Normality criteria of meromorphic functions sharing a holomorphic function*

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Abstract

Take three integers $m \geq 0$, $k \geq 1$ and $n \geq 2$. Let a ($\not\equiv 0$) be a holomorphic function in a domain D of $\mathbb{C}$ such that multiplicities of zeros of a are at most m and divisible by $n+1$. In this paper, we mainly obtain the following normality criterion: Let $\mathscr{F}$ be the family of meromorphic functions on D such that multiplicities of zeros of each $f \in \mathscr{F}$ are at least $k+m$ and such that multiplicities of poles of f are at least $m+1$. If each pair (f, g) of $\mathscr{F}$ satisfies that $f^n f^{(k)}$ and $g^n g^{(k)}$ share a (ignoring multiplicity), then $\mathscr{F}$ is normal.

1 Introduction

In this paper, we use the standard notations of the Nevanlinna theory as presented in [11, 17, 50, 52]. By definition, two meromorphic functions F and G are said to share a IM if $F-a$ and $G-a$ assume the same zeros ignoring multiplicity. When $a=\infty$ the zeros of $F-a$ mean the poles of F.

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Let D be a domain in $\mathbb{C}$ and let $\mathscr{F}$ be meromorphic functions defined in the domain D. Then $\mathscr{F}$ is said to be normal in D, in the sense of Montel, if for any sequence $\{f_n\} \subset \mathscr{F}$ there exists a subsequence $\{f_{n_j}\}$ such that f_{n_j} converges spherically locally uniformly in D, to a meromorphic function or ∞ (cf. [15, 38]). For simplicity, we take $\rightarrow$ to stand for convergence and $\rightrightarrows$ for convergence spherically locally uniformly.

Let $\mathcal{M}(D)$ (resp. $\mathcal{A}(D)$) be the set of meromorphic (resp. holomorphic) functions on D. Let n be an integer and take a positive integer k. We will study normality of the subset $\mathscr{F}$ of $\mathcal{M}(D)$ such that $f^n f^{(k)}$ satisfies some conditions for each $f \in \mathscr{F}$.

First of all, we look at some background for the case $n = 0$. Hayman [17] proved that if $F \in \mathcal{M}(\mathbb{C})$ is transcendental, then either $F^{(k)}$ assumes every finite non-zero complex number infinitely often for any positive integer k, or F assumes every finite complex number infinitely often. A normality criterion corresponding to Hayman's theorem is obtained by Gu [14] which is stated as follows: If $\mathscr{F}$ is the family in $\mathcal{M}(D)$ such that each $f \in \mathscr{F}$ satisfies $f^{(k)} \neq a$ and $f \neq b$, where a, b are two complex numbers with $a \neq 0$, then $\mathscr{F}$ is normal in the sense of Montel. In particular, if $\mathscr{F} \subset \mathcal{A}(D)$, the normality criterion was conjectured by Montel (see [38], p.125) for $k = 1$, and proved by Miranda [30]. Further, Yang [51] and Schwick [40] confirmed that the normality criterion due to Gu is true if a is replaced by a non-zero holomorphic function on D. In 2001, Jiang and Gao [22] proved that if $\mathscr{F}$ is the family in $\mathcal{A}(D)$ such that the multiplicities of zeros of each $f \in \mathscr{F}$ are least $k + m + 2$ for another non-negative integer m and such that each pair (f, g) of $\mathscr{F}$ satisfies that $f^{(k)}$ and $g^{(k)}$ share a IM (ignoring multiplicity), where $a \in \mathcal{A}(D)$ and multiplicities of zeros of a are at most m, then F is normal in D, and obtained a similar result when $\mathscr{F} \subset \mathcal{M}(D)$. For other generations, see [3], [4], [5], [10], [23], [27], [28], [43], [44] and [46].

Next we introduce some developments for the case $n \geq 1$ and $k = 1$. In 1959, Hayman [16] proposed a conjecture: If $F \in \mathcal{M}(\mathbb{C})$ is transcendental, then $F^n F'$ assumes every finite non-zero complex number infinitely often for any positive integer n. Hayman himself [16, 18] showed it is true for $n \geq 3$, and for $n = 2$, $F \in \mathcal{A}(\mathbb{C})$. Mues [31] confirmed the conjecture for $n = 2$ in 1979. Furthermore, the case of $n = 1$ was considered by Clunie [9] when $F \in \mathcal{A}(\mathbb{C})$; finally settled by Bergweiler and Eremenko [2], Chen and Fang [6]. Related to these results on value distribution, Hayman [18] conjectured that if $\mathscr{F}$ is the family of $\mathcal{M}(D)$ such that each $f \in \mathscr{F}$ satisfies $f^n f' \neq a$ for a positive integer n and a non-zero complex number a, then $\mathscr{F}$ is normal. This conjecture has been confirmed by Yang and Zhang [54] (for $n \geq 5$, and for $n \geq 2$ with $\mathscr{F} \subset \mathcal{A}(D)$), Gu [13] (for $n = 3$, 4), Pang [34] (for $n \geq 2$; cf. [12]), and Oshkin [32] (for $n = 1$ with $\mathscr{F} \subset \mathcal{A}(D)$; cf. [24]). Finally, Pang [34] (or see [6, 55, 56]) indicated that the conjecture for $n = 1$ is a consequence of his theorem and Chen-Fang's theorem [6]. Recently, based on the ideas of sharing values, Zhang [58] proved that if $\mathscr{F}$ is the family of $\mathcal{M}(D)$ such that each pair (f, g) of $\mathscr{F}$ satisfies that $f^n f'$ and $g^n g'$ share a finite non-zero complex number a IM for $n \geq 2$, then $\mathscr{F}$ is normal. There are examples showing that this result is not true for the case $n = 1$. Further, Jiang [22] concluded that if $\mathscr{F}$ is the family of $\mathcal{M}(D)$ such that each pair (f, g) of $\mathscr{F}$ satisfies that $f^n f'$ and $g^n g'$ share a IM for $n \geq 2m + 2$, where $a \in \mathcal{A}(D)$ and multiplicities of zeros of a are at most m, then $\mathscr{F}$ is normal.

Similarly, we also have analogues related to some conditions of $f\left(f^{(k)}\right)^l$ for a positive integer l. For example, Zhang and Song [60] announced that if $F \in \mathcal{M}(\mathbb{C})$ is transcendental; $a(\not\equiv 0)$ a small function of F; $l \geq 2$, then $F\left(F^{(k)}\right)^l - a$ has infinitely many zeros. A simple proof was given by Alotaibi [1]. The normality criterion corresponding to this result was obtained by Jiang and Gao [21] which is stated as follows: Let $l, k \geq 2, m \geq 0$ be three integers such that m is divisible by $l + 1$ and suppose that $a(\not\equiv 0)$ is a holomorphic function in D with zeros of multiplicity m. If $\mathscr{F}$ is the family of $\mathcal{A}(D)$ (resp. $\mathcal{M}(D)$) such that multiplicities of zeros of each $f \in \mathscr{F}$ is at least $k + m$ (resp. $\max\{k + m, 2m + 2\}$) and such that each pair (f, g) of $\mathscr{F}$ satisfies that $f\left(f^{(k)}\right)^l$ and $g\left(g^{(k)}\right)^l$ share a IM, then $\mathscr{F}$ is normal. For more results related to this topic, see Hennekemper [19], Hu and Meng [20], Li [25, 26], Schwick [39], Wang and Fang [42], C. Yang, L. Yang and Y. Wang [49].

Finally, we consider general cases of $n \geq 1$ and $k \geq 1$. In 1994, Zhang and Li [61] proved that if $F \in \mathcal{M}(\mathbb{C})$ is transcendental, then $F^n L[F] - a$ has infinitely many zeros for $n \geq 2$ and $a \neq 0, \infty$, where

$$L[F] = a_k F^{(k)} + a_{k-1} F^{(k-1)} + \cdots + a_0 F$$

in which a_i $(i = 0, 1, 2, \cdots, k)$ are small functions of F. In 1999, Pang and Zalcman [36] obtained a corresponding normality criterion as follows: If $\mathscr{F}$ is the family of $\mathcal{A}(D)$ such that zeros of each $f \in \mathscr{F}$ have multiplicities at least k and such that each $f \in \mathscr{F}$ satisfies $f^n f^{(k)} \neq a$ for a non-zero complex number a, then $\mathscr{F}$ is normal. In 2005, Zhang [59] showed that when $n \geq 2$, this result is also true if a is replaced by a non-vanishing holomorphic functions in D. For other related results, see Meng and Hu [29], Qi [37], Wang [41], Xu [45], Yang and Hu [48], L. Yang and C. Yang [53].

Take three integers $m \geq 0$, $k \geq 1$ and $n \geq 2$. Let a $(\not\equiv 0)$ be a holomorphic function in D such that multiplicities of zeros of a are at most m and divisible by $n + 1$. In this paper, we obtain the following normality criteria:

Theorem 1.1. *Let $\mathscr{F}$ be the family of $\mathcal{M}(D)$ such that multiplicities of zeros of each $f \in \mathscr{F}$ are at least $k + m$ and such that multiplicities of poles of f are at least $m + 1$ whenever f have zeros and poles. If each pair (f, g) of $\mathscr{F}$ satisfies that $f^n f^{(k)}$ and $g^n g^{(k)}$ share a IM, then $\mathscr{F}$ is normal in D.*

In special, if a has no zeros, which means $m = 0$, then Theorem 1.1 has the following form:

Corollary 1.1. *Let $\mathscr{F}$ be the family of $\mathcal{M}(D)$ such that multiplicities of zeros of each $f \in \mathscr{F}$ are at least k. If each pair (f, g) of $\mathscr{F}$ satisfies that $f^n f^{(k)}$ and $g^n g^{(k)}$ share a IM, then $\mathscr{F}$ is normal in D.*

It is easy to see that this result extends above normality criteria due to Pang and Zalcman [36], and Zhang [59]. Furthermore, we can improve partially the normality criterion due to Jiang [22] as follows:

Theorem 1.2. *If $\mathscr{F}$ is the family of $\mathcal{M}(D)$ such that each $f \in \mathscr{F}$ satisfies that $f^n f' \neq a$, then $\mathscr{F}$ is normal in D.*

The condition $a(z) \not\equiv 0$ in Theorem 1.1 and Theorem 1.2 is necessary. This fact can be illustrated by the following example:

Example 1.1. *Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Let $a(z) \equiv 0$ and*

$$\mathscr{F} = \left\{ f_j(z) = e^{j(z-\frac{1}{2})} \;\middle|\; j = 1, 2, \cdots \right\}$$

Obviously, $f_i^n f_i^{(k)}$ and $f_j^n f_j^{(k)}$ share a IM for distinct positive integers i and j (resp. $f_j^n f_j' \neq a$), but the family $\mathscr{F}$ is not normal at $z = 1/2$.

In Corollary 1.1, the condition that multiplicities of zeros of each $f \in \mathscr{F}$ are at least k is sharp. For example, we consider the following family:

Example 1.2. *Denote D as in Example 1.1. Let $a(z) = e^z$ and*

$$\mathscr{F} = \left\{ f_j(z) = j\left(z - \frac{1}{2j}\right)^{k-1} \;\middle|\; j = 1, 2, \cdots \right\}.$$

Any $f_j \in \mathscr{F}$ has only a zero of multiplicity $k-1$ in D and for distinct positive integers i and j, $f_i^n f_i^{(k)}$ and $f_j^n f_j^{(k)}$ share a IM. However, the family $\mathscr{F}$ is not normal at $z = 0$.

2 Preliminary lemmas

In order to prove our results, we require the following *Zalcman's lemma* (cf. [56]):

Lemma 2.1. *Take a positive integer k. Let $\mathscr{F}$ be a family of meromorphic functions in the unit disc $\triangle$ with the property that zeros of each $f \in \mathscr{F}$ are of multiplicity at least k. If $\mathscr{F}$ is not normal at a point $z_0 \in \Delta$, then for $0 \leq \alpha < k$, there exist a sequence $\{z_n\} \subset \Delta$ of complex numbers with $z_n \to z_0$; a sequence $\{f_n\}$ of $\mathscr{F}$; and a sequence $\{\rho_n\}$ of positive numbers with $\rho_n \to 0$ such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi)$ locally uniformly (with respect to the spherical metric) to a nonconstant meromorphic function $g(\xi)$ on $\mathbb{C}$. Moreover, the zeros of $g(\xi)$ are of multiplicity at least k, and the function $g(\xi)$ may be taken to satisfy the normalization $g^\sharp(\xi) \leq g^\sharp(0) = 1$ for any $\xi \in \mathbb{C}$. In particular, $g(\xi)$ has at most order 2.*

This result is Pang's generalization (cf. [33, 35, 47]) to the Main Lemma in [55] (where α is taken to be 0), with improvements due to Schwick [39], Chen and Gu [7]. In Lemma 2.1, the *order* of g is defined by using the Nevanlinna's characteristic function $T(r, g)$:

$$\mathrm{ord}(g) = \limsup_{r\to\infty} \frac{\log T(r, g)}{\log r}.$$

Here, as usual, $g^\sharp$ denotes the *spherical derivative*

$$g^\sharp(\xi) = \frac{|g'(\xi)|}{1 + |g(\xi)|^2}.$$

Lemma 2.2. *Let $p \geq 0$, $k \geq 1$ and $n \geq 2$ be three integers, and let a be a non-zero polynomial of degree p. If f is a non-constant rational function which has only zeros of multiplicity at least $k+p$ and has only poles of multiplicity at least $p+1$, then $f^n f^{(k)} - a$ has at least one zero.*

Proof. If f is a polynomial, then $f^{(k)} \not\equiv 0$ since f is non-constant and has only zeros of multiplicity at least $k+p$ which further means $\deg(f) \geq k+p$. Noting that $n \geq 2$, we immediately obtain that

$$\deg\left(f^n f^{(k)}\right) \geq n \deg(f) \geq n(k+p) > p = \deg(a).$$

Therefore, it follows that $f^n f^{(k)} - a$ is also a non-constant polynomial, and hence $f^n f^{(k)} - a$ has at least one zero. Next we assume that f has poles. Set

$$f(z) = \frac{A(z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\cdots(z-\alpha_s)^{m_s}}{(z-\beta_1)^{n_1}(z-\beta_2)^{n_2}\cdots(z-\beta_t)^{n_t}}, \tag{2.1}$$

where A is a non-zero constant, α_i distinct zeroes of f with $s \geq 0$, and β_j distinct poles of f with $t \geq 1$. For simplicity, we put

$$m_1 + m_2 + \cdots + m_s = M \geq (k+p)s, \tag{2.2}$$

$$n_1 + n_2 + \cdots + n_t = N \geq (p+1)t. \tag{2.3}$$

From (2.1), we obtain

$$f^{(k)}(z) = \frac{(z-\alpha_1)^{m_1-k}(z-\alpha_2)^{m_2-k}\cdots(z-\alpha_s)^{m_s-k}g(z)}{(z-\beta_1)^{n_1+k}(z-\beta_2)^{n_2+k}\cdots(z-\beta_t)^{n_t+k}}, \tag{2.4}$$

where g is a polynomial of degree $\leq k(s+t-1)$. From (2.1) and (2.4), we get

$$f^n(z)f^{(k)}(z) = \frac{A^n(z-\alpha_1)^{M_1}(z-\alpha_2)^{M_2}\cdots(z-\alpha_s)^{M_s}g(z)}{(z-\beta_1)^{N_1}(z-\beta_2)^{N_2}\cdots(z-\beta_t)^{N_t}}, \tag{2.5}$$

in which

$$M_i = (n+1)m_i - k, \ \ i = 1, 2, \cdots, s,$$

$$N_j = (n+1)n_j + k, \ \ j = 1, 2, \cdots, t.$$

Differentiating (2.5) yields

$$\left\{f^n f^{(k)}\right\}^{(p+1)}(z) = \frac{(z-\alpha_1)^{M_1-p-1}(z-\alpha_2)^{M_2-p-1}\cdots(z-\alpha_s)^{M_s-p-1}g_0(z)}{(z-\beta_1)^{N_1+p+1}\cdots(z-\beta_t)^{N_t+p+1}}, \tag{2.6}$$

where $g_0(z)$ is a polynomial of degree $\leq (p+k+1)(s+t-1)$. We assume, to the contrary, that $f^n f^{(k)} - a$ has no zero, then

$$f^n(z)f^{(k)}(z) = a(z) + \frac{C}{(z-\beta_1)^{N_1}(z-\beta_2)^{N_2}\cdots(z-\beta_t)^{N_t}}, \tag{2.7}$$

where C is a non-zero constant. Subsequently, (2.7) yields

$$\left\{f^n f^{(k)}\right\}^{(p+1)}(z) = \frac{g_1(z)}{(z-\beta_1)^{N_1+p+1}\cdots(z-\beta_t)^{N_t+p+1}}, \tag{2.8}$$

where $g_1(z)$ is a polynomial of degree $\leq (p+1)(t-1)$.

Comparing (2.6) with (2.8), we get

$$(p+1)(t-1) \geq \deg(g_1) \geq (n+1)M - ks - (p+1)s,$$

and hence

$$M < \frac{p+k+1}{n+1}s + \frac{p+1}{n+1}t. \tag{2.9}$$

From (2.5) and (2.7) we have

$$(n+1)N + kt + p = (n+1)M - ks + \deg(g).$$

Since $\deg(g) \leq k(s+t-1)$, we find

$$(n+1)N \leq (n+1)M - ks + k(s+t-1) - kt - p,$$

and thus

$$N < M. \tag{2.10}$$

By (2.9), (2.10) and noting that $M \geq (k+p)s$, $N \geq (p+1)t$, we deduce that

$$M < \frac{p+k+1}{n+1}s + \frac{p+1}{n+1}t \leq \left\{\frac{p+k+1}{(n+1)(k+p)} + \frac{1}{n+1}\right\}M. \tag{2.11}$$

Note that $n \geq 2$ implies

$$\frac{p+k+1}{(n+1)(k+p)} + \frac{1}{n+1} = \frac{2(k+p)+1}{(n+1)(k+p)} \leq 1.$$

Hence it follows from (2.11) that $M < M$, which is a contradiction. Lemma 2.2 is proved. $\square$

Lemma 2.3. *Let $p \geq 0$, $k \geq 1$ and $n \geq 2$ be three integers, and let a be a non-zero polynomial of degree p. If f is a non-constant rational function which has only zeros of multiplicity at least $k+p$ and has only poles of multiplicity at least $p+1$, then $f^n f^{(k)} - a$ has at least two distinct zeros.*

Proof. Lemma 2.2 implies that $f^n f^{(k)} - a$ has at least one zero. Assume, to the contrary, that $f^n f^{(k)} - a$ has only one zero z_0. If f is a polynomial, then we can write

$$f^n(z)f^{(k)}(z) - a(z) = A'(z-z_0)^d,$$

where A' is a non-zero constant and d is a positive integer. Since f is a non-constant polynomial which has only zeros of multiplicity at least $k+p$, we find $f^{(k)} \not\equiv 0$, and hence

$$d = \deg(f^n f^{(k)} - a) \geq \deg(f^n) \geq n(k+p) \geq 2p+2.$$

By computing we find

$$\left\{f^n f^{(k)}\right\}^{(p+1)}(z) = A'd(d-1)...(d-p)(z-z_0)^{d-p-1},$$

hence $\{f^n f^{(k)}\}^{(p+1)}$ has a unique zero z_0. Take a zero ξ_0 of f, then it is a zero of f^n with multiplicity at least $2p+2$. It follows that ξ_0 is a common zero of $\{f^n f^{(k)}\}^{(p)}$ and $\{f^n f^{(k)}\}^{(p+1)}$, which further implies that $\xi_0 = z_0$. Therefore, we obtain $\{f^n f^{(k)}\}^{(p)}(z_0) = 0$.

On the other hand, we get

$$\left\{f^n f^{(k)}\right\}^{(p)}(z) = a^{(p)}(z) + A'd(d-1)...(d-p+1)(z-z_0)^{d-p},$$

which means

$$\left\{f^n f^{(k)}\right\}^{(p)}(z_0) = a^{(p)}(z_0) \neq 0$$

since $\deg(a) = p$. This is contradictory to $\{f^n f^{(k)}\}^{(p)}(z_0) = 0$.

If f has poles, we can express f by (2.1) again, and then find

$$f^n(z)f^{(k)}(z) = a(z) + \frac{C'(z-z_0)^l}{(z-\beta_1)^{N_1}(z-\beta_2)^{N_2}\cdots(z-\beta_t)^{N_t}}, \tag{2.12}$$

where C' is a non-zero constant and l is a positive integer. We distinguish two cases to deduce contradictions.

Case 1. $p \geq l$. Since $p \geq l$, the expression (2.5) together with (2.12) implies that

$$(n+1)N + kt + p = (n+1)M - ks + \deg(g).$$

Therefore, we can also conclude (2.10), that is, $N < M$. Differentiating (2.12), we obtain

$$\left\{f^n f^{(k)}\right\}^{(p+1)}(z) = \frac{g_2(z)}{(z-\beta_1)^{N_1+p+1}\cdots(z-\beta_t)^{N_t+p+1}},$$

where $g_2(z)$ is a polynomial of degree at most $(p+1)t-(p-l+1)$, and hence

$$(p+1)t-(p-l+1) \geq \deg(g_2) \geq (n+1)M - ks - (p+1)s.$$

where the last estimate follows from (2.6). Then we have

$$\frac{p-l}{n+1} < \frac{p+k+1}{n+1}s + \frac{p+1}{n+1}t - M \leq \left\{\frac{p+k+1}{(n+1)(k+p)} + \frac{1}{n+1} - 1\right\} M \tag{2.13}$$

since $M \geq (k+p)s, N \geq (p+1)t, M > N$. It follows that

$$\frac{p+k+1}{(n+1)(k+p)} + \frac{1}{n+1} \leq 1$$

since $n \geq 2$. Therefore, from (2.13) we conclude that $p-l<0$, a contradiction with the assumption $p \geq l$.

Case 2. $l > p$. The expression (2.12) yields

$$\left\{f^n f^{(k)}\right\}^{(p+1)}(z) = \frac{(z-z_0)^{l-p-1}g_3(z)}{(z-\beta_1)^{N_1+p+1}\cdots(z-\beta_t)^{N_t+p+1}}, \tag{2.14}$$

where $g_3(z)$ is a polynomial with $\deg(g_3) \leq (p+1)t$. We claim that $z_0 \neq \alpha_i$ for each i. Otherwise, if $z_0 = \alpha_i$ for some i, then (2.12) yields

$$a^{(p)}(z_0) = \left\{f^n f^{(k)}\right\}^{(p)}(\alpha_i) = 0$$

because each α_i is a zero of $f^n f^{(k)}$ of multiplicity $\geq n(k+p) \geq 2p+2$. This is impossible since $\deg(a) = p$. Hence $(z-z_0)^{l-p-1}$ is a factor of the polynomial g_0 in (2.6). By (2.6) and (2.14), we conclude that

$$(p+1)t \geq \deg(g_3) \geq (n+1)M - ks - (p+1)s,$$

which is equivalent to

$$M \leq \frac{p+k+1}{n+1}s + \frac{p+1}{n+1}t. \tag{2.15}$$

If $l \neq (n+1)N + kt + p$, then (2.5) together with (2.12) implies

$$(n+1)N + kt + p \leq (n+1)M - ks + \deg(g),$$

so we get $N < M$ from $\deg(g) \leq k(s+t-1)$. Therefore, by using the facts $M \geq (k+p)s, N \geq (p+1)t$, (2.15) implies a contradiction

$$M < \left\{ \frac{p+k+1}{(n+1)(k+p)} + \frac{1}{n+1} \right\} M \leq M.$$

Hence $l = (n+1)N + kt + p$.

Now we must have $N \geq M$, otherwise, when $N < M$, we can deduce the contradiction $M < M$ from (2.15). Comparing (2.6) with (2.14), we find

$$(p+k+1)(s+t-1) \geq \deg(g_0) \geq l - p - 1$$

since $(z - z_0)^{l-p-1} | g_0$, and hence

$$(n+1)N + kt + p = l \leq (p+k+1)s + (p+k+1)t - k,$$

which further yields

$$N < \frac{p+k+1}{n+1} s + \frac{p+1}{n+1} t.$$

Since $M \geq (k+p)s$ and $N \geq (p+1)t$, it follows from (2.15) that

$$N < \frac{p+k+1}{(n+1)(k+p)} M + \frac{1}{n+1} N.$$

Hence $N \geq M$ yields

$$N < \left\{ \frac{p+k+1}{(n+1)(k+p)} + \frac{1}{n+1} \right\} N. \tag{2.16}$$

Since $n \geq 2$, we obtain consequently

$$\frac{p+k+1}{(n+1)(k+p)} + \frac{1}{n+1} \leq 1.$$

Hence (2.16) yields $N < N$. This is a contradiction. Proof of Lemma 2.3 is completed. $\square$

Lemma 2.4. *Let $p \geq 0$ and $n \geq 2$ be two integers such that p is divisible by $n+1$, and let a be a non-zero polynomial of degree p. If f is a non-constant rational function, then $f^n f' - a$ has at least one zero.*

Proof. If f is a non-constant polynomial, then $f' \not\equiv 0$. We consequently conclude that

$$\deg\left(f^n f'\right) = (n+1)\deg(f) - 1 \neq p$$

since p is divisible by $n+1$. It follows that $f^n f' - a$ is also a non-constant polynomial, so that $f^n f' - a$ has at least one zero.

If f has poles, we can express f by (2.1) again, and then, by differentiating (2.1), we deduce that

$$f'(z) = \frac{(z-\alpha_1)^{m_1-1}(z-\alpha_2)^{m_2-1}\cdots(z-\alpha_s)^{m_s-1}h(z)}{(z-\beta_1)^{n_1+1}(z-\beta_2)^{n_2+1}\cdots(z-\beta_t)^{n_t+1}}, \tag{2.17}$$

where $h(z)$ is a polynomial of form

$$h(z) = (M-N)z^{s+t-1} + \cdots .$$

From (2.1) and (2.17), we obtain

$$f^n f' = \frac{P}{Q},$$

in which

$$P(z) = A^n(z-\alpha_1)^{(n+1)m_1-1}(z-\alpha_2)^{(n+1)m_2-1}\cdots(z-\alpha_s)^{(n+1)m_s-1}h(z),$$

$$Q(z) = (z-\beta_1)^{(n+1)n_1+1}(z-\beta_2)^{(n+1)n_2+1}\cdots(z-\beta_t)^{(n+1)n_t+1}.$$

We suppose, to the contrary, that $f^n f' - a$ has no zero. When $M \neq N$, we have

$$f^n f' = a + \frac{B}{Q} = \frac{P}{Q},$$

where B is a non-zero constant. Therefore, we obtain

$$\deg(P) = \deg(Qa+B) = \deg(Q) + p.$$

This implies that

$$(n+1)M - s + (s+t-1) = (n+1)N + t + p,$$

or equivalently

$$M - N = \frac{p+1}{n+1},$$

in which p is divisible by $n+1$. This is impossible since $M-N$ is an integer.

If $M = N$, we can rewrite (2.1) as follows

$$f(z) = A + \frac{B'(z-\gamma_1)^{l_1}(z-\gamma_2)^{l_2}\cdots(z-\gamma_r)^{l_r}}{(z-\beta_1)^{n_1}(z-\beta_2)^{n_2}\cdots(z-\beta_t)^{n_t}},$$

where B' is a non-zero constant, γ_i are distinct with $l_i \geq 1$, $r \geq 0$ and

$$M' = l_1 + \cdots + l_r < N.$$

Thus we find

$$f'(z) = \frac{(z-\gamma_1)^{l_1-1}(z-\gamma_2)^{l_2-1}\cdots(z-\gamma_r)^{l_r-1}\hbar(z)}{(z-\beta_1)^{n_1+1}(z-\beta_2)^{n_2+1}\cdots(z-\beta_t)^{n_t+1}},$$

where $\hbar(z)$ is a polynomial of form

$$\hbar(z) = (M' - N)z^{r+t-1} + \cdots .$$

Similarly, since $\deg(P) = \deg(Q) + p$ we have

$$nM + M' - r + (r + t - 1) = (n + 1)N + t + p = (n + 1)M + t + p,$$

that is,

$$M' = M + p + 1.$$

This is impossible since $M' < N = M$. Therefore, $f^n f' - a$ has at least one zero. $\square$

The following lemma is a direct consequence of a result from [61]:

Lemma 2.5. *Let n, k be two positive integers with $n \geq 2$ and let a $(\not\equiv 0)$ be a polynomial. If f is a transcendental meromorphic function in $\mathbb{C}$, then $f^n f^{(k)} - a$ has infinitely zeros.*

3 Proof of Theorem 1.1

Without loss of generality, we may assume that $D = \{z \in \mathbb{C} \mid |z| < 1\}$. For any point z_0 in D, either $a(z_0) = 0$ or $a(z_0) \neq 0$ holds. For simplicity, we assume $z_0 = 0$ and distinguish two cases.

Case 1. $a(0) \neq 0$. To the contrary, we suppose that $\mathscr{F}$ is not normal at $z_0 = 0$. Then, by Lemma 2.1, there exist a sequence $\{z_j\}$ of complex numbers with $z_j \to 0$ $(j \to \infty)$; a sequence $\{f_j\}$ of $\mathscr{F}$; and a sequence $\{\rho_j\}$ of positive numbers with $\rho_j \to 0$ $(j \to \infty)$ such that

$$g_j(\xi) = \rho_j^{-\frac{k}{n+1}} f_j(z_j + \rho_j \xi)$$

converges uniformly to a non-constant meromorphic function $g(\xi)$ in $\mathbb{C}$ with respect to the spherical metric. Moreover, $g(\xi)$ is of order at most 2. By Hurwitz's theorem, the zeros of $g(\xi)$ have at least multiplicity $k + m$.

On every compact subset of $\mathbb{C}$ which contains no poles of g, we have uniformly

$$\begin{aligned} & f_j^n(z_j + \rho_j\xi) f_j^{(k)}(z_j + \rho_j\xi) - a(z_j + \rho_j\xi) \\ = \ & g_j^n(\xi) g_j^{(k)}(\xi) - a(z_j + \rho_j\xi) \rightrightarrows g^n(\xi) g^{(k)}(\xi) - a(0). \end{aligned} \tag{3.1}$$

If $g^n g^{(k)} \equiv a(0)$, then g has no zeros and poles. Then there exist constants c_i such that $(c_1, c_2) \neq (0, 0)$, and

$$g(\xi) = e^{c_0 + c_1\xi + c_2\xi^2}$$

since g is a non-constant meromorphic function of order at most 2. Obviously, this is contrary to the case $g^n g^{(k)} \equiv a(0)$. Hence we have $g^n g^{(k)} \not\equiv a(0)$.

By Lemma 2.3 and 2.5, the function $g^n g^{(k)} - a(0)$ has two distinct zeros ξ_0 and ξ_0^*. We choose a positive number δ small enough such that $D_1 \cap D_2 = \emptyset$ and such that $g^n g^{(k)} - a(0)$ has no other zeros in $D_1 \cup D_2$ except for ξ_0 and ξ_0^*, where

$$D_1 = \{\xi \in \mathbb{C} \mid |\xi - \xi_0| < \delta\}, \; D_2 = \{\xi \in \mathbb{C} \mid |\xi - \xi_0^*| < \delta\}.$$

By (3.1) and Hurwitz's theorem, there exist points $\xi_j \in D_1$, $\xi_j^* \in D_2$ such that

$$f_j^n(z_j + \rho_j \xi_j) f_j^{(k)}(z_j + \rho_j \xi_j) - a(z_j + \rho_j \xi_j) = 0,$$

and

$$f_j^n(z_j + \rho_j \xi_j^*) f_j^{(k)}(z_j + \rho_j \xi_j^*) - a(z_j + \rho_j \xi_j^*) = 0$$

for sufficiently large j.

By the assumption in Theorem 1.1, $f_1^n f_1^{(k)}$ and $f_j^n f_j^{(k)}$ share a IM for each j. It follows

$$f_1^n(z_j + \rho_j \xi_j) f_1^{(k)}(z_j + \rho_j \xi_j) - a(z_j + \rho_j \xi_j) = 0,$$

and

$$f_1^n(z_j + \rho_j \xi_j^*) f_1^{(k)}(z_j + \rho_j \xi_j^*) - a(z_j + \rho_j \xi_j^*) = 0.$$

By letting $j \to \infty$, and noting $z_j + \rho_j \xi_j \to 0$, $z_j + \rho_j \xi_j^* \to 0$, we obtain

$$f_1^n(0) f_1^{(k)}(0) - a(0) = 0.$$

Since the zeros of $f_1^n(\xi) f_1^{(k)}(\xi) - a(\xi)$ has no accumulation points, in fact we have

$$z_j + \rho_j \xi_j = 0, \; z_j + \rho_j \xi_j^* = 0,$$

or equivalently

$$\xi_j = -\frac{z_j}{\rho_j}, \qquad \xi_j^* = -\frac{z_j}{\rho_j}.$$

This contradicts with the facts that $\xi_j \in D_1$, $\xi_j^* \in D_2$, $D_1 \cap D_2 = \emptyset$. Thus $\mathscr{F}$ is normal at $z_0 = 0$.

Case 2. $a(0) = 0$. We assume that $z_0 = 0$ is a zero of a of multiplicity p. Then we have $p \leq m$ by the assumption. Write $a(z) = z^p b(z)$, in which $b(0) = b_p \neq 0$. Since multiplicities of all zeros of a are divisible by $n+1$, then $d = p/(n+1)$ is just a positive integer. Thus we obtain a new family of $\mathcal{M}(D)$ as follows

$$\mathscr{H} = \left\{ \frac{f(z)}{z^d} \;\middle|\; f \in \mathscr{F} \right\}.$$

We claim that $\mathscr{H}$ is normal at 0.

Otherwise, if $\mathscr{H}$ is not normal at 0, then by lemma 2.1 there exist a sequence $\{z_j\}$ of complex numbers with $z_j \to 0$ $(j \to \infty)$; a sequence $\{h_j\}$ of $\mathscr{H}$; and a sequence $\{\rho_j\}$ of positive numbers with $\rho_j \to 0$ $(j \to \infty)$ such that

$$g_j(\xi) = \rho_j^{-\frac{k}{n+1}} h_j(z_j + \rho_j \xi) \tag{3.2}$$

converges uniformly to a non-constant meromorphic function $g(\xi)$ in $\mathbb{C}$ with respect to the spherical metric, where $g^\sharp(\xi) \le 1$, $\mathrm{ord}(g) \le 2$, and h_j has the following form

$$h_j(z) = \frac{f_j(z)}{z^d}.$$

We will deduce contradiction by distinguishing two cases.

Subcase 2.1. There exists a subsequence of z_j/ρ_j, for simplicity we still denote it as z_j/ρ_j, such that $z_j/\rho_j \to c$ as $j \to \infty$, where c is a finite number. Thus we have

$$F_j(\xi) = \frac{f_j(\rho_j \xi)}{\rho_j^{\frac{k}{n+1}+d}} = \frac{(\rho_j\xi)^d h_j(z_j + \rho_j(\xi - \frac{z_j}{\rho_j}))}{(\rho_j)^d (\rho_j)^{\frac{k}{n+1}}} \rightrightarrows \xi^d g(\xi - c) = h(\xi),$$

and

$$F_j^n(\xi) F_j^{(k)}(\xi) - \frac{a(\rho_j\xi)}{\rho_j^p} = \frac{f_j^n(\rho_j\xi) f_j^{(k)}(\rho_j\xi) - a(\rho_j\xi)}{\rho_j^p} \rightrightarrows h^n(\xi)h^{(k)}(\xi) - b_p\xi^p. \tag{3.3}$$

Noting $p \le m$, it follows from Lemma 2.3 and 2.5 that $h^n(\xi)h^{(k)}(\xi) - b_p\xi^p$ has two distinct zeros at least. Additionally, with similar discussion to the proof of Case 1, we can conclude that $h^n(\xi)h^{(k)}(\xi) - b_p\xi^p \not\equiv 0$. Let ξ_1 and ξ_1^* be two distinct zeros of $h^n(\xi)h^{(k)}(\xi) - b_p\xi^p$. We choose a positive number γ properly, such that $D_3 \cap D_4 = \emptyset$ and such that $h^n(\xi)h^{(k)}(\xi) - b_p\xi^p$ has no other zeros in $D_3 \cup D_4$ except for ξ_1 and ξ_1^*, where

$$D_3 = \{\xi \in \mathbb{C} \mid |\xi - \xi_1| < \gamma\},\ D_4 = \{\xi \in \mathbb{C} \mid |\xi - \xi_1^*| < \gamma\}.$$

By (3.3) and Hurwitz's theorem, there exist points $\zeta_j \in D_3$, $\zeta_j^* \in D_4$ such that

$$f_j^n(\rho_j\zeta_j) f_j^{(k)}(\rho_j\zeta_j) - a(\rho_j\zeta_j) = 0,$$

and

$$f_j^n(\rho_j\zeta_j^*) f_j^{(k)}(\rho_j\zeta_j^*) - a(\rho_j\zeta_j^*) = 0$$

for sufficiently large j. By the similar arguments in Case 1, we obtain a contradiction.

Subcase 2.2. There exists a subsequence of z_j/ρ_j, for simplicity we still denote it as z_j/ρ_j, such that $z_j/\rho_j \to \infty$ as $j \to \infty$. Then

$$\begin{aligned} f_j^{(k)}(z_j+\rho_j\xi) &= \Big\{(z_j+\rho_j\xi)^d h_j(z_j+\rho_j\xi)\Big\}^{(k)} \\ &= (z_j+\rho_j\xi)^d h_j^{(k)}(z_j+\rho_j\xi) + \sum_{i=1}^{k} a_i (z_j+\rho_j\xi)^{d-i} h_j^{(k-i)}(z_j+\rho_j\xi) \\ &= (z_j+\rho_j\xi)^d \rho_j^{-\frac{nk}{n+1}} g_j^{(k)}(\xi) + \sum_{i=1}^{k} a_i (z_j+\rho_j\xi)^{d-i} \rho_j^{-\frac{nk}{n+1}+i} g_j^{(k-i)}(\xi), \end{aligned}$$

in which $a_i (i=1,2,...,k)$ are all constants. Since $z_j/\rho_j \to \infty$, $b(z_j+\rho_j\xi) \to b_p$ as $j \to \infty$, it follows that

$$\begin{aligned} & b_p \frac{f_j^n(z_j+\rho_j\xi) f_j^{(k)}(z_j+\rho_j\xi)}{a(z_j+\rho_j\xi)} - b_p \\ = \;& b_p \frac{(z_j+\rho_j\xi)^p g_j^n(\xi) g_j^{(k)}(\xi)}{b(z_j+\rho_j\xi)(z_j+\rho_j\xi)^p} + \sum_{i=1}^{k} b_p \frac{(z_j+\rho_j\xi)^p g_j^n(\xi) g_j^{(k-i)}(\xi)}{b(z_j+\rho_j\xi)(z_j+\rho_j\xi)^p} \left(\frac{\rho_j}{z_j+\rho_j\xi}\right)^i - b_p \\ \rightrightarrows \;& g^n(\xi) g^{(k)}(\xi) - b_p \end{aligned} \tag{3.4}$$

on every compact subset of $\mathbb{C}$ which contains no poles of g. Since all zeros of $f_j \in \mathscr{F}$ have at least multiplicity $k+m$, then multiplicities of zeros of g are at least k. Then from Lemma 2.3 and 2.5, the function $g^n(\xi)g^{(k)}(\xi) - b_p$ has at least two distinct zeros. With similar discussion to the proof of Case 1, we can get a contradiction.

Hence the claim is proved, that is, $\mathscr{H}$ is normal at $z_0 = 0$. Therefore, for any sequence $\{f_t\} \subset \mathscr{F}$ there exist $\Delta_r = \{z : |z| < r\}$ and a subsequence $\{h_{t_k}\}$ of $\{h_t(z) = f_t(z)/z^d\} \subset \mathscr{H}$ such that $h_{t_k} \rightrightarrows I$ or ∞ in Δ_r, where I is a meromorphic function. Next we distinguish two cases.

Case A. Assume $f_{t_k}(0) \neq 0$ when k is sufficiently large. Then $I(0) = \infty$, and hence for arbitrary $R > 0$, there exists a positive number δ with $0 < \delta < r$ such that $|I(z)| > R$ when $z \in \Delta_\delta$. Hence when k is sufficiently large, we have $|h_{t_k}(z)| > R/2$, which means that $1/f_{t_k}$ is holomorphic in Δ_δ. In fact, when $|z| = \delta/2$,

$$\left|\frac{1}{f_{t_k}(z)}\right| = \left|\frac{1}{h_{t_k}(z) z^d}\right| \le M = \frac{2^{d+1}}{R\delta^d}.$$

By applying maximum principle, we have

$$\left|\frac{1}{f_{t_k}(z)}\right| \le M$$

for $z \in \Delta_{\delta/2}$. It follows from Motel's normal criterion that there exists a convergent subsequence of $\{f_{t_k}\}$, that is, $\mathscr{F}$ is normal at 0.

Case B. There exists a subsequence of f_{t_k}, for simplicity we still denote it as f_{t_k}, such that $f_{t_k}(0) = 0$. Then we get $I(0) = 0$ since $h_{t_k}(z) = f_{t_k}(z)/z^d \rightrightarrows I(z)$, and hence there exists a positive number ρ with $0 < \rho < r$ such that $I(z)$ is holomorphic in Δ_ρ and has a unique zero $z = 0$ in Δ_ρ. Therefore, we have $f_{t_k}(z) \rightrightarrows z^d I(z)$ in Δ_ρ since h_{t_k} converges spherically locally uniformly to a holomorphic function I in Δ_ρ. Thus $\mathscr{F}$ is normal at 0.

Similarly, we can prove that $\mathscr{F}$ is normal at arbitrary $z_0 \in D$, hence $\mathscr{F}$ is normal in D.

4 Proof of Corollary 1.1

By using Lemma 2.3 and 2.5, we find that if f is a non-constant meromorphic function which has only zeros of multiplicity at least k, then $f^n f^{(k)} - a$ has at least two distinct zeros for a non-zero complex number a. Therefore, noting that a has no zeroes, we can verify that $\mathscr{F}$ is normal in D by utilizing the same method in the proof of Theorem 1.1.

5 Proof of Theorem 1.2

Without loss of generality, we assume that $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and $z_0 = 0$. Now we distinguish two cases by either $a(0) = 0$ or $a(0) \neq 0$.

Case 1. $a(0) \neq 0$. To the contrary, we suppose that $\mathscr{F}$ is not normal at 0. By using the notations in the proof of Theorem 1.1, we also obtain

$$\begin{aligned} & f_j^n(z_j + \rho_j\xi) f_j'(z_j + \rho_j\xi) - a(z_j + \rho_j\xi) \\ = \; & g_j^n(\xi) g_j'(\xi) - a(z_j + \rho_j\xi) \rightrightarrows g^n(\xi) g'(\xi) - a(0), \end{aligned} \tag{5.1}$$

where $g^n g^{(k)} \not\equiv a(0)$.

By Lemma 2.4 and 2.5, the function $g^n g' - a(0)$ has a zero ξ_2. By (5.1) and Hurwitz's theorem, there exist points $\eta_j \to \xi_2$ $(j \to \infty)$ such that for sufficiently large j, $z_j + \rho_j \eta_j \in D$ and

$$f_j^n(z_j + \rho_j\eta_j) f_j'(z_j + \rho_j\eta_j) - a(z_j + \rho_j\eta_j) = 0,$$

which contradicts the assumption that $f^n f' \neq a$.

Case 2. $a(0) = 0$. By using the notations in the proof of Theorem 1.1, we also get the formulas (3.1)–(3.4). Therefore, with the similar method in Case 1, we can prove that $\mathscr{F}$ is normal at z_0, and hence $\mathscr{F}$ is normal in D.

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