{R}_\alpha(g)(r,x) = \begin{cases} \sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{2^\alpha\,\Gamma(\alpha+1)}\displaystyle\int_r^{+\infty}\int_{-\sqrt{u^2-r^2}}^{\sqrt{u^2-r^2}} g(u,x+v) \\ \times(u^2-v^2-r^2)^{\alpha-1}u\,du\,dv, \text{ if } \alpha > 0, \\ \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{\mathbb{R}} g\big(\sqrt{r^2+(x-y)^2}, y\big)dy, \text{ if } \alpha = 0; \end{cases} \tag{1.4}$$

where $g$ is any continuous function on $\mathbb{R}^2$, even with respect to the first variable and with compact support.
In particular, for $\alpha = 0$ and by a change of variables, we get

$$\mathscr{R}_0(f)(r,x) = \frac{1}{2\pi}\int_0^{2\pi} f(r\cos\theta, x + r\sin\theta)d\theta.$$

This means that $\mathscr{R}_0(f)(r,x)$ is the mean value of $f$ on the circle centered at $(0,x)$ and radius $r$.
The mean operator $\mathscr{R}_0$ and its dual ${}^t\mathscr{R}_0$ play an important role and have many applications, for example, in image processing of the so-called synthetic aperture radar (SAR) data [19, 20] or in the linearized inverse scattering problem in acoustics [13].
The operators $\mathscr{R}_\alpha$ and its dual ${}^t\mathscr{R}_\alpha$ have the same properties as the Radon transform [18], for this reason, $\mathscr{R}_\alpha$ is called sometimes the generalized Radon transform.
The Fourier transform $\mathscr{F}_\alpha$ associated with the operator $\mathscr{R}_\alpha$ is defined by

$$\begin{aligned} \forall(\mu,\lambda) \in \Upsilon,\ \mathscr{F}_\alpha(f) &= \int_0^\infty\int_{\mathbb{R}} f(r,x)\mathscr{R}_\alpha\big(\cos(\mu\cdot)e^{-i\lambda\cdot}\big)(r,x)d\nu_\alpha(r,x) \\ &= \int_0^\infty\int_{\mathbb{R}} f(r,x)\ j_\alpha(r\sqrt{\mu^2+\lambda^2})e^{-i\lambda x}\big)(r,x)d\nu_\alpha(r,x), \end{aligned} \tag{1.5}$$

where

- $\Upsilon$ is the set given by

$$\Upsilon = \mathbb{R}^2 \cup \big\{(i\mu,\lambda);\ (\mu,\lambda) \in \mathbb{R}^2;\ |\mu| \leqslant |\lambda|\big\}. \tag{1.6}$$

- $d\nu_\alpha(r,x)$ is the measure defined on $[0,+\infty[\times\mathbb{R}$ by

$$d\nu_\alpha(r,x)=\frac{r^{2\alpha+1}dr}{2^\alpha\Gamma(\alpha+1)}\otimes\frac{dx}{\sqrt{2\pi}}.$$

- $j_\alpha$ is the modified Bessel function that will be defined in the second section.

Many harmonic analysis results have been established for the Fourier transform $\mathscr{F}_\alpha$ [1, 2, 3, 28]. Also, many uncertainty principles related to the Fourier transform $\mathscr{F}_\alpha$ have been proved [23, 26, 27].
Our purpose in this work is to establish the uncertainty principle in terms of entropy for the Fourier transform $\mathscr{F}_\alpha$, from which we derive the Heisenberg-Pauli-Weyl uncertainty principle.
More precisely, we prove first the following Hausdorff-Young inequality

**Theorem 1.1.** *(Hausdorff-Young) The Fourier transform $\mathscr{F}_\alpha$ can be extended to a continuous operator from $L^p(d\nu_\alpha)$; $p\in\ [1,2]$, into $L^{p'}(d\gamma_\alpha)$; $p'=p/(p-1)$, and for every $f\in\ L^p(d\nu_\alpha)$,*

$$\left|\left|\mathscr{F}_\alpha(f)\right|\right|_{p',\gamma_\alpha}\ \leqslant\ A_p^{\alpha+3/2}\ ||f||_{p,\nu_\alpha}.$$

Where

- $A_p=\dfrac{p^{1/p}}{p'^{1/p'}}=\dfrac{p^{1/p}}{\left(\frac{p}{p-1}\right)^{(p-1)/p}}.$
- $L^p(d\nu_\alpha)$; $p\in\ [1,+\infty]$, is the Lebesgue space formed by the measurable functions $f$ on $[0,+\infty[\times\mathbb{R}$ such that $||f||_{p,\nu_\alpha}<+\infty$, with

$$||f||_{p,\nu_\alpha}=\begin{cases}\left(\displaystyle\int_0^{+\infty}\int_{\mathbb{R}}\left|f(r,x)\right|^p d\nu_\alpha(r,x)\right)^{1/p}, & \text{if } p\in\ [1,+\infty[,\\ \operatorname*{ess\ sup}_{(r,x)\in\ [0,+\infty[\times\mathbb{R}}\left|f(r,x)\right|, & \text{if } p=+\infty.\end{cases}$$

- $L^p(d\gamma_\alpha)$; $p\in\ [1,+\infty]$, is the Lebesgue space of measurable functions $g$ on the set

$$\Upsilon_+=\mathbb{R}_+\times\mathbb{R}\cup\left\{(it,x);\ (t,x)\in\ \mathbb{R}^2;\ 0\leqslant t\leqslant|x|\right\},$$

such that

$$||g||_{p,\gamma_\alpha}=\begin{cases}\left(\displaystyle\iint_{\Upsilon_+}\left|g(\mu,\lambda)\right|^p d\gamma_\alpha(\mu,\lambda)\right)^{1/p}<+\infty, & \text{if } p\in\ [1,+\infty[;\\ \operatorname*{ess\ sup}_{(\mu,\lambda)\in\ \Upsilon_+}\left|g(\mu,\lambda)\right|<+\infty, & \text{if } p=+\infty,\end{cases}$$

where $d\gamma_\alpha$ is the Plancherel measure on $\Upsilon_+$ that will be defined in the second section.
Using Theorem 1.1, we will demonstrate the main result of this paper, that is

**Theorem 1.2.** *(Entropy) Let* $f \in L^2(d\nu_\alpha)$ *such that* $\|f\|_{2,\nu_\alpha} = 1$. *We assume that*

$$\int_0^{\infty}\int_{\mathbb{R}} \big|f(r,x)\big|^2 \Big|\ln\big(\big|f(r,x)\big|\big)\Big| d\nu_\alpha(r,x) < +\infty,$$

*and*

$$\iint_{\Upsilon_+} \big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^2 \Big|\ln\big(\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|\big)\Big| d\gamma_\alpha(\mu,\lambda) < +\infty.$$

*Then, we have*

$$E_{\nu_\alpha}\big(|f|^2\big) + E_{\gamma_\alpha}\big(|\mathscr{F}_\alpha(f)|^2\big) \geqslant (2\alpha+3)(1-\ln 2),$$

*where* $E_{\nu_\alpha}\big(|f|^2\big)$ *(respectively* $E_{\gamma_\alpha}\big(|\mathscr{F}_\alpha(f)|^2\big)$*) is the entropy of* $|f|^2$ *(respectively* $|\mathscr{F}_\alpha(f)|^2$*).*

Theorem 1.2 allows us to prove the Heisenberg-Pauli-Weyl inequality.

**Theorem 1.3.** *(Heisenberg-Pauli-Weyl) For every function* $f \in L^2(d\nu_\alpha)$, *we have*

$$\left(\int_0^{\infty}\int_{\mathbb{R}} (r^2+x^2)\big|f(r,x)\big|^2 d\nu_\alpha(r,x)\right)^{1/2}\left(\iint_{\Upsilon_+} (\mu^2+2\lambda^2)\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^2 d\gamma_\alpha(\mu,\lambda)\right)^{1/2}$$
$$\geqslant \big(\alpha+3/2\big)\|f\|^2_{2,\nu_\alpha}.$$

## 2. The Riemann-Liouville transform

In this section, we recall some harmonic analysis results related to the convolution product and the Fourier transform associated with Riemann-Liouville operator. For more details see [1, 2, 3, 28].

Let $D$ and $\Xi$ be the singular partial differential operators defined by

$$\begin{cases} D = \dfrac{\partial}{\partial x}; \\ \Xi = \dfrac{\partial^2}{\partial r^2} + \dfrac{2\alpha+1}{r}\dfrac{\partial}{\partial r} - \dfrac{\partial^2}{\partial x^2}; \quad (r,x) \in \,]0,+\infty[\times\mathbb{R}, \ \alpha \geqslant 0. \end{cases}$$

The partial differential operators $D$ and $\Xi$ satisfy the intertwining properties with the Riemann-Liouville operator and its dual

$${}^t\mathscr{R}_\alpha \Xi(f) = \frac{\partial^2}{\partial r^2}\,{}^t\mathscr{R}_\alpha(f), \qquad {}^t\mathscr{R}_\alpha D(f) = D\,{}^t\mathscr{R}_\alpha(f),$$

$$\Xi\mathscr{R}_\alpha(f) = \mathscr{R}_\alpha \frac{\partial^2}{\partial r^2}(f), \qquad D\mathscr{R}_\alpha(f) = \mathscr{R}_\alpha D(f),$$

where $f$ is a sufficiently smooth function.

On the other hand, for all $(\mu,\lambda)\in\mathbb{C}^2$, the system

$$\begin{cases} Du(r,x)=-i\lambda u(r,x);\\ \Xi u(r,x)=-\mu^2 u(r,x);\\ u(0,0)=1,\quad \dfrac{\partial u}{\partial r}(0,x)=0;\ \forall x\in\mathbb{R}, \end{cases}$$

admits a unique solution $\varphi_{\mu,\lambda}$ given by

$$\forall (r,x)\in [0,+\infty[\times\mathbb{R},\quad \varphi_{\mu,\lambda}(r,x) = j_\alpha\big(r\sqrt{\mu^2+\lambda^2}\big)\ e^{-i\lambda x}, \tag{2.1}$$

where $j_\alpha$ is the modified Bessel function defined by

$$j_\alpha(z)=2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha}=\Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\Big(\frac{z}{2}\Big)^{2k},$$

and $J_\alpha$ is the Bessel function of first kind and index $\alpha$ [11, 12, 24, 32] . The modified Bessel function $j_\alpha$ has the integral representation

$$j_\alpha(z) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_{-1}^{1}(1-t^2)^{\alpha-1/2}\exp(-izt)dt. \tag{2.2}$$

Consequently, for every $k\in\mathbb{N}$ and $z\in\mathbb{C}$, we have

$$\big|j_\alpha^{(k)}(z)\big| \leqslant e^{|\mathrm{Im}(z)|}. \tag{2.3}$$

The eigenfunction $\varphi_{\mu,\lambda}$ satisfies the following properties

. The function $\varphi_{\mu,\lambda}$ is bounded on $\mathbb{R}^2$ if, and only if $(\mu,\lambda)\in\Upsilon$, where $\Upsilon$ is the set defined by relation (1.6), and in this case

$$\sup_{(r,x)\in\mathbb{R}^2}\big|\varphi_{\mu,\lambda}(r,x)\big| = 1. \tag{2.4}$$

. The function $\varphi_{\mu,\lambda}$ has the following Mehler integral representation

$$\varphi_{\mu,\lambda}(r,x)=\begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1}\cos\big(\mu rs\sqrt{1-t^2}\big)\exp\big(-i\lambda(x+rt)\big)\\ \times(1-t^2)^{\alpha-1/2}(1-s^2)^{\alpha-1}dtds; & \text{if } \alpha>0,\\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1}\cos\big(r\mu\sqrt{1-t^2}\big)\exp\big(-i\lambda(x+rt)\big)\\ \times\dfrac{dt}{\sqrt{1-t^2}}, & \text{if } \alpha=0. \end{cases}$$

. The precedent integral representation of the eigenfunction $\varphi_{\mu,\lambda}$ and relation (1.3) show that

$$\forall (r,x)\in [0,+\infty[\times\mathbb{R},\quad \varphi_{\mu,\lambda}(r,x)=\mathscr{R}_\alpha\big(\cos(\mu\cdot)e^{-i\lambda\cdot}\big)(r,x).$$

The eigenfunction $\varphi_{\mu,\lambda}$ satisfies the product formula

$$\varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y)=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_0^{\pi}\varphi_{\mu,\lambda}\big(\sqrt{r^2+s^2+2rs\cos\theta},x+y\big)\sin^{2\alpha}\theta d\theta.$$

This formula allows us to define the translation operators and the convolution product.

**Definition 2.1.** i) For every $(r,x)\in\ [0,+\infty[\times\mathbb{R}$, the translation operator $\tau_{(r,x)}$ associated with Riemann-Liouville operator is defined on $L^p(d\nu_\alpha);\ p\in\ [1,+\infty]$, by

$$\begin{aligned}&\tau_{(r,x)}f(s,y)\\ &=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_0^{\pi}f\big(\sqrt{r^2+s^2+2rs\cos\theta},x+y\big)\sin^{2\alpha}(\theta)d\theta.\end{aligned} \tag{2.5}$$

ii) The convolution product of $f,g\ \in\ \ L^1(d\nu_\alpha)$ is defined for every $(r,x)\ \in$ $[0,+\infty[\times\mathbb{R}$, by

$$f*g(r,x)\ =\ \int_o^{+\infty}\int_{\mathbb{R}}\tau_{(r,-x)}(\check{f})(s,y)g(s,y)d\nu_\alpha(s,y), \tag{2.6}$$

where $\check{f}(s,y)=f(s,-y)$.

The set $[0,+\infty[\times\mathbb{R}$ equipped with the convolution product $*$ is an hypergroup in the sense of [8].

Moreover, we have the following properties
. The eigenfunction $\varphi_{\mu,\lambda}$ satisfies the product formula

$$\tau_{(r,x)}(\varphi_{\mu,\lambda})(s,y)=\varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y).$$

. For every $f\in\ L^p(d\nu_\alpha);\ 1\leqslant p\leqslant+\infty$, and for every $(r,x)\in\ [0,+\infty[\times\mathbb{R}$, the function $\tau_{(r,x)}(f)$ belongs to $L^p(d\nu_\alpha)$ and we have

$$\big|\big|\tau_{(r,x)}(f)\big|\big|_{p,\nu_\alpha}\ \leqslant\ ||f||_{p,\nu_\alpha}. \tag{2.7}$$

. For every $f\in\ L^1(d\nu_\alpha)$ and $(r,x)\in\ [0,+\infty[\times\mathbb{R}$,

$$\int_0^{\infty}\int_{\mathbb{R}}\tau_{(r,x)}(f)(s,y)d\nu_\alpha(s,y)\ =\ \int_0^{\infty}\int_{\mathbb{R}}f(s,y)d\nu_\alpha(s,y). \tag{2.8}$$

. For every $f\in\ L^p(d\nu_\alpha);\ p\in\ [1,+\infty[$, we have

$$\lim_{(r,x)\to(0,0)}\big|\big|\tau_{(r,x)}(f)-f\big|\big|_p\ =\ 0\ . \tag{2.9}$$

. Let $\varphi$ be a nonnegative measurable function on $\mathbb{R}\times\mathbb{R}$, even with respect to the first variable, such that

$$\int_0^{+\infty}\int_{\mathbb{R}}\varphi(r,x)d\nu_\alpha(r,x)=1.$$

Then by the relation (2.9), the sequence $(\varphi_k)_{k\in\mathbb{N}^*}$ defined by

$$\forall (r,x)\in\mathbb{R}\times\mathbb{R},\ \varphi_k(r,x)=k^{2\alpha+3}\varphi(kr,kx)$$

is an approximation of the identity in $L^p(d\nu_\alpha)$; $p\in[1,+\infty[$, that is for every $f\in L^p(d\nu_\alpha)$, we have

$$\lim_{k\to+\infty}\big|\big|\varphi_k*f-f\big|\big|_{p,\nu_\alpha} = 0. \tag{2.10}$$

. For $f,\ g\in L^1(d\nu_\alpha)$, the function $f*g$ belongs to $L^1(d\nu_\alpha)$, the convolution product is commutative, associative and we have

$$||f*g||_{1,\nu_\alpha}\leqslant||f||_{1,\nu_\alpha}||g||_{1,\nu_\alpha}.$$

Moreover, if $1\leqslant p,q,r\leqslant+\infty$ are such that $1/r=1/p+1/q-1$ and if $f\in L^p(d\nu_\alpha)$, $g\in L^q(d\nu_\alpha)$, then the function $f*g$ belongs to $L^r(d\nu_\alpha)$, and we have the Young's inequality

$$||f*g||_{r,\nu_\alpha} \leqslant ||f||_{p,\nu_\alpha}||g||_{q,\nu_\alpha}. \tag{2.11}$$

In the sequel, we need the following notations
. $\mathscr{B}_{\Upsilon_+}$ is the σ-algebra defined on $\Upsilon_+$ by

$$\mathscr{B}_{\Upsilon_+}=\left\{\theta^{-1}(B),\ B\in\mathscr{B}_{\text{or}}\big([0,+\infty[\times\mathbb{R}\big)\right\},$$

where $\theta$ is the bijective function defined on the set $\Upsilon_+$ by

$$\theta(\mu,\lambda) = \big(\sqrt{\mu^2+\lambda^2},\lambda\big), \tag{2.12}$$

and $\mathscr{B}_{\text{or}}\big([0,+\infty[\times\mathbb{R}\big)$ is the usual Borel σ-algebra on $[0,+\infty[\times\mathbb{R}$.
. $d\gamma_\alpha$ is the measure defined on $\mathscr{B}_{\Upsilon_+}$ by

$$\forall A\in\mathscr{B}_{\Upsilon_+},\ \gamma_\alpha(A)=\nu_\alpha\big(\theta(A)\big).$$

**Proposition 2.2.** i. *For all nonnegative measurable function* $g$ *on* $\Upsilon_+$, *we have*

$$\begin{aligned}&\iint_{\Upsilon_+}g(\mu,\lambda)d\gamma_\alpha(\mu,\lambda)\\&=\frac{1}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\Big(\int_0^{+\infty}\int_{\mathbb{R}}g(\mu,\lambda)(\mu^2+\lambda^2)^\alpha\mu d\mu d\lambda\\&+\int_{\mathbb{R}}\int_0^{|\lambda|}g(i\mu,\lambda)(\lambda^2-\mu^2)^\alpha\mu d\mu d\lambda\Big)\ .\end{aligned}$$

ii. *For all nonnegative measurable function* $f$ *on* $[0,+\infty[\times\mathbb{R}$ *(respectively integrable on* $[0,+\infty[\times\mathbb{R}$ *with respect to the measure* $d\nu_\alpha$ *),* $f\circ\theta$ *is a nonnegative measurable function on* $\Upsilon_+$ *(respectively integrable on* $\Upsilon_+$ *with respect to the measure* $d\gamma_\alpha$ *) and we have*

$$\int\int_{\Upsilon_+}(f\circ\theta)(\mu,\lambda)d\gamma_\alpha(\mu,\lambda) \;=\; \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)d\nu_\alpha(r,x). \tag{2.13}$$

Now, using the eigenfunction $\varphi_{\mu,\lambda}$ given by the relation (2.1), we can define the Fourier transform.

**Definition 2.3.** The Fourier transform associated with the Riemann-Liouville operator is defined on $L^1(d\nu_\alpha)$ by

$$\forall(\mu,\lambda)\in\Upsilon,\qquad \mathscr{F}_\alpha(f)(\mu,\lambda)=\int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\varphi_{\mu,\lambda}(r,x)d\nu_\alpha(r,x).$$

We have the following properties

. From the relation (2.4), we deduce that for $f\in L^1(d\nu_\alpha)$, the function $\mathscr{F}_\alpha(f)$ belongs to the space $L^\infty(d\gamma_\alpha)$ and we have

$$\big|\big|\mathscr{F}_\alpha(f)\big|\big|_{\infty,\gamma_\alpha} \;\leqslant\; ||f||_{1,\nu_\alpha}. \tag{2.14}$$

. For $f\in L^1(d\nu_\alpha)$, we have

$$\forall(\mu,\lambda)\in\Upsilon,\qquad \mathscr{F}_\alpha(f)(\mu,\lambda) \;=\; \widetilde{\mathscr{F}_\alpha}(f)\circ\theta(\mu,\lambda), \tag{2.15}$$

where for every $(\mu,\lambda)\in\mathbb{R}^2$,

$$\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda) \;=\; \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)j_\alpha(r\mu)\exp(-i\lambda x)d\nu_\alpha(r,x), \tag{2.16}$$

and $\theta$ is the function defined by the relation (2.12).
. Let $f\in L^1(d\nu_\alpha)$ such that the function $\mathscr{F}_\alpha(f)$ belongs to the space $L^1(d\gamma_\alpha)$, then we have the following inversion formula for $\mathscr{F}_\alpha$, for almost every $(r,x)\in[0,+\infty[\times\mathbb{R}$,

$$f(r,x) \;=\; \int\int_{\Upsilon_+}\mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma_\alpha(\mu,\lambda). \tag{2.17}$$

. Let $f\in L^1(d\nu_\alpha)$. For every $(r,x)\in[0,+\infty[\times\mathbb{R}$, we have

$$\forall(\mu,\lambda)\in\Upsilon,\qquad \mathscr{F}_\alpha\big(\tau_{(r,x)}(f)\big)(\mu,\lambda)=\overline{\varphi_{\mu,\lambda}(r,x)}\mathscr{F}_\alpha(f)(\mu,\lambda).$$

. For $f,g\in L^1(d\nu_\alpha)$, we have

$$\forall(\mu,\lambda)\in\Upsilon,\qquad \mathscr{F}_\alpha(f*g)(\mu,\lambda)=\mathscr{F}_\alpha(f)(\mu,\lambda)\mathscr{F}_\alpha(g)(\mu,\lambda).$$

. Let $p \in [1, +\infty]$. From the relation (2.13), the function $f$ belongs to $L^p(d\nu_\alpha)$ if, and only if the function $f \circ \theta$ belongs to the space $L^p(d\gamma_\alpha)$ and we have

$$\left|\left|f \circ \theta\right|\right|_{p,\gamma_\alpha} = ||f||_{p,\nu_\alpha}. \tag{2.18}$$

Since the mapping $\widetilde{\mathscr{F}_\alpha}$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ onto itself [22], then the relations (2.15) and (2.18) show that the Fourier transform $\mathscr{F}_\alpha$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ into $L^2(d\gamma_\alpha)$. Namely, for every $f \in L^2(d\nu_\alpha)$, the function $\mathscr{F}_\alpha(f)$ belongs to the space $L^2(d\gamma_\alpha)$ and we have

$$\left|\left|\mathscr{F}_\alpha(f)\right|\right|_{2,\gamma_\alpha} = ||f||_{2,\nu_\alpha}. \tag{2.19}$$

. Using the relations (2.14), (2.19) and the Riesz-Thorin theorem's [30, 31], we deduce that for every $f \in L^p(d\nu_\alpha)$; $p \in [1, 2]$, the function $\mathscr{F}_\alpha(f)$ lies in $L^{p'}(d\gamma_\alpha)$; $p' = p/(p-1)$, and we have

$$\left|\left|\mathscr{F}_\alpha(f)\right|\right|_{p',\gamma_\alpha} \leqslant ||f||_{p,\nu_\alpha}. \tag{2.20}$$

However, the inequality (2.20) is not optimal and we have

**Theorem 2.4.** *(Hausdorff-Young) The Fourier transform $\mathscr{F}_\alpha$ can be extended to a continuous operator from $L^p(d\nu_\alpha)$; $p \in [1, 2]$, into $L^{p'}(d\gamma_\alpha)$; $p' = p/(p-1)$, and for every $f \in L^p(d\nu_\alpha)$,*

$$\left|\left|\mathscr{F}_\alpha(f)\right|\right|_{p',\gamma_\alpha} \leqslant A_p^{\alpha+3/2}\, ||f||_{p,\nu_\alpha}, \tag{2.21}$$

*where* $A_p = \dfrac{p^{1/p}}{p'^{1/p'}} = \dfrac{p^{1/p}}{\left(\frac{p}{p-1}\right)^{(p-1)/p}}$ *is the Babenko-Beckner constant.*

*Proof.* Let $\mathscr{H}_\alpha$ be the Hankel transform with respect to the first variable defined by

$$\mathscr{H}_\alpha(f)(r, x) = \int_0^\infty f(s, x) j_\alpha(rs) d\omega_\alpha(s),$$

where $d\omega_\alpha$ is the measure defined on $[0, +\infty[$ by

$$d\omega_\alpha(s) = \frac{1}{2^\alpha \Gamma(\alpha+1)}\, s^{2\alpha+1} ds.$$

Then, for every $f \in L^p(d\nu_\alpha)$ and for almost every $x \in \mathbb{R}$, the function $f(., x)$ belongs to $L^p(d\omega_\alpha)$ and from [14], we get

$$\left(\int_0^\infty \left|\mathscr{H}_\alpha(f)(r, x)\right|^{p'} d\omega_\alpha(r)\right)^{1/p'} \leqslant A_p^{\alpha+1} \left(\int_0^\infty \left|f(r, x)\right|^p d\omega_\alpha(r)\right)^{1/p}. \tag{2.22}$$

Also, we define the usual Fourier transform with respect to the second variable by setting

$$\Lambda(f)(r,x)=\int_{\mathbb{R}} f(r,y)\ e^{-ixy}dm(y),$$

where $dm$ is the measure defined on $\mathbb{R}$ by $dm(y)=\dfrac{dy}{\sqrt{2\pi}}$ . Then, for every $f\in\ L^p(d\nu_\alpha)$ and for almost every $r\in\ [0,+\infty[$, the function $f(r,.)$ belongs to $L^p(dm)$ and from [4], we get

$$\Big(\int_{\mathbb{R}}\big|\Lambda(f)(r,x)\big|^{p'}dm(x)\Big)^{1/p'}\ \leqslant\ A_p^{1/2}\ \Big(\int_{\mathbb{R}}\big|f(r,x)\big|^{p}dm(x)\Big)^{1/p}. \quad (2.23)$$

Now, from the relations (2.13), (2.15) and by Fubini's theorem, we have

$$\begin{aligned}&\Big(\int\int_{\Upsilon_+}\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^{p'}d\gamma_\alpha(\mu,\lambda)\Big)^{1/p'}\\ &=\Big(\int_0^\infty\int_{\mathbb{R}}\big|\widetilde{\mathscr{F}_\alpha}(f)(r,x)\big|^{p'}d\nu_\alpha(r,x)\Big)^{1/p'}\\ &=\Big[\int_0^\infty\Big(\int_{\mathbb{R}}\Big|\Lambda\big(\mathscr{H}_\alpha(f)\big)(r,x)\Big|^{p'}dm(x)\Big)d\omega_\alpha(r)\Big]^{1/p'},\end{aligned}$$

and by the relation (2.23), we get

$$\begin{aligned}&\Big(\int\int_{\Upsilon_+}\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^{p'}d\gamma_\alpha(\mu,\lambda)\Big)^{1/p'}\\ &\leqslant\Big[\int_0^\infty A_p^{p'/2}\ \Big(\int_{\mathbb{R}}\big|\mathscr{H}_\alpha(f)(r,x)\big|^{p}dm(x)\Big)^{p'/p}d\omega_\alpha(r)\Big]^{1/p'}\\ &\leqslant A_p^{1/2}\ \Big\{\Big[\int_0^\infty\Big(\int_{\mathbb{R}}\big|\mathscr{H}_\alpha(f)(r,x)\big|^{p}dm(x)\Big)^{p'/p}d\omega_\alpha(r)\Big]^{p/p'}\Big\}^{1/p}.\end{aligned}$$

From Minkowski's inequality [15], we obtain

$$\begin{aligned}&\Big(\int\int_{\Upsilon_+}\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^{p'}d\gamma_\alpha(\mu,\lambda)\Big)^{1/p'}\\ &\leqslant A_p^{1/2}\ \Big\{\int_0^\infty\Big(\int_{\mathbb{R}}\big|\mathscr{H}_\alpha(f)(r,x)\big|^{p'}d\omega_\alpha(r)\Big)^{p/p'}dm(x)\Big\}^{1/p},\end{aligned}$$

and by the relation (2.22), it follows that

$$\begin{aligned}&\Big(\int\int_{\Upsilon_+}\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^{p'}d\gamma_\alpha(\mu,\lambda)\Big)^{1/p'}\\ &\leqslant A_p^{\frac{1}{2}+\alpha+1}\Big\{\int_0^\infty\Big(\int_0^\infty\big|f(r,x)\big|^p d\omega_\alpha(r)\Big)dm(x)\Big\}^{1/p}\\ &= A_p^{\alpha+3/2}\ ||f||_{p,\nu_\alpha}.\end{aligned}$$

□

## 3. Entropy uncertainty principle

This section is devoted to establish the main result of this paper, that is the entropy uncertainty principle.
We start this section by some intermediated results.

**Lemma 3.1.** *Let $x$ be a positive real number. Then,*

i. *For every $p\in\ [1,2[,$ we have*

$$x^2-x\ \ \leqslant\ \ \frac{x^p-x^2}{p-2}\leqslant x^2\ln x. \tag{3.1}$$

ii. *For every $p\in\ ]2,3]$, we have*

$$x^2\ln x\ \ \leqslant\ \ \frac{x^p-x^2}{p-2}\leqslant x^3-x^2. \tag{3.2}$$

*Proof.* Let $\vartheta$ be the function defined by

$$\vartheta(p)=\frac{x^p-x^2}{p-2}.$$

The function $\vartheta$ is differentiable on $[1,2[$ and $]2,3]$ and we have

$$\vartheta'(p)=\frac{(p-2)x^p\ \ln(x)+x^2-x^p}{(p-2)^2}.$$

Let $h(p)=(p-2)x^p\ \ln(x)+x^2-x^p$. We have

$$h'(p)=(p-2)x^p\ (\ln(x))^2,$$

which means that the function $h$ is decreasing on $[1,2]$ and increasing on $[2,3]$. Since $h(2)=0$, we deduce that for every $p\geqslant 1;\ h(p)\geqslant 0$ and that the function $\vartheta$ is increasing on $[1,2[$ and $]2,3]$.
Consequently,

$$\forall p\in\ [1,2[,\quad \vartheta(1)\leqslant\vartheta(p)\leqslant\lim_{p\longrightarrow 2-}\vartheta(p),$$

$$\forall p\in\ ]2,3],\quad \lim_{p\longrightarrow 2-}\vartheta(p)\leqslant\vartheta(p)\leqslant\vartheta(3).$$

This proves the lemma.

□

**Definition 3.2.** i. For every nonnegative measurable function $f$ on $[0,+\infty[\times\mathbb{R}$ such that

$$\int_0^{\infty}\int_{\mathbb{R}} f(r,x)\,\Big|\ln\big(f(r,x)\big)\Big| d\nu_\alpha(r,x) < +\infty,$$

the weighted entropy of $f$ is defined by

$$E_{\nu_\alpha}(f) = -\int_0^{\infty}\int_{\mathbb{R}} f(r,x)\ \ln\big(f(r,x)\big) d\nu_\alpha(r,x).$$

ii. For every nonnegative measurable function $g$ on $\Upsilon_+$ such that

$$\int\int_{\Upsilon_+} g(\mu,\lambda)\,\Big|\ln\big(g(\mu,\lambda)\big)\Big| d\gamma_\alpha(\mu,\lambda) < +\infty,$$

the weighted entropy of $g$ is defined by

$$E_{\gamma_\alpha}(g) = -\int\int_{\Upsilon_+} g(\mu,\lambda)\ \ln\big(g(\mu,\lambda)\big) d\gamma_\alpha(\mu,\lambda).$$

The first important result of this section is the following theorem, that is the uncertainty principle in terms of entropy for a function $f \in L^1(d\nu_\alpha)\cap L^2(d\nu_\alpha)$.

**Theorem 3.3.** *Let* $f \in L^1(d\nu_\alpha)\cap L^2(d\nu_\alpha)$ ; $\|f\|_{2,\nu_\alpha} = 1$, *such that*

$$\int_0^{\infty}\int_{\mathbb{R}} \big|f(r,x)\big|^2\,\Big|\ln\big(\big|f(r,x)\big|\big)\Big| d\nu_\alpha(r,x) < +\infty,$$

*and*

$$\int\int_{\Upsilon_+} \big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^2\,\Big|\ln\big(\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|\big)\Big| d\gamma_\alpha(\mu,\lambda) < +\infty.$$

*Then, we have*

$$E_{\nu_\alpha}\big(|f|^2\big) + E_{\gamma_\alpha}\big(|\mathscr{F}_\alpha(f)|^2\big) \geqslant (2\alpha+3)(1-\ln 2). \tag{3.3}$$

*Proof.* Let $f \in L^1(d\nu_\alpha)\cap L^2(d\nu_\alpha)$ such that $\|f\|_{2,\nu_\alpha} = 1$. By a convexity argument; for every $p \in [1,2]$, the function $f$ belongs to the space $L^p(d\nu_\alpha)$ and $\mathscr{F}_\alpha(f)$ belongs to $L^{p'}(d\gamma_\alpha)$; $p' = p/(p-1)$.

Let $\varphi$ be the function defined on $]1,2]$ by

$$\begin{aligned}\varphi(p) &= \int\int_{\Upsilon_+} \left|\mathscr{F}_\alpha(f)(\mu,\lambda)\right|^{p'} d\gamma_\alpha(\mu,\lambda) \\ &- A_p^{p'(\alpha+3/2)} \left(\int_0^\infty \int_{\mathbb{R}} \left|f(r,x)\right|^p d\nu_\alpha(r,x)\right)^{p'/p} \\ &= \int\int_{\Upsilon_+} \left|\mathscr{F}_\alpha(f)(\mu,\lambda)\right|^{p/(p-1)} d\gamma_\alpha(\mu,\lambda) \\ &- \left(\frac{p^{1/p}}{\left(\frac{p}{p-1}\right)^{(p-1)/p}}\right)^{\frac{p}{p-1}(\alpha+3/2)} \left(\int_0^\infty \int_{\mathbb{R}} \left|f(r,x)\right|^p d\nu_\alpha(r,x)\right)^{1/(p-1)}.\end{aligned}$$

From Theorem 2.4 and the relation (2.19), we deduce that for every $p \in ]1,2]$; $\varphi(p) \leqslant 0$ and $\varphi(2) = 0$, which implies that

$$\begin{aligned}\lim_{p\to 2^-} \frac{\varphi(p)-\varphi(2)}{p-2} &= \lim_{p\to 2^-} \frac{\varphi(p)}{p-2} \\ &= \varphi'(2^-) \geqslant 0. \end{aligned} \tag{3.4}$$

Now,

$$\frac{d}{dp}\left(\int_0^\infty \int_{\mathbb{R}} \left|f(r,x)\right|^p d\nu_\alpha(r,x)\right)\Big|_{p=2} = \lim_{p\to 2^-} \int_0^\infty \int_{\mathbb{R}} \frac{\left|f(r,x)\right|^p - \left|f(r,x)\right|^2}{p-2}\, d\nu_\alpha(r,x),$$

and from Lemma 3.1,

$$\left|\frac{\left|f(r,x)\right|^p - \left|f(r,x)\right|^2}{p-2}\right| \leqslant \left|f(r,x)\right|^2 + \left|f(r,x)\right| + \left|f(r,x)\right|^2 \left|\ln\left|f(r,x)\right|\right|.$$

Since $f \in L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)$ and

$$\int_0^\infty \int_{\mathbb{R}} \left|f(r,x)\right|^2 \left|\ln\left|f(r,x)\right|\right| d\nu_\alpha(r,x) < +\infty,$$

then, by the dominated convergence theorem, we get

$$\begin{aligned}\frac{d}{dp}\left(\int_0^\infty \int_{\mathbb{R}} \left|f(r,x)\right|^p d\nu_\alpha(r,x)\right)\Big|_{p=2} &= \int_0^\infty \int_{\mathbb{R}} \left|f(r,x)\right|^2 \ln\left(\left|f(r,x)\right|\right) d\nu_\alpha(r,x) \\ &= -\frac{1}{2} E_{\nu_\alpha}\left(|f|^2\right).\end{aligned} \tag{3.5}$$

As the same way,

$$\begin{aligned}
&\frac{d}{dp}\Big(\int\int_{\Upsilon_+}\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^{p/p-1}d\gamma_\alpha(\mu,\lambda)\Big)\Big|_{p=2^-}\\
&=-\frac{d}{dp'}\Big(\int\int_{\Upsilon_+}\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^{p'}d\gamma_\alpha(\mu,\lambda)\Big)\Big|_{p'=2^+}\\
&=-\lim_{p'\to 2^+}\int\int_{\Upsilon_+}\frac{\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^{p'}-\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^{2}}{p'-2}d\gamma_\alpha(\mu,\lambda).
\end{aligned}$$

Applying Lemma 3.1 ii), we deduce that for $p'\in]2,3]$,

$$\begin{aligned}
\Big|\frac{\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^{p'}-\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^{2}}{p'-2}\Big| &\leqslant \big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^{3}+\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^{2}\\
&+ \big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^{2}\Big|\ln\Big(\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|\Big)\Big|.
\end{aligned}$$

Since $\mathscr{F}_\alpha(f)$ belongs to $L^2(d\gamma_\alpha)\cap L^3(d\gamma_\alpha)$ and since

$$\int\int_{\Upsilon_+}\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^{2}\ \Big|\ln\Big(\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|\Big)\Big|d\gamma_\alpha(\mu,\lambda)<+\infty,$$

then, again by the dominated convergence theorem, we obtain

$$\begin{aligned}
&\frac{d}{dp}\Big(\int\int_{\Upsilon_+}\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^{p/(p-1)}d\gamma_\alpha(\mu,\lambda)\Big)\Big|_{p=2^-}\\
&=-\int\int_{\Upsilon_+}\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^{2}\ \ln\Big(\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|\Big)d\gamma_\alpha(\mu,\lambda)\\
&=\frac{1}{2}\ E_{\gamma_\alpha}\Big(\big|\mathscr{F}_\alpha(f)\big|^2\Big).
\end{aligned}\tag{3.6}$$

Finally, we have

$$\frac{d}{dp}\Big\{\Big(\frac{p^{1/p}}{\big(\frac{p}{p-1}\big)^{(p-1)/p}}\Big)^{\frac{p}{p-1}(\alpha+3/2)}\Big\}\Big|_{p=2} = \big(\alpha+3/2\big)(1-\ln 2).\tag{3.7}$$

Applying the relations (3.4), (3.5), (3.6) and (3.7), we get

$$E_{\nu_\alpha}\big(|(f)|^2\big)+E_{\gamma_\alpha}\Big(\big|\mathscr{F}_\alpha(f)\big|^2\Big)\geqslant(2\alpha+3)(1-\ln 2).$$

□

**Lemma 3.4.** *Let $f$ be a measurable function on $[0,+\infty[\times\mathbb{R}$ and let*

$$\omega:\ [0,+\infty[\longrightarrow[0,+\infty[$$

*be a nondecreasing convex function such that the function $\omega\big(|f|\big)$ belongs to $L^1(d\nu_\alpha)$. Let $(f_k)$ be a sequence of measurable nonnegative functions on*

$[0,+\infty[\times\mathbb{R}$ *such that for every* $k\in\mathbb{N}$; $||f_k||_{1,\nu_\alpha}=1$, *and the sequence* $(f_k*f)_k$ *converges pointwise to* $f$.
*Then, for every* $k\in\mathbb{N}$, *the function* $\omega\big(|f_k*f|\big)$ *belongs to* $L^1(d\nu_\alpha)$ *and we have*

$$\lim_{k\to+\infty}\int_0^\infty\int_{\mathbb{R}}\omega\big(|f_k*f|\big)(r,x)d\nu_\alpha(r,x)=\int_0^\infty\int_{\mathbb{R}}\omega\big(|f|\big)(r,x)d\nu_\alpha(r,x).$$

*Proof.* From the relation (2.8), it follows that for every $k\in\mathbb{N}$ and $(s,y)\in[0,+\infty[\times\mathbb{R}$,

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}}\tau_{(s,-y)}(\check{f_k})(r,x)d\nu_\alpha(r,x) &= \int_0^\infty\int_{\mathbb{R}}\check{f_k}(r,x)d\nu_\alpha(r,x)\\ &= \int_0^\infty\int_{\mathbb{R}}f_k(r,x)d\nu_\alpha(r,x)=1.\end{aligned}$$

This means that for every $k\in\mathbb{N}$, $(s,y)\in[0,+\infty[\times\mathbb{R}$; $\tau_{(s,-y)}(\check{f_k})(r,x)d\nu_\alpha(r,x)$ is a probability measure on $[0,+\infty[\times\mathbb{R}$.
Applying Jensen's inequality and the fact that the function $\omega$ is convex, we get

$$\begin{aligned}\omega\big(|f_k*f|\big)(s,y) &= \omega\Big(\Big|\int_0^\infty\int_{\mathbb{R}}f(r,x)\tau_{(s,-y)}(\check{f_k})(r,x)d\nu_\alpha(r,x)\Big|\Big)\\ &\leqslant \omega\Big(\int_0^\infty\int_{\mathbb{R}}\big|f(r,x)\big|\tau_{(s,-y)}(\check{f_k})(r,x)d\nu_\alpha(r,x)\Big)\\ &\leqslant \int_0^\infty\int_{\mathbb{R}}\omega\big(|f|\big)(r,x)\tau_{(s,-y)}(\check{f_k})(r,x)d\nu_\alpha(r,x)\\ &= \omega\big(|f|\big)*f_k(s,y).\end{aligned}\tag{3.8}$$

From the relations (2.11) and (3.8), we deduce that for every $k\in\mathbb{N}$, the function $\omega\big(|f_k*f|\big)$ belongs to $L^1(d\nu_\alpha)$ and we have

$$\begin{aligned}\big|\big|\omega\big(|f_k*f|\big)\big|\big|_{1,\nu_\alpha} &\leqslant \big|\big|\omega\big(|f|\big)\big|\big|_{1,\nu_\alpha}||f_k||_{1,\nu_\alpha}\\ &= \big|\big|\omega\big(|f|\big)\big|\big|_{1,\nu_\alpha}.\end{aligned}$$

This implies that

$$\limsup_{k\to+\infty}\big|\big|\omega\big(|f_k*f|\big)\big|\big|_{1,\nu_\alpha}\ \leqslant\ \big|\big|\omega\big(|f|\big)\big|\big|_{1,\nu_\alpha}.\tag{3.9}$$

On the other hand, by Fatou's lemma,

$$\begin{aligned}\big|\big|\omega\big(|f|\big)\big|\big|_{1,\nu_\alpha} &= \int_0^\infty\int_{\mathbb{R}}\lim_{k\to+\infty}\omega\big(|f_k*f|\big)(r,x)d\nu_\alpha(r,x)\\ &\leqslant \liminf_{k\to+\infty}\big|\big|\omega\big(|f_k*f|\big)\big|\big|_{1,\nu_\alpha}.\end{aligned}\tag{3.10}$$

The proof is complete by combining the relations (3.9) and (3.10).

□

We denote by $\mathscr{S}_e(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, rapidly decreasing together with all their derivatives, even with respect to the first variable.
The space $\mathscr{S}_e(\mathbb{R}^2)$ is endowed with the topology generated by the family of norms

$$\rho_m(\varphi) \quad = \quad \sup_{\substack{(r,x)\in\ [0,+\infty[\times\mathbb{R} \\ k+|\beta|\leqslant m}} \big(1+r^2+x^2\big)^k \big|D^\beta(\varphi)(r,x)\big|. \tag{3.11}$$

Now, we are able to prove the uncertainty principle in terms of entropy in its final form.

**Theorem 3.5.** *(Entropy) Let* $f \in L^2(d\nu_\alpha)$ *such that* $||f||_{2,\nu_\alpha} = 1$. *We assume that*

$$\int_0^\infty \int_{\mathbb{R}} \big|f(r,x)\big|^2 \ \Big|\ln\big(\big|f(r,x)\big|\big)\Big| d\nu_\alpha(r,x) < +\infty,$$

*and*

$$\iint_{\Upsilon_+} \big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^2 \ \Big|\ln\big(\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|\big)\Big| d\gamma_\alpha(\mu,\lambda) < +\infty.$$

*Then, we have*

$$E_{\nu_\alpha}\big(|f|^2\big) + E_{\gamma_\alpha}\big(|\mathscr{F}_\alpha(f)|^2\big) \geqslant (2\alpha+3)(1-\ln 2).$$

*Proof.* Let $f$ be a function satisfying the hypothesis. We will construct a sequence $(f_k)_k \subset L^1(d\nu_\alpha)\cap L^2(d\nu_\alpha)$ such that

$$\lim_{k\to+\infty} ||f_k||_{2,\nu_\alpha} = ||f||_{2,\nu_\alpha},$$

$$\lim_{k\to+\infty} E_{\nu_\alpha}\big(|f_k|^2\big) = E_{\nu_\alpha}\big(|f|^2\big),$$

and

$$\lim_{k\to+\infty} E_{\gamma_\alpha}\big(|\mathscr{F}_\alpha(f_k)|^2\big) = E_{\gamma_\alpha}\big(|\mathscr{F}_\alpha(f)|^2\big).$$

Let $(g_k)_{k\in\ \mathbb{N}}$ be the sequence defined by

$$g_k(r,x) = 2^{\alpha+3/2}\ k^{2\alpha+3}\ e^{-k^2(r^2+x^2)}.$$

From the relation (2.10), the sequence $(g_k)_k$ is an approximation of the identity, in particular, for every $f \in L^2(d\nu_\alpha)$,

$$\lim_{k\to+\infty} ||g_k * f - f||_{2,\nu_\alpha} \quad = \quad 0. \tag{3.12}$$

Now, for every $\varphi \in \mathscr{S}_e(\mathbb{R}^2)$ and $f \in L^2(d\nu_\alpha)$, the function $\varphi.f$ belongs to $L^1(d\nu_\alpha)\cap L^2(d\nu_\alpha)$ and we have

$$\widetilde{\mathscr{F}_\alpha}(\varphi.f) \quad = \quad \widetilde{\mathscr{F}_\alpha}(\varphi) * \widetilde{\mathscr{F}_\alpha}(f). \tag{3.13}$$

Let $h_k = \widetilde{\mathscr{F}_\alpha}^{-1}(g_k) = \widetilde{\mathscr{F}_\alpha}(g_k)$. For every $(r,x) \in [0,+\infty[\times\mathbb{R}$, we have

$$h_k(r,x) = e^{-\frac{r^2+x^2}{4k^2}}.$$

We define the sequence $(\varphi_k)_k$ by setting $\varphi_k = h_k f$. Then, from the relation (3.13), we get

$$\widetilde{\mathscr{F}_\alpha}(\varphi_k) = \widetilde{\mathscr{F}_\alpha}(h_k) * \widetilde{\mathscr{F}_\alpha}(f) = g_k * \widetilde{\mathscr{F}_\alpha}(f).$$

On the other hand, for every $k \in \mathbb{N}$, the function $\varphi_k$ belongs to $L^1(d\nu_\alpha)\cap L^2(d\nu_\alpha)$ and from dominated convergence theorem, the sequence $(\varphi_k)_k$ converges to $f$ in $L^2(d\nu_\alpha)$ and the sequence

$$\widetilde{\mathscr{F}_\alpha}(\varphi_k) = g_k * \widetilde{\mathscr{F}_\alpha}(f)$$

converges to $\widetilde{\mathscr{F}_\alpha}(f)$ in $L^2(d\nu_\alpha)$. So, there is a subsequence $\left(g_{\theta(k)} * \widetilde{\mathscr{F}_\alpha}(f)\right)_k$ which converges pointwise almost every where to $\widetilde{\mathscr{F}_\alpha}(f)$.
Let

$$f_k = h_{\theta(k)}.f = \varphi_{\theta(k)}.$$

Then, $(f_k)_k$ converges to $f$ in $L^2(d\nu_\alpha)$ and $\left(\widetilde{\mathscr{F}_\alpha}(f_k)\right)_k$ converges in $L^2(d\nu_\alpha)$ and pointwise to $\widetilde{\mathscr{F}_\alpha}(f)$. Applying the relation (3.3) to $\dfrac{f_k}{||f_k||_{2,\nu_\alpha}}$, we deduce that for every $k \in \mathbb{N}$,

$$\begin{aligned} &-\int_0^\infty \int_{\mathbb{R}} \left|f_k(r,x)\right|^2 \ln\left(\left|f_k(r,x)\right|\right) d\nu_\alpha(r,x) \\ &-\iint_{\Upsilon_+} \left|\mathscr{F}_\alpha(f_k)(\mu,\lambda)\right|^2 \ln\left(\left|\mathscr{F}_\alpha(f_k)(\mu,\lambda)\right|\right) d\gamma_\alpha(\mu,\lambda) \\ &\geqslant (\alpha+3/2)(1-\ln(2))||f_k||^2_{2,\nu_\alpha} - ||f_k||^2_{2,\nu_\alpha} \ln\left(||f_k||^2_{2,\nu_\alpha}\right). \end{aligned} \tag{3.14}$$

As said above, we have

$$\lim_{k\to+\infty} ||f_k||_{2,\nu_\alpha} = ||f||_{2,\nu_\alpha}. \tag{3.15}$$

On the other hand, there exists $C > 0$ such that for every $k \in \mathbb{N}$,

$$\left|\left|f_k(r,x)\right|^2 . \ln\left(\left|f_k(r,x)\right|^2\right)\right| \leqslant C \left(\left|f(r,x)\right|^2 + \left|f(r,x)\right|^2 \left|\ln\left(\left|f(r,x)\right|^2\right)\right|\right).$$

Again, by dominated convergence theorem,

$$\begin{aligned} &\lim_{k\to+\infty} \int_0^\infty \int_{\mathbb{R}} \left|f_k(r,x)\right|^2 \ln\left(\left|f_k(r,x)\right|^2\right) d\nu_\alpha(r,x) \\ &= \int_0^\infty \int_{\mathbb{R}} \left|f(r,x)\right|^2 \ln\left(\left|f(r,x)\right|^2\right) d\nu_\alpha(r,x). \end{aligned} \tag{3.16}$$

Let us checking

$$\begin{aligned}&\int\int_{\Upsilon_+}\left|\mathscr{F}_\alpha(f_k)(\mu,\lambda)\right|^2\ \ln\left(\left|\mathscr{F}_\alpha(f_k)(\mu,\lambda)\right|\right)d\gamma_\alpha(\mu,\lambda)\\&=\int_0^{\infty}\int_{\mathbb{R}}\left|\widetilde{\mathscr{F}}_\alpha(f_k)(r,x)\right|^2\ \ln\left(\left|\widetilde{\mathscr{F}}_\alpha(f_k)(r,x)\right|\right)d\nu_\alpha(r,x).\end{aligned}$$

Let $\omega_1,\ \omega_2:\ [0,+\infty[\longrightarrow[0,+\infty[$, defined by

$$\omega_1(t)=\begin{cases}t^2\ln t; & \text{if } t\geqslant 1,\\ 0; & \text{if } t\leqslant 1,\end{cases}$$

and

$$\omega_2(t)=\begin{cases}2t^2; & \text{if } t\geqslant 1,\\ -t^2\ln t+2t^2; & \text{if } t\leqslant 1.\end{cases}$$

The functions $\omega_1$ and $\omega_2$ are nondecreasing convex functions on $[0,+\infty[$, and we have

$$\begin{aligned}&\int_0^{\infty}\int_{\mathbb{R}}\omega_1\left(\left|\widetilde{\mathscr{F}}_\alpha(f)\right|\right)(r,x)d\nu_\alpha(r,x)\\&=\int\int_{\left|\widetilde{\mathscr{F}}_\alpha(f)(r,x)\right|\geqslant 1}\left|\widetilde{\mathscr{F}}_\alpha(f)(r,x)\right|^2\ \ln\left(\left|\widetilde{\mathscr{F}}_\alpha(f)(r,x)\right|\right)d\nu_\alpha(r,x)\\&\leqslant\int_0^{\infty}\int_{\mathbb{R}}\left|\widetilde{\mathscr{F}}_\alpha(f)(r,x)\right|^2\ \left|\ln\left(\left|\widetilde{\mathscr{F}}_\alpha(f)(r,x)\right|\right)\right|d\nu_\alpha(r,x)<+\infty.\end{aligned}$$

As the same way,

$$\begin{aligned}&\int_0^{\infty}\int_{\mathbb{R}}\omega_2\left(\left|\widetilde{\mathscr{F}}_\alpha(f)\right|\right)(r,x)d\nu_\alpha(r,x)\\&=\int\int_{\left|\widetilde{\mathscr{F}}_\alpha(f)(r,x)\right|\leqslant 1}\left(2\left|\widetilde{\mathscr{F}}_\alpha(f)(r,x)\right|^2-\left|\widetilde{\mathscr{F}}_\alpha(f)(r,x)\right|^2\ \ln\left(\left|\widetilde{\mathscr{F}}_\alpha(f)(r,x)\right|\right)\right)d\nu_\alpha(r,x)\\&+\int\int_{\left|\widetilde{\mathscr{F}}_\alpha(f)(r,x)\right|\geqslant 1}2\left|\widetilde{\mathscr{F}}_\alpha(f)(r,x)\right|^2d\nu_\alpha(r,x)<+\infty.\end{aligned}$$

On the other hand, we have

$$\widetilde{\mathscr{F}}_\alpha(f_k)=g_{\theta(k)}*\widetilde{\mathscr{F}}_\alpha(f),$$

with $\|g_{\theta(k)}\|_{1,\nu_\alpha} = 1$ for every $k \in \mathbb{N}$. From Lemma 3.4, it follows that for every $i \in \{1,2\}$,

$$\begin{aligned}
&\lim_{k\to+\infty}\int_0^\infty\int_{\mathbb{R}} \omega_i\big(|\widetilde{\mathscr{F}}_\alpha(f_k)|\big)(r,x)d\nu_\alpha(r,x)\\
&=\lim_{k\to+\infty}\int_0^\infty\int_{\mathbb{R}} \omega_i\Big(\big|g_{\theta(k)} * \widetilde{\mathscr{F}}_\alpha(f)\big|\Big)(r,x)d\nu_\alpha(r,x)\\
&=\int_0^\infty\int_{\mathbb{R}} \omega_i\big(|\widetilde{\mathscr{F}}_\alpha(f)|\big)(r,x)d\nu_\alpha(r,x). \qquad (3.17)
\end{aligned}$$

However, for every $t \in [0,+\infty[$, we have

$$t^2\ \ln t \ = \ \omega_1(t)-\omega_2(t)+2t^2. \qquad (3.18)$$

From the relations (3.17) and (3.18), it follows that

$$\begin{aligned}
&\lim_{k\to+\infty}\int_0^\infty\int_{\mathbb{R}} \big|\widetilde{\mathscr{F}}_\alpha(f_k)(r,x)\big|^2\ \ln\big(|\widetilde{\mathscr{F}}_\alpha(f_k)(r,x)|\big)d\nu_\alpha(r,x)\\
&=\int_0^\infty\int_{\mathbb{R}} \big|\widetilde{\mathscr{F}}_\alpha(f)(r,x)\big|^2\ \ln\big(|\widetilde{\mathscr{F}}_\alpha(f)(r,x)|\big)d\nu_\alpha(r,x). \qquad (3.19)
\end{aligned}$$

Using the relations (3.14), (3.15), (3.16) and (3.19) and the fact that $\|f\|_{2,\nu_\alpha} = 1$, we get

$$E_{\nu_\alpha}\big(|f|^2\big)+E_{\gamma_\alpha}\big(|\mathscr{F}_\alpha(f)|^2\big) \geqslant (2\alpha+3)(1-\ln 2),$$

which achieves the proof. □

## 4. Heisenberg-Pauli-Weyl uncertainty principle

In this section; we will show that from the uncertainty principle in terms of entropy, we can find the well known Heisenberg-Pauli-Weyl inequality for the Fourier transform $\mathscr{F}_\alpha$. We recall that this inequality has been proved by the second author and the other in [27], where we have used Hermite and Laguerre orthogonal polynomials.

**Theorem 4.1.** *(Heisenberg-Pauli-Weyl) For every function* $f \in L^2(d\nu_\alpha)$, *we have*

$$\Big(\int_0^\infty\int_{\mathbb{R}}(r^2+x^2)\big|f(r,x)\big|^2 d\nu_\alpha(r,x)\Big)^{1/2}\Big(\int\int_{\Upsilon_+}(\mu^2+2\lambda^2)\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big|^2 d\gamma_\alpha(\mu,\lambda)\Big)^{1/2}$$
$$\geqslant \big(\alpha+3/2\big)\|f\|^2_{2,\nu_\alpha}.$$

*Proof.* For every $s > 0$, we denote by $g_s$ the Gauss Kernel associated with Riemann-Liouville operator, defined by

$$g_s(r,x) = \frac{e^{\frac{-(r^2+x^2)}{2s^2}}}{s^{2\alpha+3}}.$$

Then, for every $s > 0$, we have

$$\int_0^{\infty}\int_{\mathbb{R}} g_s(r,x)d\nu_\alpha(r,x) = 1.$$

This shows that for every $s > 0$,

$$d\mu_{\alpha,s}(r,x) = g_s(r,x)d\nu_\alpha(r,x)$$

is a probability measure on $[0,+\infty[\times\mathbb{R}$. Since $\omega(t) = t\ \ln t$ is a convex function on $[0,+\infty[$, then by Jensen's inequality, for every $f \in\ L^2(d\nu_\alpha)$; $||f||_{2,\nu_\alpha} = 1$, we get

$$\omega\Big(\int_0^{\infty}\int_{\mathbb{R}} \frac{|f(r,x)|^2}{g_s(r,x)} d\mu_{\alpha,s}(r,x)\Big) \ \leqslant\ \int_0^{\infty}\int_{\mathbb{R}} \omega\Big(\frac{|f(r,x)|^2}{g_s(r,x)}\Big) d\mu_{\alpha,s}(r,x),$$

which means that

$$\int_0^{\infty}\int_{\mathbb{R}} |f(r,x)|^2\ \ln\Big(\frac{|f(r,x)|^2}{g_s(r,x)}\Big) d\nu_{\alpha,s}(r,x) \geqslant 0.$$

So,

$$\begin{aligned} -\int_0^{\infty}\int_{\mathbb{R}} |f(r,x)|^2\ \ln\big(|f(r,x)|^2\big) d\nu_\alpha(r,x) \ &\leqslant\ \ln(s^{2\alpha+3})||f||^2_{2,\nu_\alpha} \\ &+\ \frac{1}{2s^2}\int_0^{\infty}\int_{\mathbb{R}} (r^2+x^2)|f(r,x)|^2 d\nu_\alpha(r,x). \end{aligned}$$

Since $||f||_{2,\nu_\alpha} = 1$ and by Definition 3.2 , we get

$$E_{\nu_\alpha}(|f|^2) \ \leqslant\ \ln(s^{2\alpha+3}) + \frac{1}{2s^2}\int_0^{\infty}\int_{\mathbb{R}} (r^2+x^2)|f(r,x)|^2 d\nu_\alpha(r,x). \tag{4.1}$$

On the other hand, the function $\widetilde{\mathscr{F}_\alpha}(f)$ belongs to $L^2(d\nu_\alpha)$ and

$$||\widetilde{\mathscr{F}_\alpha}(f)||_{2,\nu_\alpha} = ||f||_{2,\nu_\alpha} = 1,$$

then the relation (4.1) implies that

$$\begin{aligned} E_{\nu_\alpha}\big(|\widetilde{\mathscr{F}_\alpha}(f)|^2\big) \ &=\ E_{\gamma_\alpha}\big(|\mathscr{F}_\alpha(f)|^2\big) \\ &\leqslant\ \ln(s^{2\alpha+3}) + \frac{1}{2s^2}\int_0^{\infty}\int_{\mathbb{R}} (\mu^2+\lambda^2)|\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)|^2 d\nu_\alpha(\mu,\lambda) \\ &=\ \ln(s^{2\alpha+3}) + \frac{1}{2s^2}\int\int_{\Upsilon_+} (\mu^2+2\lambda^2)|\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda). \end{aligned} \tag{4.2}$$

The relations (4.1) and (4.2) lead to

$$\begin{aligned} E_{\nu_\alpha}\big(|f|^2\big) + E_{\gamma_\alpha}\big(|\mathscr{F}_\alpha(f)|^2\big) & \\ \leqslant 2\ln(s^{2\alpha+3}) + \frac{1}{2s^2}\Big[\int_0^\infty\int_{\mathbb{R}}(r^2+x^2)|f(r,x)|^2 d\nu_\alpha(r,x) & \\ + \int\int_{\Upsilon_+}(\mu^2+2\lambda^2)|\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda)\Big]. & \end{aligned}$$

Using Theorem 3.5, we deduce that for every $s > 0$,

$$\begin{aligned} &\int_0^\infty\int_{\mathbb{R}}(r^2+x^2)|f(r,x)|^2 d\nu_\alpha(r,x) + \int\int_{\Upsilon_+}(\mu^2+2\lambda^2)|\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) \\ &\geqslant 2s^2\big[(2\alpha+3)(1-\ln 2) - 2\ln(s^{2\alpha+3})\big] \\ &= 2s^2\Big[(2\alpha+3) - \ln\big((2s^2)^{2\alpha+3}\big)\Big]. \end{aligned}$$

In particular, for $s = \sqrt{2}/2$, it follows that for every $f \in L^2(d\nu_\alpha)$; $||f||_{2,\nu_\alpha} = 1$, we have

$$\begin{aligned} &\int_0^\infty\int_{\mathbb{R}}(r^2+x^2)|f(r,x)|^2 d\nu_\alpha(r,x) + \int\int_{\Upsilon_+}(\mu^2+2\lambda^2)|\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) \\ &\geqslant 2\alpha+3. \end{aligned} \tag{4.3}$$

Replacing $f$ by $\dfrac{f}{||f||_{2,\nu_\alpha}}$ with $f \in L^2(d\nu_\alpha)$, we get

$$\begin{aligned} &\int_0^\infty\int_{\mathbb{R}}(r^2+x^2)|f(r,x)|^2 d\nu_\alpha(r,x) + \int\int_{\Upsilon_+}(\mu^2+2\lambda^2)|\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) \\ &\geqslant (2\alpha+3)||f||^2_{2,\nu_\alpha}. \end{aligned} \tag{4.4}$$

The inequality (4.4) is sometimes called Heisenberg summation formula.
Now, for every $f \in L^2(d\nu_\alpha)$ and $t > 0$, we define the dilated $f_t$ of $f$ by

$$f_t(r,x) = f(tr,tx).$$

Then, $||f_t||^2_{2,\nu_\alpha} = \frac{1}{t^{2\alpha+3}}||f||^2_{2,\nu_\alpha}$ and for every $(\mu,\lambda) \in \Upsilon$, we have

$$\mathscr{F}_\alpha(f_t)(\mu,\lambda) = \frac{1}{t^{2\alpha+3}}\mathscr{F}_\alpha(f)\big(\frac{\mu}{t},\frac{\lambda}{t}\big).$$

Replacing $f$ by $f_t$ in the relation (4.4), we deduce that for every $f \in L^2(d\nu_\alpha)$ and every real $t > 0$, we have

$$\begin{aligned} &\frac{1}{t^2}\int_0^\infty\int_{\mathbb{R}}(r^2+x^2)|f(r,x)|^2 d\nu_\alpha(r,x) + t^2\int\int_{\Upsilon_+}(\mu^2+2\lambda^2)|\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) \\ &\geqslant (2\alpha+3)||f||^2_{2,\nu_\alpha}. \end{aligned}$$

In particular, if we pick

$$t = \frac{\left(\int_0^\infty \int_{\mathbb{R}} (r^2 + x^2)|f(r,x)|^2 d\nu_\alpha(r,x)\right)^{1/4}}{\left(\int\int_{\Upsilon_+} (\mu^2 + 2\lambda^2)|\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda)\right)^{1/4}},$$

we get

$$\left(\int_0^\infty \int_{\mathbb{R}} (r^2 + x^2)\left|f(r,x)\right|^2 d\nu_\alpha(r,x)\right)^{1/2}\left(\int\int_{\Upsilon_+} (\mu^2 + 2\lambda^2)\left|\mathscr{F}_\alpha(f)(\mu,\lambda)\right|^2 d\gamma_\alpha(\mu,\lambda)\right)^{1/2}$$
$$\geqslant \left(\alpha + 3/2\right)\|f\|^2_{2,\nu_\alpha}.$$

□

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