Topologies generated by nested collections

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Abstract

We study binary relations and topologies induced by means of nested collections of subsets of a given nonempty set. Dually, given a topological space we characterize properties of the topology in terms of suitable nested families of open subsets. By means of this relationship between nested collections of sets and nested topologies, we study the representability of total preorders via (semi-)continuous real-valued order-isomorphisms.

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1 Introduction

In order to study properties of semicontinuous (respectively, continuous) representability of total preorders defined on a topological space, in section 3.4 of [5], the concept of a *semicontinuous* (respectively, a *continuous*) *ordinal family* was introduced. Basically, an ordinal family consists of a suitable nested collection of subsets of a given nonempty set where a topology has been defined. Through these families we may induce semicontinuous (respectively, continuous) total preorders on the given set and analyze their numerical representability by means of semicontinuous (respectively, continuous) real-valued order-preserving functions (see also [8]).

Using also these kinds of families, in [9] topological spaces whose topology is induced by a total preorder (namely, the so-called *preorderable topologies*) were also characterized.

On the other hand, in [28, 29, 30, 31, 32, 1, 17] topologies induced by binary relations defined on a given set have been considered and analyzed in depth. In particular, preorderable topologies (among others) have been characterized in these new terms, as topologies induced by some suitable binary relation with additional properties.

Remark 1.1. As a matter of fact, topologies induced in various ways by binary relations of certain kinds had already been studied in the specialized literature (see e.g. [19, 28, 29, 30, 21, 31, 32, 1, 27, 17]). A suitable introduction to this particular topic is [28]. In this direction, in [29] (see also [30, 31, 32]) the notion of a *relator* was defined on a nonempty set as a nonempty family of reflexive binary relations on X, immediately giving rise to the definition of a topology on X. The restriction of the relations to be reflexive was dropped in [21] and other subsequent papers. Moreover, in what follows we consider *only one* binary relation on the given set X. Needless to say, a single binary relation corresponds to a special kind of relator. More recently, researchers using rough sets (see [23, 1, 27]) have also had an interest in this topic: for instance, in [1] is essentially studied the bitopological space generated by the upper and by the lower threads of a relation, as well as the related closure operators.

The aim of the present paper is twofold. First of all, given a nonempty set we would like to analyze binary relations and topologies that are defined on that set through nested collections of subsets. Secondly, starting now from a topological space, we intend to characterize properties of the given topology in terms of the existence of suitable nested collections of open subsets.

The structure of the paper goes as follows: After the Introduction and Preliminaries, in section 3 we study binary relations and topologies induced by nested collections of subsets of a given set. In section 4 we analyze properties of nested topologies. In section 5 we study properties of a topological space that depend on the semicontinuous (or continuous) representability of total preorders. This study leans on suitable nested collections of open sets, called ordinal families. A final section 6 of further comments closes the paper.

2 Preliminaries

From now on X will denote a nonempty set.

Definition 2.1. A binary relation R on X is a subset of the cartesian product $X^2 = X \times X$. Given two elements $x, y \in X$, we will use the standard notation xRy to express that the pair (x, y) belongs to R. We denote $\Delta_X = \{(x, x) : x \in X\}$. Given two binary relations R, S on X, its composition $S \circ R$ is defined as follows: $S \circ R = \{(x, y) \in X \times X : (x, z) \in S, (z, y) \in R \text{ for some } z \in R\}$.

Associated to a binary relation R on a set X, we consider the binary relations R^c and R^{-1} on X, respectively defined by $R^c = X^2 \setminus R$, and by $R^{-1} = \{(y, x) : (x, y) \in R\}$.

A binary relation R defined on a set X is said to be

(1) *reflexive* if $\Delta_X \subseteq R$,

(2) *irreflexive* if $\Delta_X \cap R = \emptyset$,

(3) *symmetric* if $R = R^{-1}$,

(4) *antisymmetric* if $R \cap R^{-1} \subseteq \Delta_X$,

(5) *asymmetric* if $R \cap R^{-1} = \emptyset$,

(6) *total* if $R \cup R^{-1} = X^2$,

(7) *transitive* if $R \circ R \subseteq R$.

Remark 2.2. In the particular case of *ordered* or *preordered* structures, the standard notation is different. We include it here for sake of completeness.

A *preorder* $\precsim$ on X is a binary relation on X which is reflexive and transitive.

An antisymmetric preorder is said to be a *partial order*. A *total preorder* $\precsim$ on a set X is a preorder such that if $x, y \in X$ then $x \precsim y$ or $y \precsim x$ holds. An antisymmetric total preorder is said to be a *total order*. A total order is also called a *linear order*, and a totally ordered set $(X, \precsim)$ is also said to be a *chain* (see e.g. [11, 12]).

If $\precsim$ is a preorder on X, then as usual we denote the associated *asymmetric* relation by $\prec$ and the associated *equivalence* relation by $\sim$ and these are defined, respectively, by $x \prec y \iff (x \precsim y) \wedge \neg(y \precsim x)$ and by $x \sim y \iff (x \precsim y) \wedge (y \precsim x)$. Also, the associated *dual* preorder $\precsim_d$ is defined by $x \precsim_d y \iff y \precsim x$. The asymmetric part of a linear order is said to be a *strict linear order*.

Remark 2.3. When $\precsim$ is a linear order or, even more generally, when $\precsim$ is reflexive and antisymmetric, we can also define $\prec$ as follows: $x \prec y \iff (x \precsim y) \wedge (x \neq y)$.

Definition 2.4. A *total preorder* $\precsim$ on a set X is said to be *representable* if there exists a real-valued map $u : X \to \mathbb{R}$ such that, for any $x, y \in X$, we have $x \precsim y \Leftrightarrow u(x) \leq u(y)$. The map u is said to be an *order-monomorphism*.

Definition 2.5. Let R be a binary relation on X.

Following [1], given an element $x \in X$, the set $Rx = \{y \in X : yRx\}$ (respectively, the set $xR = \{z \in X : xRz\}$) is said to be the *forset* (respectively, the *afterset*) of the element $x \in X$ as regards the binary relation R.

The family $\{Rx\}_{x \in X}$ (respectively, the family $\{xR\}_{x \in X}$) of all the forsets (respectively, aftersets) of the elements of X with respect to R constitutes the subbasis of a topology, that we denote τ^l_R (respectively, τ^r_R). It is said to be the *left R-topology* (respectively, the *right R-topology*) *induced by the binary relation* R on the set X.

Finally, the topology τ_R whose subbasis is $\{Rx\}_{x \in X} \bigcup \{xR\}_{x \in X}$ is called the *two-sided R-topology*, to which we will refer in the sequel as the *topology induced by the binary relation R*. (See [17] for further information).

Remark 2.6. About the notation and nomenclature used in Definition 2.5 we point out that in some papers (see e.g. [14, 9, 17]) the left R-topology (respectively, the right R-topology) is called "*upper topology*" (respectively "*lower topology*"). However, that nomenclature may cause confusion when compared to the traditional one in the domain theory (see [15]), where, if a partial order $\precsim$ is fixed on a set X, then usually the upper topology is the least topology such that all the *principal lower sets* $\{y \in X : y \precsim x\}$, $x \in X$, are *closed*. Hence such sets form a *closed subbase* of the upper topology, therefore the open sets are *upper sets*. If given a binary operation $R \subseteq X^2$ we had called "upper topology induced by R" to the topology with the (open) subbase $\{y \in X : yRx\}_{x \in X}$, then this upper topology consists of *lower* sets, which probably is not what the readers would expect. By this reason, in the present manuscript we have decided to switch to the terminology "left R-topology" and "right R-topology". In addition, we also point out that some authors (see e.g. [32]) use the notation $R(x)$ (respectively, $R^{-1}(x)$) instead of xR (respectively, instead of Rx).

Definition 2.7. Let X be endowed with a topology τ.

The topology τ is said to be *preorderable* if there exists a total preorder $\precsim$ on X whose associated asymmetric relation $\prec$ induces τ. In addition, τ is said to be *lower preorderable* if there exists a total preorder $\precsim$ on X such that τ coincides with the right $\prec$-topology induced by the asymmetric binary relation $\prec$ on X. (See also [9]).

Remark 2.8. Classical results concerning different kinds of orderability of topologies defined on a set X can be seen in [34, 24, 22, 20, 9]. In particular, in [24] the topologies for which there exists a linear order $\precsim$ on X whose associated asymmetric relation $\prec$ induces the topology, namely the so-called *orderable topologies*, were already characterized. At this stage, we may observe that given a total preorder $\precsim$ defined on a set X, since the associated binary relation $\sim$ is indeed an equivalence, we have that $\precsim$ immediately induces a *linear order* on the quotient space $X/\sim$. Thus, after taking quotients, total preorders become linear orders, and topologies satisfy the separation axiom T_1. Thus, a topology τ on X is *preorderable* if and only if there is an equivalence relation on X such that the associated quotient topology is indeed orderable. Having this in mind, we could think that the characterization of *preorderable* topologies achieved in [9] is a corollary of some of the classical results concerning *orderability* of topologies. However, *this is not the case*. The reason is the following crucial fact: given a topology of which we wonder whether it is preorderable or not, we do not know *a priori* which is the suitable quotient that forces the corresponding quotient topology to be orderable. Therefore, we are obliged to characterize preorderability of topologies *in a totally independent way*, without using the classical results on orderability. (See [9, 17] for a further discussion on these important nuances and subtleties concerning preorderability vs. orderability of a given topology).

Definition 2.9. If X is a set endowed with a total preorder $\precsim$ and τ is a topology on X, then the preorder $\precsim$ is said to be *τ-continuous* if for each $x \in X$ the sets $\{a \in X : x \precsim a\}$ and $\{b \in X : b \precsim x\}$) are τ-closed. In addition, the preorder $\precsim$ is said to be *τ-upper* (respectively, *τ-lower*) *semicontinuous* if for each $x \in X$ the set $\{a \in X : x \precsim a\}$ (respectively, the set $\{b \in X : b \precsim x\}$) is τ-closed.

The topology τ is said to have the *continuous representability property (CRP)* if every continuous total preorder $\precsim$ defined on X admits a representation by means of a continuous real-valued order-monomorphism. (These topologies were studied in [16, 7, 10, 13]). Similarly, τ has the *semicontinuous representability property (SRP)* if every lower (upper) semicontinuous total preorder defined on X is representable by means of a lower (upper) semicontinuous real-valued order-monomorphism. (See e.g. [3, 7, 8]).

Finally, we introduce the main concept of a *nested family*, key of the present paper.

Definition 2.10. A nonempty family $\mathcal{N}$ of subsets of a set X, is said to be *nested* if for any $A, B \in \mathcal{N}$, we have $A \subseteq B$ or $B \subseteq A$. That is, with respect to set-inclusion $\mathcal{N}$ is a totally (linearly) ordered subset of the power set $\mathcal{P}(X)$ of X.

Remark 2.11. Nested families could also be briefly called *chains*, using the terminology introduced in Remark 2.2. By the way, in p. 32 of [18], the term used is "*nests*".

The nested family $\mathcal{N}$ is a basis for a topology on X. This topology, denoted $\tau_{\mathcal{N}}$, is said to be the *natural topology* associated to $\mathcal{N}$.

Given a nested family $\mathcal{N}$ of subsets of X, the relation $R_{\mathcal{N}}$ given by $xR_{\mathcal{N}}y \Leftrightarrow \forall O \in \mathcal{N}$ $(y \in O \Rightarrow x \in O)$ $(x, y \in X)$ is said to be the *natural reflexive binary relation* associated to the nested family $\mathcal{N}$.

Example 2.12. Let U be a nonempty set, usually called *universe*. A *fuzzy subset* X is a map $\mu_X : U \to [0, 1]$. The function μ_X is also said to be the *characteristic function* (or *indicator*) of the fuzzy subset X. (The notations X and μ_X, as well as the terms "fuzzy subset" vs. "characteristic function of a fuzzy subset" are often used interchangeably). Given $\alpha \in [0, 1]$, we define the α-cut of X, denoted X_α, as the (crisp) subset of U given by $X_\alpha = \{t \in U : \mu_X(t) \geq \alpha\}$. Thus, the fuzzy subset X can also be interpreted as a nested family $\{X_\alpha\}_{\alpha \in [0,1]}$ of subsets of U.

Definition 2.13. The binary relation $R^a_{\mathcal{N}}$ defined on X by $zR^a_{\mathcal{N}}t \Leftrightarrow \exists O \in \mathcal{N};\ z \in O, t \notin O$ $(z, t \in X)$ is said to be the *natural irreflexive binary relation* associated to the nested family $\mathcal{N}$. (In [17], $R^a_{\mathcal{N}}$ is also said to be the *adjoint* of the binary relation $R_{\mathcal{N}}$).

Remark 2.14. The binary relation $R^{-1}_{\mathcal{N}}$ defined on X by $xR^{-1}_{\mathcal{N}}y \Leftrightarrow \forall O \in \mathcal{N}$ $(x \in O \Rightarrow y \in O)$ $(x, y \in X)$ is indeed the natural reflexive binary relation associated to the nested family $\{X \setminus O : O \in \mathcal{N}\}$.

Through the concept of a nested collection, just introduced in Definition 2.10, we consider a particular kind of topologies, namely those whose family of open sets is nested.

Definition 2.15. Let X be endowed with a topology τ. The topology $\tau \subseteq \mathcal{P}(X)$ is said to be *nested* if it is a nested collection of subsets of X.

Remark 2.16. The notion of a "*nested topology*" has a disparate meaning in some contexts coming from Enginering. See e.g. [25].

3 Nested collections of subsets of a given set

Let $\mathcal{N} = \{O_\alpha : \alpha \in A\}$ denote a nested family on a set X, where A stands for a set of indices. The following property comes from the definition of the natural topology $\tau_{\mathcal{N}}$.

Proposition 3.1. *The natural topology $\tau_{\mathcal{N}}$ is nested.*

Proof. Given two $\tau_{\mathcal{N}}$-open subsets $\mathcal{U}$ and $\mathcal{V}$, it is clear that $\mathcal{U} = \bigcup_{\alpha \in A_1} O_\alpha$ and $\mathcal{V} = \bigcup_{\beta \in A_2} O_\beta$ for some subsets of indices $A_1, A_2 \subseteq A$. If $\mathcal{V}$ does not contain $\mathcal{U}$, there exists $x \in X$ such that $x \in \mathcal{U} \setminus \mathcal{V}$. Hence, by the definitions of $\mathcal{U}$ and $\mathcal{V}$, it follows that $x \in O_\gamma$ for some $\gamma \in A_1$ and $x \notin O_\beta$ for all $\beta \in A_2$. Thus $O_\gamma \nsubseteq O_\beta$ holds for every $\beta \in A_2$. Since $\mathcal{N}$ is a nested collection, it follows that $O_\beta \subseteq O_\gamma \subseteq \mathcal{U}$ holds for every $\beta \in A_2$. Therefore $\mathcal{V} \subseteq \mathcal{U}$. □

Remark 3.2. Proposition 3.1 is also a direct consequence of the following easy fact: if $\mathcal{N}$ is a nested family, then the families $\{\bigcap \mathcal{A} : \mathcal{A} \subseteq \mathcal{N}\}$ and $\{\bigcup \mathcal{A} : \mathcal{A} \subseteq \mathcal{N}\}$ are both nested.

Let us analyze now the main properties of the natural binary relation $R_{\mathcal{N}}$.

Proposition 3.3. *The binary relation $R_{\mathcal{N}}$ is a total preorder on X.*

Proof. It is clear that $R_{\mathcal{N}}$ is reflexive. But, by definition, it is also transitive: observe that if $xR_{\mathcal{N}}y$ and $yR_{\mathcal{N}}z$ then, for every $\alpha \in A$, we have $z \in O_\alpha \Rightarrow y \in O_\alpha \Rightarrow x \in O_\alpha$, so that $xR_{\mathcal{N}}z$.

Let us prove now that $R_{\mathcal{N}}$ is total: Given $x, y \in X$ with $xR^c_{\mathcal{N}}y$, there exists $\alpha \in A$ such that $y \in O_\alpha,\ x \notin O_\alpha$. Suppose that there exists also an index $\beta \in A$ such that $x \in O_\beta,\ y \notin O_\beta$. It is then clear that neither O_α is contained in O_β nor O_β is contained in O_α. But this contradicts the hypothesis, since the family $\mathcal{N}$ is nested. □

Remark 3.4. Conversely to Proposition 3.3, given a total preorder $\precsim$ on a set X, it is obvious that the family of forsets (aftersets) relative to $\precsim$, as well as the family of forsets (aftersets) relative to $\prec$, are nested. Thus, we may associate, in a natural way, several nested families to a given total preorder.

Once we know that $R_{\mathcal{N}}$ is a total preorder on X, that we may also denote $\precsim_{\mathcal{N}}$, it could be helpful to describe the forsets and aftersets of an element with respect to $\precsim_{\mathcal{N}}$, as well as the forsets and aftersets that correspond to $\prec_{\mathcal{N}}$. This is made through the following Proposition 3.5, whose proof is an immediate consequence of Definition 2.10 and Proposition 3.3.

Proposition 3.5. *Let $\mathcal{N}$ be a nested family on X. Then, for a given $x \in X$ we have that $y \precsim_{\mathcal{N}} x \Leftrightarrow y \in \bigcap\{O \in \mathcal{N}, x \in O\}$. Also, $y \prec_{\mathcal{N}} x \Leftrightarrow y \in \bigcup\{O \in \mathcal{N} :\ x \notin O\}$.*

We may wonder when a given nested family $\mathcal{N}$ on a set X coincides with a family of the forsets of either $\precsim$ or $\prec$, where $\precsim$ is some total preorder defined on X.

The following Theorem 3.6 and Theorem 3.7 answer this question.

Theorem 3.6. *Let $\mathcal{N}$ be a nested family on X. Then there exists a total preorder $\precsim$ on X such that $\mathcal{N}$ coincides with the collection of forsets of $\prec$, that is $\{\prec x\}_{x\in X} = \mathcal{N}$ if and only if the following conditions hold:*

i) For $O \in \mathcal{N},\ \bigcap\{O' \in \mathcal{N} :\ O \subsetneq O'\} \setminus O \neq \emptyset$.

ii) $\bigcup_{O\in\mathcal{N}}(\bigcap\{O' \in \mathcal{N}, O \subsetneq O'\} \setminus O) = X$.

Proof. $(\Rightarrow)$ Suppose that there exists a total preorder $\precsim$ on X such that $\{\prec x\}_{x\in X} = \mathcal{N}$. Given $O \in \mathcal{N}$, choose $x \in X$ such that $\prec x = O$. By definition of $\prec$ and the forset $\prec x$ it is obvious that $x \notin O$. Suppose that there exists $O' \in \mathcal{N}$ such that $O \subsetneq O'$ and $x \notin O'$. Choose $y \in O' \setminus O$. Since $y \notin \prec y$, $\prec y \subsetneq O'$. Then $x \precsim y \precsim x$. Since $\precsim$ is transitive, for $z \in X$ it holds that $z \prec x \Leftrightarrow z \prec y$ and also $x \prec z \Leftrightarrow y \prec z$. Therefore, given $O^* \in \mathcal{N}$ it holds that $y \in O^* \Rightarrow x \in O^*$, which contradicts $x \notin O',\ y \in O'$. Thus, the assumption $O' \in \mathcal{N}$, $O \subsetneq O'$ and $x \notin O'$ has lead to a contradiction. Hence $x \in \bigcap\{O' \in \mathcal{N} :\ O \subsetneq O'\} \setminus O$, which establishes condition i).

Given $x \in X$, as above, for $O \in \mathcal{N}$ such that $O = \prec x$ we have that $x \in \bigcap\{O' \in \mathcal{N}, O \subsetneq O'\} \setminus O$. Hence condition ii) also holds.

$(\Leftarrow)$ Conversely, assuming that conditions i) and ii) hold, we define the preorder $\precsim$ on X as follows: given $x \in X$, by ii), there exists $O \in \mathcal{N}$ such that $x \in (\bigcap\{O' :\ O \subsetneq O'\}) \setminus O$. Let $\prec x = O$. Define another binary relation $\precsim$ on X by $y \precsim x \Leftrightarrow \prec y \subseteq \prec x$. The notation "$\prec$" and "$\precsim$" is justified: first, since $\mathcal{N}$ is nested $\precsim$ is total and transitive and therefore a total preorder (see Proposition 3.3); and second, $\prec$ is the asymmetric part of $\precsim$.

Next, $\{\prec x\}_{x\in X} \subseteq \mathcal{N}$ by the definition of $\prec$. If $O \in \mathcal{N}$, by condition i) there exists $x \in X$ such that $O = \prec x$. Therefore, the collection of forsets $\{\prec x\}_{x\in X}$ coincides with the nested family $\mathcal{N}$. □

Theorem 3.7. *Let $\mathcal{N}$ be a nested family on X. Then there exists a total preorder $\precsim$ on X such that $\mathcal{N}$ coincides with the collection of forsets of $\precsim$, that is $\{\precsim x\}_{x\in X} = \mathcal{N}$ if and only if the following conditions hold:*

i) For $O \in \mathcal{N}$, $O \setminus (\bigcup\{O' \in \mathcal{N} : O' \subsetneq O\}) \neq \emptyset$.

ii) $\bigcup_{O\in\mathcal{N}}[O \setminus (\bigcup\{O' \in \mathcal{N} : O' \subsetneq O\})] = X$.

Proof. It is analogous to the proof of Theorem 3.6. □

Remark 3.8. Obviously, if a nested family $\mathcal{N}$ satisfies the conditions in the statement of Theorem 3.6 (respectively, of Theorem 3.7) then the natural topology $\tau_{\mathcal{N}}$ coincides with the left $\prec$-topology $\tau^l_{\prec}$ (respectively, $\tau^l_{\precsim}$). In the first situation, since $\tau^l_{\prec}$ coincides with the right $\prec^d$-topology $\tau^r_{\prec^d}$ associated to the dual $\prec^d$ of $\prec$ (i.e.: $x \prec^d y \Leftrightarrow y \prec x$, for every $x, y \in X$), it follows that $\tau_{\mathcal{N}}$ is lower preorderable. (See also [8, 9] for further details).

4 Nested topologies

In this section we analyze properties of *nested* topologies (see Definition 2.15 above). To start with, we study what happens when a nested family is indeed a topology.

Proposition 4.1. *Let X be a set endowed with a nested topology τ. Then, considering τ as a nested family of subsets of X, the natural topology associated to τ is again τ.*

Proof. Considered as a nested collection, denote $\tau = \mathcal{N}$. Since $\mathcal{N}$ is a subbasis of $\tau_{\mathcal{N}}$ it follows that $\tau \subseteq \tau_{\mathcal{N}}$. In addition, given a subbasis of a given topology, it is well-known the the topology generated by the subbasis is the coarsest one for which every element of the subbasis is open. Therefore, we also have that $\tau_{\mathcal{N}} \subseteq \tau$. □

Matching Proposition 3.1 and Proposition 4.1 we immediately obtain the following corollary.

Corollary 4.2. *A topology τ is nested if and only if it is the natural topology associated to a nested collection of sets.*

Remark 4.3. If (X, τ) is a topological space, the analysis of properties of τ that could be characterized through suitable nested collections of τ-open sets is indeed equivalent, by Proposition 3.1, to the analysis of properties of τ that could lean on nested subtopologies. of τ. (By a *subtopology* of τ we mean here a topology on X that is coarser than τ.)

The last Remark 4.3 is a new motivation for the study of properties of nested topologies. Throughout this section, we provide further results in this direction. Nested topologies have also been used to get some sufficient conditions for a topology to be induced by a binary relation in [17], section 6. Indeed, next Theorem 4.4 already appeared in [17] (Theorem 6.2). We furnish here a different alternative proof, and include it for the sake of completeness.

Theorem 4.4. *A sufficient condition for a nested topology τ on a set X to be induced by a binary relation R is the existence of collections $\{U_\alpha\}_{\alpha\in A}$ and $\{V_\alpha\}_{\alpha\in A}$ of τ-open sets (where A stands for a set of indices), satisfying the following properties:*

i) $\{V_\alpha\}_{\alpha\in A}$ *is a subbasis for* τ.

ii) *Given* $\alpha,\beta\in A$ *it holds that* $U_\alpha\subseteq U_\beta \Leftrightarrow V_\beta\subseteq V_\alpha$.

iii) *Given* $\alpha\in A$, *either* $U_\alpha\setminus\bigcup_{U_\beta\subsetneq U_\alpha} U_\beta\neq\emptyset$ *or both* $\bigcap_{U_\alpha\subsetneq U_\beta} U_\beta\setminus U_\alpha\neq\emptyset$ *and* $V_\alpha=\bigcup_{V_\beta\subsetneq V_\alpha} V_\beta$.

Proof. Given τ on X, and $\{U_\alpha\}_{\alpha\in A}$, $\{V_\alpha\}_{\alpha\in A}$ satisfying the hypotheses of the statement, we define the binary relation R on X by declaring that xRy if there exists $\alpha\in A$ such that $x\in U_\alpha$ and $y\in V_\alpha$ $(x,y\in X)$.

Observe first that $xR=\bigcup_{x\in U_\alpha} V_\alpha$ and similarly $Rx=\bigcup_{x\in V_\alpha} U_\alpha$. Thus both xR and Rx are τ-open sets, for every $x\in X$.

Let us prove now that the collection $\{xR\}_{x\in X}$ is a subbasis for τ. Fix $\alpha\in A$. If $U_\alpha\setminus\bigcup_{U_\beta\subsetneq U_\alpha} U_\beta\neq\emptyset$ holds, choose an element $x\in U_\alpha\setminus\bigcup_{U_\beta\subsetneq U_\alpha} U_\beta$. Then $xR=\bigcup_{x\in U_\beta} V_\beta=\bigcup_{U_\alpha\subseteq U_\beta} V_\beta$ which by condition ii) is $\bigcup_{V_\beta\subseteq V_\alpha} V_\beta=V_\alpha$. If, otherwise, $\bigcap_{U_\alpha\subsetneq U_\beta} U_\beta\setminus U_\alpha\neq\emptyset$ and $V_\alpha=\bigcup_{V_\beta\subsetneq V_\alpha} V_\beta$ holds, then choose $x\in\bigcap_{U_\alpha\subsetneq U_\beta} U_\beta\setminus U_\alpha$. Then $xR=\bigcup_{x\in U_\beta} V_\beta=\bigcup_{U_\alpha\subsetneq U_\beta} V_\beta$ which, again by condition ii), is $\bigcup_{V_\beta\subsetneq V_\alpha} V_\beta$. But $\bigcup_{V_\beta\subsetneq V_\alpha} V_\beta=V_\alpha$ by condition iii). Since $V_\alpha\in\{xR\}_{x\in X}$ for every $\alpha\in A$ and we have that both xR and Rx are τ-open, for every $x\in X$, the collection $\{xR\}_{x\in X}$ is indeed a subbasis for τ. Therefore $\tau=\tau_R$, in the sense of Definition 2.5. □

The following Example 4.5 shows that the condition introduced in Theorem 4.4 fails to be a necessary condition, in general.

Example 4.5. Let ω denote the first countable ordinal. Let $\mathbb{N}$ be the set of natural numbers including 0. Consider now the set $X=\{-n:n\in\mathbb{N}\}\cup\{k:k\in\mathbb{N}\}\cup\{\omega\}\cup\{\omega+k:k\in\mathbb{N}\}$, endowed with its natural linear order $\precsim$ given by $\ldots-2\prec-1\prec 0\prec 1\prec 2\prec\ldots\prec\omega\prec\omega+1\prec\omega+2\prec\ldots$, and the topology $\tau=\{O_x:x\in X\}\cup\{\emptyset\}\cup\{X\}\cup\{O_\omega\setminus\{\omega\}\}$ where $O_x=\{y\in X:y\precsim x\}$.

Define the binary relation R on X, as follows: $Rx=O_{\omega+(-x)}$ if $x\precsim 0$, $Rx=O_{-x}$ if $0\prec x\prec\omega$ and finally $Rx=\emptyset$ if $\omega\precsim x$.

Then $xR=O_{-x}$ if $x\precsim 0$, $xR=O_0$ if $0\prec x\prec\omega$ and, also, the afterset $(\omega+x)R$ is O_{-x} if $0\precsim x\prec\omega$.

Therefore $\{Rx:x\in X\}=\{\ldots,O_{-2},O_{-1},O_\omega,O_{\omega+1},\ldots\}\cup\{\emptyset\}$ and $\{xR:x\in X\}=\{\ldots,O_{-2},O_{-1},O_0,O_1,O_2,\ldots\}$, so that R induces τ.

Next suppose that there exist collections $\{U_\alpha\}_{\alpha\in A}$ and $\{V_\alpha\}_{\alpha\in A}$ of τ-open sets such that

i) $\{V_\alpha\}_{\alpha\in A}$ is a subbasis for τ and

ii) if $\alpha,\beta\in A$ then $U_\alpha\subseteq U_\beta\Leftrightarrow V_\beta\subseteq V_\alpha$.

By i) and the definition of the topology τ it follows that $\{O_x:x\in X\}\subseteq\{V_\alpha\}_{\alpha\in A}$. Hence, reindexing the subcollection $\{V_\alpha\}_{\alpha\in A}\setminus\{\emptyset,\ X,\ O_\omega\setminus\{\omega\}\}$, we get $\ldots V_{-2}\subsetneq V_{-1}\subsetneq V_0\subsetneq V_1\subsetneq V_2\subsetneq\ldots\subsetneq V_\omega\subsetneq V_{\omega+1}\subsetneq V_{\omega+2}\subsetneq\ldots$, and by ii), this implies $\ldots U_{\omega+2}\subsetneq U_{\omega+1}\subsetneq U_\omega\subsetneq\ldots\subsetneq U_2\subsetneq U_1\subsetneq U_0\subsetneq U_{-1}\subsetneq U_{-2}\subsetneq\ldots$. But there can be no such sequence of inclusions in τ.

Example 4.6. Consider the set $\mathbb{N}$ of natural numbers endowed with the topology $\tau = \{\mathbb{N} \setminus \{0, 1, 2, \ldots, k : k \in \mathbb{N}\}\} \cup \{\emptyset\} \cup \{\mathbb{N}\}$, that is obviously nested. As proved in Example 3.8 in [17], the topology τ is not induced by any binary relation R on $\mathbb{N}$. Observe also that conditions i) and ii) of Theorem 4.4 are incompatible for this topology τ.

Remark 4.7. Observe that this Example 4.6 proves, in particular, that given a nested family $\mathcal{N}$ on a set X, the natural topology $\tau_{\mathcal{N}}$ may fail to coincide with the topology $\tau_{R_{\mathcal{N}}}$ induced by the natural reflexive binary relation $R_{\mathcal{N}}$ associated to $\mathcal{N}$.

Example 4.8. Let $X = [0, 1]$ be endowed with the nested topology $\tau = \{[0, x) : x \in X\} \bigcup \{X\}$. Let $A = (0, 1)$. Given $x \in A$, define $U_x = [0, x)$ and $V_x = [0, 1 - x)$. It is straightforward to see that the collection $\{U_x\}_{x \in A} \bigcup \{V_x\}_{x \in A}$ satisfies the conditions i), ii) and the second option of iii), that appear in the statement of Theorem 4.4. Therefore τ is induced by a binary relation R.

Example 4.9. Let $X = \{C, D, E, F\}$ be endowed with the topology τ defined as follows: $\tau = \{\{C, D, E, F\}, \{D, E, F\}, \{E, F\}, \{F\}, \emptyset, \}$. Let $A = \{1, 2, 3\}$ and $U_1 = \{D, E, F\}, V_1 = \{F\}, U_2 = \{E, F\}, V_2 = \{E, F\}, U_3 = \{F\}, V_3 = \{D, E, F\}$. Then $\{U_\alpha\}_{\alpha \in A} \bigcup \{V_\alpha\}_{\alpha \in A}$ satisfies the conditions i), ii) and the first option of iii), which appear in the statement of Theorem 4.4. Hence τ is induced by a binary relation R.

Remark 4.10. In the conditions of Theorem 4.4, for a nested topology τ on a set X that is induced by a binary relation R, we may *not* expect that R coincides with R_τ, where R_τ stands for the natural reflexive binary relation associated to the nested topology τ considered as a nested collection on X. (See Definition 2.10). As a matter of fact, by Proposition 3.3, the relation R_τ is a total preorder, so that if X is finite as in Example 4.9, the topology induced by the binary relation R_τ (see Definition 2.5) is the discrete one. In other words, in Example 4.9 we have that $R \neq R_\tau$ and $\tau = \tau_R \neq \tau_{R_\tau}$.

Definition 4.11. Let X be a set endowed with a total preorder $\precsim$. Let S be a nonempty subset of X. An element $s \in S$ is said to be *maximal of the subset S as regards the total preorder $\precsim$* if for every $t \in S$ it holds that $t \precsim s$.

Definition 4.12. Let τ be a nested topology on a set X endowed with a total preorder $\precsim$.

A map $f : X \to \tau$ is said to be *monotone* if for $x, y \in X$, $y \precsim x \Rightarrow f(x) \subseteq f(y)$.

For $x \in X$ we define the set $U_x = \bigcap\{U \in \tau : x \in U\}$ if $\bigcap\{U \in \tau : x \in U\} \in \tau$, and $U_x = \bigcup\{U \in \tau : x \notin U\}$ otherwise. The subset U_x is said to be the *characteristic* set of the element $x \in X$ with respect to the nested topology τ.

Proposition 4.13. *Let τ be a nested topology on a set X endowed with the total preorder $\precsim$ defined by τ. The following statements are equivalent:*

i) There exists a binary relation R on X that induces τ and such that $\{xR\}$ is a basis for τ.

ii) There exists a monotone function $f : X \to \tau$ such that $f(X)$ is a basis for τ and such that for every $m \in X$ it holds that $m \in U_m$ if m is a maximal element of the set $\{x \in X : z \in f(x)\}$ as regards $\precsim$, for some $z \in f(m)$.

Proof. ($\Rightarrow$) Define $f : X \to \tau$ by $f(x) = xR$ for $x \in X$. Suppose $x, y, z \in X$ are such that $y \precsim x$ and $z \in f(x)$. Then $z \in xR$, $x \in Rz$ and $y \in Rz$ by the definition of $\precsim$. Thus, $z \in yR$ and $z \in f(y)$. Therefore $f(x) \subseteq f(y)$ and f is monotone. Observe that $f(X) = \{xR : x \in X\}$, which is a basis of τ by hypothesis. Finally, assume that $m, z \in X$ are such that m is a maximal element, as regards $\precsim$, of the set $\{x \in X : z \in f(x)\}$. Then $z \in mR$ and m is a maximal with respect to $\precsim$ of the set Rz. Therefore $Rz = \{y \in X : y \precsim m\}$. By Proposition 3.5, $Rz = \bigcap\{U : U \in \tau,\ x \in U\}$ and since $Rz \in \tau$, it holds that $Rz = U_m$ (characteristic set of the element $m \in X$), by Definition 4.11. In addition, since $m \in Rz$ we have that $m \in U_m$.

($\Leftarrow$) Define R on X by $xR = f(x)$ for $x \in X$. Then $\{f(x) = xR : x \in X\} = f(X)$ is a basis of τ by hypothesis.

It remains to show that $Rz \in \tau$ for all $z \in X$. By definition, given $z \in X$ we have that $Rz = \{x \in X : z \in f(x)\} = \bigcup_{z \in f(x)} \{y \in X : y \precsim x\}$ since f is monotone.

We distinguish the following two cases:

- *Case 1.* There exists at least one maximal element, as regards $\precsim$, of the set Rz. In this case we choose a maximal element m of that set. It is straightforward to see that m is maximal,with respect to $\precsim$, for $\{x \in X : z \in f(x)\}$. Thus $Rz = \{y \in X : y \precsim m\} = \bigcap\{U : m \in U,\ U \in \tau\}$, by Proposition 3.5. By the maximality property of m, it follows by hypothesis that $m \in U_m$. Therefore, by Definition 4.12 the characteristic set U_m is actually $U_m = \bigcap\{U : m \in U,\ U \in \tau\}$, so that $Rz \in \tau$.
- *Case 2.* The set Rz has no maximum with respect to $\precsim$. Then $Rz = \bigcup_{z \in f(x)} \{y \in X : y \precsim x\} = \bigcup_{z \in f(x)} \{y \in X : y \prec x\} = \bigcup_{z \in f(x)} U_x$ since $\{y \in X : y \prec x\} \subseteq U_x \subseteq \{y \in X : y \precsim x\}$ by the hypothesis and Definition 4.12. Therefore $Rz \in \tau$.

$\square$

5 Ordinal families

When looking for topological conditions that could characterize the fact of a topological space (X, τ) satisfying the continuous (respectively, semicontinuous) representability property CRP (respectively, SRP) we pay attention to the idea that any continuous (respectively, semicontinuous) totally preordered structure that we could define on that topological space would have associated in a natural way a nested collection of τ-open sets satisfying suitable additional properties. In this way, we introduce the concept of a *semicontinuous* (respectively, a *continuous*) *ordinal family*. (See also [9]).

Definition 5.1. Let (X, τ) be a topological space. A collection $\mathcal{O} = \{O_\alpha\}_{\alpha \in A}$ of subsets of X (where A denotes a set of indices), is said to be a *semicontinuous ordinal family* in (X, τ) if it satisfies the following conditions:

(i) $O_\alpha \in \tau$, for every $\alpha \in A$. (In other words, $\mathcal{O}$ consists of τ-open sets).

(ii) For every $\alpha, \beta \in A$, $O_\alpha \subseteq O_\beta$ or $O_\beta \subseteq O_\alpha$. (In other words, $\mathcal{O}$ is nested).

(iii) For every $\alpha \in A$, $(\bigcap_{\gamma \in A, O_\alpha \subsetneq O_\gamma} O_\gamma \setminus O_\alpha) \neq \emptyset$.

In addition, $\mathcal{O}$ is said to be a *continuous ordinal family* in (X, τ) if it is a semicontinuous ordinal family and satisfies:

(iv) For every $x \in X$, $\bigcup_{\alpha \in A, x \in O_\alpha} (X \setminus O_\alpha) \in \tau$.

Let (X, τ) be a topological space. We analyze now some questions related to the existence of semicontinuous ordinal families. First of all, given a semicontinuous ordinal family we could consider some associated preorders that satisfy certain special properties.

Proposition 5.2. *Let (X,τ) be a topological space. let $\mathcal{O} = \{O_\alpha\}_{\alpha\in A}$ a semicontinuous ordinal family. The binary relation $\precsim_{\mathcal{O}}$ given by $x \precsim_{\mathcal{O}} y \Leftrightarrow \forall \alpha \in A\ (y \in O_\alpha \Rightarrow x \in O_\alpha)$ is an upper semicontinuous total preorder on (X,τ).*

Proof. The fact that $\precsim_{\mathcal{O}}$ is a total preorder has already been established in Proposition 3.3. Let us see that $\precsim_{\mathcal{O}}$ is upper semicontinuous. Given $x \in X$ we have that the forset $\prec_{\mathcal{O}} x$ of the element x with respect to the binary relation $\prec_{\mathcal{O}}$ is $\bigcup_{\alpha\in A; x\notin O_\alpha} O_\alpha$. Thus $\prec_{\mathcal{O}} x \in \tau$, since each $O_\alpha \in \tau$. Therefore the preorder is upper semicontinuous. □

Remark 5.3. Let us see some properties of the preorder $\precsim_{\mathcal{O}}$ defined from a semicontinuous ordinal family $\mathcal{O} = \{O_\alpha\}_{\alpha\in A}$. The properties can be checked straightforwardly. (Compare to Proposition 3.5).

(*i*) For every $\alpha \in A$ we have that O_α is the forset $\prec_{\mathcal{O}} x$, where the element x belongs to $(\bigcap_{\gamma\in A, O_\alpha \subsetneq O_\gamma} O_\gamma \setminus O_\alpha)$.

Indeed, it is easy to see that if $x \in (\bigcap_{\gamma\in A; O_\alpha \subsetneq O_\gamma} O_\gamma \setminus O_\alpha)$, then $[x]_{\mathcal{O}} = \{y \in X : y \precsim_{\mathcal{O}} x \precsim_{\mathcal{O}} y\} = (\bigcap_{\gamma\in A; O_\alpha \subsetneq O_\gamma} O_\gamma \setminus O_\alpha)$.

(*ii*) Although we are considering the upper semicontinuous case, so that in general the aftersets that correspond to $\prec_{\mathcal{O}}$ may fail to be open sets, we still can say how those aftersets could be described. This goes as follows:

(a) For each $x \in X$, the afterset $x\prec_{\mathcal{O}}$ of the element x with respect to the binary relation $\prec_{\mathcal{O}}$ is $X \setminus \bigcap_{\alpha\in A; x\in O_\alpha} O_\alpha$.

(b) In addition, given an element $x \in (\bigcap_{\gamma\in A; O_\alpha \subsetneq O_\gamma} O_\gamma \setminus O_\alpha)$, we have that the afterset $x\prec_{\mathcal{O}}$ is $X \setminus \bigcap_{\gamma\in A; O_\alpha \subsetneq O_\gamma} O_\gamma$.

We have seen that given a semicontinuous ordinal family we may consider an associated upper semicontinuous total preorder of which we know its structure and main properties. Now, we will directly start from an upper semicontinuous total preorder, and we see that we can also define a semicontinuous ordinal family associated in a natural way to the given preorder.

Proposition 5.4. *Let (X,τ) be a topological space and $\precsim$ an upper semicontinuous total preorder defined on it. Then, the family of forsets of $\prec$, namely $\mathcal{F}_{\precsim} = \{\prec x\}_{x\in X}$ is a semicontinuous ordinal family on (X,τ).*

Proof. We know that, for each $x \in X$, the forset $\prec x$ is τ-open, since $\precsim$ is upper semicontinuous. Moreover, for each $x \in X$ we have that $x \in (\bigcap_{z\in X; \prec x \subsetneq \prec z} \prec z \setminus \prec x)$, so that $(\bigcap_{z\in X; \prec x \subsetneq \prec z} \prec z \setminus \prec x) \neq \emptyset$. Also, for every $x, y \in X$ we have that $x \precsim y$ or $y \precsim x$ holds. Hence it follows $\prec x \subseteq \prec y$, or else $\prec y \subseteq \prec x$. Therefore $\mathcal{F}_{\precsim}$ is a semicontinuous ordinal family. □

Remark 5.5. Let (X,τ) be a topological space:

i) Given $\precsim$, an upper semicontinuous total preorder, we consider the semicontinuous ordinal family associated to $\precsim$, that we denote $\mathcal{F}_{\precsim}$. Now we consider the corresponding preorder associated in a natural way to that ordinal family. We denote this new preorder by $\precsim_{\mathcal{F}_{\precsim}}$. That is, we follow this scheme: $\precsim \longmapsto \mathcal{F}_{\precsim} \longmapsto \precsim_{\mathcal{F}_{\precsim}}$. As a matter of fact, these two preorders coincide: $x \precsim y \iff \forall z \in X(y \in \prec z \Rightarrow x \in \prec z) \iff x \precsim_{\mathcal{F}_{\precsim}} y$. Therefore, any total preorder can be understood, in particular, as the total preorder associated to an ordinal family, $\mathcal{F}_{\precsim}$.

ii) Now, we start considering a semicontinuous ordinal family $\mathcal{O} = \{O_\alpha\}_{\alpha \in A}$ and we consider its associated total preorder $\precsim_{\mathcal{O}}$. From this preorder, we build the associated semicontinuous ordinal family $\mathcal{F}_{\precsim_{\mathcal{O}}}$. Now we follow the scheme: $\mathcal{O} = \{O_\alpha\}_{\alpha \in A} \longmapsto \precsim_{\mathcal{O}} \longmapsto \mathcal{F}_{\precsim_{\mathcal{O}}} = \{\prec_{\mathcal{O}} x\}_{x \in X}$. In this second (dual) situation, the coincidence could *fail* to be true, in general. However, when the ordinal family that we had ab initio is stable under unions, the coincidence is established. This is true because, for every $x \in X$, the forset $\prec_{\mathcal{O}} x$ is $\bigcup_{\alpha \in A, x \notin O_\alpha} O_\alpha$. What is always true (in the general case) is that $\mathcal{O} \subseteq \mathcal{F}_{\precsim_{\mathcal{O}}}$ (see Remark 5.3.(i)). Consequently, we can not say that any ordinal family is associated to some total preorder, but we can say, at least, that it is *contained* in the ordinal family that is induced by its own associated total preorder.

Remark 5.6. We can also work with *lower semicontinuous* total preorders, in an entirely analogous way. The proofs are indeed similar to the ones given for the upper semicontinuous case in Proposition 5.2 and Proposition 5.4.

We provide now (see Theorem 5.8 below) a topological characterization of the semicontinuous representability property SRP in terms of ordinal families, already issued in [9].

Definition 5.7. Given a topological space (X, τ) and a semicontinuous ordinal family $\mathcal{O} = \{O_\alpha\}_{\alpha \in A}$ we define, for every $x \in X$, the set $O^*_x = \bigcup_{\alpha \in A, x \notin O_\alpha} O_\alpha$. This set is said to be the *vanishing set* of the element x with respect to $\mathcal{O}$. (See also [9]).

Theorem 5.8. *Let (X, τ) be a topological space. The following are equivalent:*

(i) *τ satisfies SRP.*

(ii) *For every semicontinuous ordinal family $\mathcal{O} = \{O_\alpha\}_{\alpha \in A}$, there exists a countable collection $\{x_n\}_{n \in \mathbb{N}} \subseteq X$, such that if $O_\alpha \subset O_\beta$, then there exists $n \in \mathbb{N}$ such that $O_\alpha \subseteq O^*_{x_n} \subseteq O_\beta$.*

Proof. See Theorem 5.3.(ii) in [9]. □

We have already seen that there exists a close relationship between ordinal families and certain total preorders. Important particular cases of that relationship appear when the ordinal family is indexed in $[0, \Omega)$. In these cases, we can establish a link between SRP and two important topological properties, namely to be hereditarily separable and to be hereditarily Lindelöf. (See Theorem 5.11 below).

Definition 5.9. Given a topological space (X, τ), an uncountable collection of elements of X, say $(x_\alpha)_{\alpha < \Omega}$, is said to be *right-separated* (respectively, *left-separated*), if for every ordinal $\alpha < \Omega$ we have that x_α does not belong to the τ-closure $\overline{B_\alpha}$ of the set $B_\alpha = \{x_\beta \mid \alpha < \beta < \Omega\}$ (respectively, if for every ordinal $\alpha < \Omega$ we have that x_α does not belong to the τ-closure $\overline{C_\alpha}$ of the set $C_\alpha = \{x_\beta \mid \beta < \alpha\}$).

Lemma 5.10. *A topological space* (X,τ) *is hereditarily Lindelöf (respectively, hereditarily separable) if and only if it does not contain any uncountable right-separated family (respectively, if and only if it does not contain any uncountable left-separated family).*

Proof. See e.g. Theorem 3.1 in [26]. □

Theorem 5.11. *Let* (X,τ) *be a topological space that satisfies SRP. Then* τ *is both hereditarily separable and hereditarily Lindelöf.*

Proof. (See also Theorem 4.8 in [8], as well as Lemma 4.1 and Proposition 4.2 in [3] and Lemma 2,.3 in [6]). The proof we provide here has the particular feature of leaning on the concepts of right-separated and left-separated families.

First of all, assume that (X,τ) is not hereditarily Lindelöf. In this case, by Lemma 5.10, (X,τ) contains a right-separated family $(x_\alpha)_{\alpha<\Omega}$, such that for every $\alpha<\Omega$ we have that x_α does not belong to the τ-closure $\overline{B_\alpha}$, where $B_\alpha=\{x_\beta \mid \alpha<\beta<\Omega\}$. Denote, for each $\alpha<\Omega$, $U_\alpha = X\setminus \overline{B_\alpha}$. We observe that, for each $\alpha<\Omega$, we have that $x_\alpha\in U_\alpha$ and, in addition, $U_\alpha\in\tau$. Calling $O_0=\emptyset$ and $O_\alpha=\bigcup_{\gamma<\alpha}U_\gamma$ $(0<\alpha<\Omega)$, it is straightforward to see that $\mathcal{O}=\{O_\alpha\}_{\alpha\in[0,\Omega)}$ is a semicontinuous ordinal family, such that for each $\alpha,\beta\in[0,\Omega)$ it holds that $\alpha\prec\beta\Leftrightarrow O_\alpha\subsetneq O_\beta$. As in Proposition 5.2, now we may endow (X,τ) with an upper semicontinuous total preorder associated to that semicontinuous ordinal family. Given $x,y\in X$ we define: $x\precsim_{\mathcal{O}} y\Leftrightarrow\forall\alpha\in A\ (y\in O_\alpha\Rightarrow x\in O_\alpha)$. By construction, the preordered set $(X,\precsim_{\mathcal{O}})$ contains a copy that is order-isomorphic to $[0,\Omega)$, hence it is not representable (see [2]). Thus, in particular, there is no upper semicontinuous utility function representing $\precsim_{\mathcal{O}}$.

Assume now that (X,τ) is not hereditarily separable. In this case, by Theorem 5.10, (X,τ) contains a left-separated family $(x_\alpha)_{\alpha<\Omega}$, such that for every ordinal $\alpha<\Omega$ we have that x_α does not belong to the τ-closure $\overline{C_\alpha}$, where $C_\alpha=\{x_\beta\mid\beta<\alpha\}$. Denote, for each $\alpha<\Omega$, $U_\alpha=X\setminus\overline{C_\alpha}$. We observe that, for each $\alpha<\Omega$, we have that $x_\alpha\in U_\alpha$. In addition, $U_\alpha\in\tau$. Calling $O_\alpha=U_{\alpha+1}$ $(0\leq\alpha<\Omega)$, it is easy to check that $\mathcal{O}=\{O_\alpha\}_{\alpha\in[0,\Omega)}$ is a semicontinuous ordinal family, satisfying also that for every $\alpha,\beta\in[0,\Omega)$ it holds that $\alpha\prec\beta\Leftrightarrow O_\beta\subsetneq O_\alpha$. Again as in Proposition 5.2 (see also Remark 5.6) we may define on (X,τ) a lower semicontinuous total preorder associated to that semicontinuous ordinal family. Given $x,y\in X$, we define: $x\precsim'_{\mathcal{O}} y\Leftrightarrow\forall\alpha\in A\ (x\in O_\alpha\Rightarrow y\in O_\alpha)$. By construction, the preordered set $(X,\precsim'_{\mathcal{O}})$ contains an order-isomorphic copy of $[0,\Omega)$, so that it is not representable. In particular, there is no lower semicontinuous utility function that represents $\precsim'_{\mathcal{O}}$. □

Remark 5.12.

i) In [3] a similar result appears, but without mentioning that the topological property of hereditary separability and that of being hereditarily Lindelöf can be linked to the existence of certain particular preorders.

ii) As already mentioned in [3, 8] the converse of Theorem 5.11 is not true in general. However, in some special topologies (e.g., some classical topologies on a Banach space) it is indeed true that SRP is equivalent to being hereditarily separable and hereditarily Lindelöf. (See e.g. Theorem 5.3 in [8]).

iii) From Theorem 5.11 it follows that SRP implies separability. The analogous result for CRP is *not* true (see [7])). Therefore Theorem 5.11 cannot be extended to the continuous case.

Now we analyze some questions related to *continuous* ordinal families.

Proposition 5.13. *Let (X, τ) be a topological space. Let $\mathcal{O} = \{O_\alpha\}_{\alpha \in A}$ be a continuous ordinal family. The binary relation $\precsim_{\mathcal{O}}$ given by $x \precsim_{\mathcal{O}} y \Leftrightarrow \forall \alpha \in A$ $(y \in O_\alpha \Rightarrow x \in O_\alpha)$ is a continuous total preorder defined on (X, τ).*

Proof. By Proposition 5.2, it is enough to see that $\precsim_{\mathcal{O}}$ is lower semicontinuous. And this follows because, by Remark 5.3.(ii), for each $x \in X$, the afterset $x \prec_{\mathcal{O}}$ is $X \setminus \bigcap\limits_{\alpha \in A, x \in O_\alpha} O_\alpha = \bigcup_{\alpha \in A, x \in O_\alpha} (X \setminus O_\alpha)$. Therefore the afterset $x \prec_{\mathcal{O}}$ belongs to the topology τ because the ordinal family is continuous. □

Proposition 5.14. *Let (X, τ) be a topological space and $\precsim$ a continuous total preorder defined on it. Then the family $\mathcal{F}_{\precsim} = \{\prec x\}_{x \in X}$ is a continuous ordinal family on (X, τ).*

Proof. By Proposition 5.4, it is enough to check the last condition arising in the definition of a continuous ordinal family. Thus, given an element $x \in X$ we have that $\bigcup_{y \in X, x \in \prec y} (X \setminus \prec y)$ is the afterset $x \prec$ of the element X as regards the binary relation $\prec$. And we know that the afterset $x \prec$ is τ-open because the given preorder is continuous. □

Remark 5.15. As in the semicontinuous case, we could do a study similar to that in Remark 5.5.

Now we characterize *CRP* in terms of continuous ordinal families.

Theorem 5.16. *Let (X, τ) a topological space. The following are equivalent:*

- (i) *τ satisfies CRP.*
- (ii) *For every continuous ordinal family $\mathcal{O} = \{O_\alpha\}_{\alpha \in A}$, there exists a countable collection $\{x_n\}_{n \in \mathbb{N}} \subseteq X$, such that if $O_\alpha \subset O_\beta$, there exists $n \in \mathbb{N}$ with $O_\alpha \subseteq O^*_{x_n} \subseteq O_\beta$.*

Proof. It is analogous to Theorem 5.8. □

Remark 5.17. From Theorem 5.8 and Theorem 5.16 we immediately get that SRP implies CRP. The converse is not true. (See also Remark 5.12 (iii)).

6 Further comments

In [9] the key properties CRP and SRP were also characterized in terms of, respectively, lower preorderable topologies and preorderable topologies. Namely, a topological space (X, τ) satisfies the continuous representability property CRP (respectively, satisfies the semicontinuous representability property SRP) if and only if all its preorderable (respectively, lower preorderable) subtopologies satisfy the second countability axiom. (See [9], Theorem 5.1). In addition, both preorderable and lower preorderable topologies were characterized there, completing the panorama on *orderability* of topologies initiated in classical works (see e.g. [33, 34, 24]).

Actually, in [9], Theorem 3.1, given a topological space (X, τ) it is proved that the topology τ is lower preorderable if and only if it has a basis $\mathcal{B} = \{O_\alpha \subseteq X : \alpha \in A\}$, satisfying the following two conditions: (a) The family $\mathcal{B}$ is nested, and (b) for every

$\alpha \in A$, $(\bigcap_{\gamma\in A, O_\alpha \subsetneq O_\gamma} O_\gamma \setminus O_\alpha) \neq \emptyset$. Moreover, the topology τ is preorderable if and only if it has a subbasis $\mathcal{S} = \{O_\alpha \subseteq X : \alpha \in A\} \bigcup \{P_x : x \in X\}$, where the part $\mathcal{B} = \{O_\alpha \subsetneq X : \alpha \in A\}$ satisfies the above conditions (a) and (b), and $\mathcal{S}$ also satisfies the new condition (c), namely for every $x \in X$, $P_x = \bigcup_{\alpha\in A, x\in O_\alpha} (X \setminus O_\alpha)$. (Here A stands for a set of indices).

As a matter of fact, if we compare this result with Definition 5.1 we immediately realize again the close relationship between ordinal families and the satisfaction of CRP and SRP (characterized in terms of preorderable and lower preorderable topologies). Moreover, we see that these latter concepts are characterized in terms of *nested* collections of subsets, so motivating again the ideas of the present manuscript. In this direction, the use of nested subtopologies to characterize orderability properties of a given topological space had already appeared, implicitly, in [33].

(By the way, other alternative characterizations of preorderable and lower preorderable topologies, established in terms of topologies induced by binary relations, appear in [17], Corollary 5.2).

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