THE UNDIRECTED POWER GRAPH OF A FINITE GROUP

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Abstract. The power graph $P(G)$ of a group G is the graph which has a vertex set of the group elements and two elements are adjacent if one is a power of the other. Chakrabarty, Ghosh and Sen [Undirected power graphs of semigroups, Semigroup Forum **78** (2009) 410-426] proved the main properties of the power graph of a finite group. The aim of this paper is to generalize some results of the mentioned paper and presenting some counterexamples for one of the problems raised by these authors. It is also proved that the power graph of a $p-$group is $2-$connected if and only if the group is a cyclic or generalized quaternion group and if G is a nilpotent group which is not of prime power order then the power graph $P(G)$ is $2-$connected. We also prove that the number of edges of the power graph of the simple groups is less than or equal to the number of edges in the power graph of the cyclic group of the same order. This partially answers to a question in an earlier paper. Finally, we give a complete classification of groups in which the power graph is a union of complete graphs sharing a common vertex.

Keywords: Power graph, Hamiltonian graph, $2-$connected graph, simple group.

1. Introduction

All groups and graphs in this paper are assumed to be finite. Throughout the paper, we follow the terminology and notation of [10] for groups and [11] for graphs. The power graphs are a new representation of groups using graphs. These graphs were first used by Chakrabarty et al. [6] using semigroups. We also encourage interested readers to consult papers [3, 4, 8, 9] on power graphs constructed from finite groups.

Suppose G is a finite group. The *power graph* $P(G)$ is a graph in which $V(P(G)) = G$ and two distinct elements x and y are adjacent if and only if one of them is a power of the other. If G is a finite group then it can be easily investigated that the power graph $P(G)$ is a connected graph of diameter at most 2. In [6], it is proved that for a finite group G, $P(G)$ is complete if and only if G is a cyclic group of order 1 or p^m, for some prime number p and positive integer m.

Suppose G is a finite group and $x, y \in G$. The "cyclic subgroup of G generated by x" is denoted by $\langle x\rangle$ and $deg(x)$ denotes the degree of x in $P(G)$. The distance between x and y in $P(G)$ is defined as the length of a minimal path connecting them. The girth of Γ, $g(\Gamma)$, is the length of a shortest cycle within the graph.

We need to know some graph theoretical properties of power graphs of finite groups to be able to handle them comprehensively. To fulfill this goal the results of the removal of identity element and the graph remaining connected have been studied. We have classified $p-$ groups based on these assumptions.

A cut vertex in a graph Γ is any vertex whose removal increases the number of $\Gamma-$components. The graph Γ is said to be $2-$connected if Γ does not have a cut vertex. A trail in Γ containing every edge exactly once is called an Eulerian trail. Γ is called Eulerian, if it has an Eulerian cycle, or equivalently, every vertex of Γ is of even degree. A Hamiltonian path in Γ is a path that visits each vertex exactly once. A Hamiltonian cycle is a Hamiltonian path that is a cycle. A graph containing a Hamiltonian cycle is called a Hamiltonian graph.

If G is a finite group and $x \in G$ then obviously,

$$deg(x) = |\{g \in G \mid \langle x\rangle \leq \langle g\rangle \ or \ \langle g\rangle \leq \langle x\rangle\}|.$$

In the following result, a characterization of Eulerian power graphs is presented.

Lemma 1. The power graph $P(G)$ is Eulerian if and only if $|G|$ is odd.

Proof. Choose $x \in G$. The number of elements g such that $\langle g\rangle \leq \langle x\rangle$ is $o(x)$ and the number of elements g such that $\langle x\rangle \leq \langle g\rangle$ is $\sum_{\langle x\rangle<\langle g\rangle} \varphi(o(g))$. From the fact that the graph is simple, we have

$$deg(x) = o(x) + \sum_{\langle x\rangle<\langle g\rangle} \varphi(o(g)) - 1.$$

Since for each odd positive integer n, $\varphi(n)$ is even, $o(x) - 1 + \sum_{\langle x\rangle<\langle g\rangle} \varphi(o(g))$ is even. Therefore, the degree of each vertex is even and so $P(G)$ is Eulerian. Since $deg(1) = |G| - 1$, the converse is obvious. □

Lemma 2. The power graph of a finite group G is a tree if and only if G is an elementary abelian $2-$group.

Lemma 3. $g(P(G)) = 3$ if and only if G is not an elementary abelian $2-$group. Moreover, if $P(G)$ is 2-connected then $g(P(G) - e) = 3$.

Proof. If $g(P(G)) = 3$ then $P(G)$ has a cycle of length 3 and so it is not an elementary abelian $2-$group. Conversely, if G is not an elementary abelian $2-$group then it has at least one element of order 4 or an odd prime p, as desired. To prove the second part, we notice that G has an element of order ≥ 4. □

2. Main Results

In this section we first present some counterexamples to a conjecture given in [6]. Then we focus on the classification problem of 2-connected power graphs and the power graphs which is a union of complete graphs share the identity.

2.1. **Counterexamples.** Suppose U_n denotes the unit group of the ring Z_n. In [6], Chakrabarty, Ghosh and Sen, have asked about the values of n for which $P(U_n)$ is Hamiltonian. They have written: in this context our conjecture is that $P(U_n)$ is Hamiltonian for all values of $n \geq 3$ except for $n = 2^m p_1 p_2 \cdots p_k$, where p_1, p_2, $\cdots$, p_k are distinct Fermat primes, m and k are nonnegative integers, $m \geq 2$ for $k = 0, 1$ and $k \geq 2$ for $m = 0, 1$. We now present some sequences of counterexamples to prove that this conjecture is incorrect.

By [11, Proposition 7.2.3], if a graph G has a vertex subset S such that $V(G) - S$ has at least $|S| + 1$ components then G is not Hamiltonian. We apply this result to provide a counterexample for the mentioned conjecture.

Counterexample 4. If $n = 2^v \times 3^2$, $v \geq 3$, then $P(U_n)$ does not have a Hamiltonian cycle.

Proof. By a classical result in number theory [7], $U_n \cong Z_2 \times Z_2 \times Z_3 \times Z_{2^{v-2}}$. Suppose x, y, t and z are elements of U_n such that $o(x) = 2, o(y) = 2, o(t) = 3$, x and y are distinct, and do not lie in $\langle z \rangle$ and $o(z) = 2^{v-2}$. If $S = \{1, t, t^2\}$ then $\{x, xt, xt^2\}$, $\{y, yt, yt^2\}$, $\{xy, xyt, xyt^2\}$ and $\{xyz^{2^{(v-3)}}, xyz^{2^{(v-3)}}t, xyz^{2^{(v-3)}}t^2\}$ are connected components of $P(U_n) - S$. □

Counterexample 5. If $n = 2^t \times 7$, $t \geq 2$, then $P(U_n)$ does not have a Hamiltonian cycle.

Proof. The proof is similar to Counterexample 4. □

Counterexample 6. If $n = 2^2 \times 3^2 \times p$, where p is a Fermat prime, then $P(U_n)$ does not have a Hamiltonian cycle.

Proof. The proof follows from the same argument as Counterexample 4. $\square$

Other counterexamples exist as well, such as: $n = 2^2 \times 3^2 \times 17$, $2^2 \times 3 \times 13$ and more counterexamples of this form can be constructed using the same method. By the previous theorem, the main question of Chakrabarty, Ghosh and Sen [6] to determine those values of n for which $P(U_n)$ is Hamiltonian, remains still open.

2.2. **2-Connectivity of Power Graphs.** Suppose G is a finite group such that $P(G) - \{e\}$ is connected. Then one can easily see that $P(G)$ is $2-$connected. In what follows, we investigate the structure of $2-$connected power graphs. This is a generalized approach to what was proved by Chakrabarty et al. in [6].

Theorem 7. Suppose G is a $p-$group. The power graph $P(G)$ is $2-$connected if and only if G is a cyclic or generalized quaternion group.

Proof. Suppose G is a $p-$group and $P(G)$ is $2-$connected. We first prove that for each pair of distinct elements $x, y \in G$ there exists a cycle of length 3 or 4 containing x and y. Since G is $2-$connected there exists a cycle containing x and y. Suppose C is a minimal cycle containing these elements. If $xy \in E(P(G))$ then we have the cycle e, x, y, e, as desired. So, we can assume that $d(x, y) > 1$. Set

$$C : e, x, u_1, \cdots, u_k, y, e.$$

By definition $\langle x\rangle \leq \langle u_1\rangle$ or $\langle u_1\rangle \leq \langle x\rangle$. Suppose $\langle x\rangle \leq \langle u_1\rangle$. If $\langle u_1\rangle \leq \langle u_2\rangle$ then x and u_2 are adjacent, contradicting the minimality of C. Thus, $\langle u_2\rangle \leq \langle u_1\rangle$. Since $\langle u_1\rangle$ is cyclic, $\langle x\rangle \leq \langle u_2\rangle$ or $\langle u_2\rangle \leq \langle x\rangle$, which is a contradiction. This shows that $g(P(G)) = 3$. We now assume that $\langle u_1\rangle \leq \langle x\rangle$. Again from the minimality of C, we have: $\langle u_1\rangle \leq \langle u_2\rangle$. If $k \geq 3$ then a similar argument as above leads to a contradiction. So, we have a cycle of length 4 containing x and y. Hence, G has a cycle of form e, x, y, e or e, x, u, y, e.

Next, we prove that G has a unique non-trivial subgroup contained in every subgroup of G. Assume x, y, z are three non-identity elements of G. We show that $\langle x\rangle \cap \langle y\rangle \cap \langle z\rangle$ is non-trivial. We consider four different cases in our main proof, as follows:

Case 1. *There is a cycle of length 3 containing x, y and a cycle of length 3 containing y, z.* Suppose $\langle z\rangle \leq \langle y\rangle$. Since $\langle x\rangle \leq \langle y\rangle$ or $\langle y\rangle \leq \langle x\rangle$, we have: $\langle z\rangle, \langle x\rangle \leq \langle y\rangle$ or $\langle z\rangle \leq \langle y\rangle \leq \langle x\rangle$. In the fourth case, by cyclicity of $\langle y\rangle$, $\langle x\rangle \leq \langle z\rangle \leq \langle y\rangle$ or

$\langle z\rangle \leq \langle x\rangle \leq \langle y\rangle$. The case of $\langle y\rangle \leq \langle z\rangle$ is similar. This means that one of the elements of $\{x, y, z\}$ is contained in cyclic subgroups generated by the other two elements.

Case 2. *There is a cycle of length 3 containing x, y and a cycle of length 4 containing y, z.* Choose an element t such that $\langle t\rangle \leq \langle y\rangle$,$\langle z\rangle$. If $\langle x\rangle \leq \langle y\rangle$ then $\langle t\rangle$, $\langle x\rangle \leq \langle y\rangle$. Therefore $\langle t\rangle \leq \langle x\rangle$ or $\langle x\rangle \leq \langle t\rangle$. Thus, one of the elements of $\{x, y, z, t\}$ is contained in cyclic subgroups generated by the other three elements.

Case 3. *There is a cycle of length 4 containing x, y and a cycle of length 3 containing y, z.* This case is similar to the case 2.

Case 4. *There is a cycle of length 4 containing x, y and a cycle of length 4 containing y, z.* Choose elements u and t such that $\langle u\rangle \leq \langle x\rangle$, $\langle y\rangle$ and $\langle t\rangle \leq \langle y\rangle$, $\langle z\rangle$. Then $\langle u\rangle$, $\langle t\rangle$, $\leq \langle y\rangle$. Subsequently, $\langle u\rangle \leq \langle x\rangle$, $\langle y\rangle$, $\langle z\rangle$, $\langle t\rangle$ or $\langle t\rangle \leq \langle x\rangle$, $\langle y\rangle$, $\langle z\rangle$, $\langle u\rangle$.

Therefore, by an inductive argument, we can find a non-trivial subgroup of G contained in each non-trivial subgroup of G. Now by [2, Theorems 8.5, 8.6], G is cyclic or generalized quaternion. Conversely, from the fact that in the power graphs of cyclic groups all vertices are joined and in the generalized quaternion group Q_{2^n} there exists a unique involution that is joined to all other vertices, the result follows. □

As a consequence of what was proven above, we now know that:

Corollary 8. Suppose G is a finite $p-$group. Then $P(G)$ has a Hamiltonian cycle if and only if $|G| \neq 2$ and G is cyclic.

Suppose $G = Q_{2^n}$ and t is the unique element of G of order 2. Since the number of connected components of $P(G) - \{e, t\}$ is greater than 2, it is not Hamiltonian. If p is odd, we have the following characterization of Hamiltonian power graphs of finite $p-$groups.

Corollary 9. If p is a odd prime then the power graph of a $p-$group is $2-$connected if and only if it is Hamiltonian.

We are now ready to investigate the structure of $2-$connected power graphs in general. We start by nilpotent groups.

Theorem 10. Suppose G is a nilpotent group. If G is not a $p-$group then the power graph $P(G)$ is $2-$connected.

Proof. Suppose $G = P_1 \times P_2 \times \cdots \times P_k$, where P_i's are p_i-group and $p_1, \cdots, p_k$, $k \geq 2$, are distinct primes. Choose two non-identity arbitrary elements $x, y \in G$. Obviously, $o(x) = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and $o(y) = p_1^{\beta_1} \cdots p_k^{\beta_k}$, where α_i's and β_i's are non-negative integers. We have two different cases as follows:

Case 1: *There exists i, $1 \leq i \leq k$, such that $\alpha_i \neq 0$ and $\beta_i = 0$.* Assume that $o(x) = p_i^{\alpha_i} t$ and $o(y) = r$. Then $o(x^t) = p_i^{\alpha_i}$ and $(r, p_i^{\alpha_i}) = 1$. Since G is nilpotent, $[x^t, y] = 1$ and so $x^t = (x^t y)^{vr}$ and $y = (x^t y)^{up_i^{\alpha_i}}$, for some $u, v \in Z$. Hence $x, x^t, x^t y, y$ is a path connecting x and y in $P(G)$, as desired.

Case 2: *$o(x)$ and $o(y)$ have the same prime divisors.* We can assume that $o(x) = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ and $o(y) = p_1^{\beta_1} \cdots p_s^{\beta_s}$, where α_i's and β_i's are positive integers. Then $o(x^{p_2^{\alpha_2} \cdots p_s^{\alpha_s}}) = p_1^{\alpha_1}$ and $o(y^{p_2^{\beta_2} \cdots p_s^{\beta_s}}) = p_1^{\beta_1}$. Choose $t \in G$ of order p_2 and define

$$u = x^{p_2^{\alpha_2} \cdots p_s^{\alpha_s}} \quad ; \quad v = y^{p_2^{\beta_2} \cdots p_s^{\beta_s}}.$$

Therefore, $(vt)^{o(v)} = t^{o(v)}$ and $(ut)^{o(u)} = t^{o(u)}$. So, we have the following path connecting x and y in $P(G)$:

$$x, x^{p_2^{\alpha_2} \cdots p_s^{\alpha_s}}, x^{p_2^{\alpha_2} \cdots p_s^{\alpha_s}} t, t, y^{p_2^{\beta_2} \cdots p_s^{\beta_s}} t, y^{p_2^{\beta_2} \cdots p_s^{\beta_s}}, y,$$

proving the theorem. □

The converse of Theorem 10, is not necessarily correct. To investigate this, it is enough to consider the group $G = Z_5 \times S_3$. Then G is $2-$connected non-nilpotent group. Moreover, the dihedral group D_{2p}, p is prime, is a solvable group that its power graph is not $2-$connected. On the other hand, we have seen that if G is a nilpotent group with at least two prime factors in its order then $P(G)$ is $2-$connected, but nilpotency of groups does not yield that their power graphs are Hamiltonian. The group $G = Z_3 \times D_8$ is an example of such groups.

Lemma 11. If A and B are groups of coprime orders such that A is cyclic of prime order then $P(A \times B)$ is $2-$connected.

Proof. The proof is similar to those given in Theorem 10. □

The previous lemma shows that $2-$connectivity does not imply solvability or nilpotency of groups.

2.3. **Power Graphs of Simple Groups.** In [8], the authors conjectured that the power graph $P(Z_n)$ has the maximum number of edges among all power graphs of groups of order n. In this section the power graphs of simple groups are taken into account. As a consequence, the mentioned conjecture will be proved in the class of all simple groups.

Lemma 12. Let G be a finite group that is not of prime order. If Max $\omega(G) = p$, where p is prime and $\omega(G)$ is the set of all element orders of G, then $P(G)$ is not $2-$connected.

Proof. Suppose G is $2-$connected. Then $P(G) - 1$ is connected. Choose an element g of order p. Since G is not of a prime order, there is an element of a coprime order to p in G, say t. Now the connectivity of $P(G) - 1$ implies that g and t lie on a path. But, $\langle g \rangle$ does not have a non-trivial subgroup and since g has the maximum order, g is not contained in any cyclic subgroup of G. Therefore, there is no path from g to t in $P(G) - 1$, which is a contradiction. $\square$.

Simple groups A_p, $p = 5, 6, 7$, $L_2(p)$, for prime p, $p \geq 5$, and the sporadic groups, M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, $O'N$, Ly, J_3, Ru, are examples of groups that satisfy the condition of Lemma 12 and so their power graphs are not $2-$connected.

Our calculations with finite groups of small orders suggests the following conjecture:

Question 13. Does there exist a non-abelian simple group with a $2-$connected power graph?

In the following theorem we prove that the number of edges in the power graphs of a simple group of order n is at most the number of edges in the power graph of a cyclic group of order n.

Theorem 14. If G is a finite simple group of order n then $|E(P(G))| \leq |E(P(Z_n))|$.

Proof. We first prove that if $x \in Z_n$ then

$$deg(x) = o(x) + \sum_{ko(x)|n \ \& \ k\neq 1} \varphi(ko(x)) - 1.$$

To do this, it is enough to note that the number of the subgroups of Z_n containing $\langle x \rangle$ as a proper subgroup is $\sum_{ko(x)|n \ \& \ k\neq 1} \varphi(ko(x))$. Next we show that $\delta(Z_n) \geq \frac{n}{p}$, where

p is the largest prime factor of n. For a proof, we notice that $\varphi(ko(x)) \geq \varphi(o(x))\varphi(k)$ and so $\sum_{ko(x)|n \ \& \ k\neq 1} \varphi(ko(x)) \geq \sum_{ko(x)|n \ \& \ k\neq 1} \varphi(o(x))\varphi(k)$. Thus,

$$\begin{aligned} deg(x) &\geq o(x) + \varphi(o(x)) \sum_{k|\frac{n}{o(x)} \ \& \ k\neq 1} \varphi(k) - 1 \\ &= o(x) + \varphi(o(x))|G : \langle x\rangle| - \varphi(o(x)) - 1. \end{aligned}$$

If q is the largest prime divisor of $o(x)$ then $p \geq q$. By [1, Lemma C], if n is a positive integer and p is the largest prime factor of n then $\varphi(n) \geq \frac{n}{p}$. Hence, $deg(x) \geq o(x) - \varphi(o(x)) - 1 + \frac{n}{p}$. Since $o(x) \geq \varphi(o(x)) + 1$, $\delta(Z_n) \geq \frac{n}{p}$. Define

$$s = Max\{deg(g) \mid g \in P(G) - e\}.$$

From the Euler's theorem in elementary graph theory, if $\Delta(P(G)-e) \leq \delta(P(Z_n))$ then $|E(P(G))| \leq |E(P(Z_n))|$.

To complete our argument, we prove that if G is a finite group and $x \in G$ such that $deg_{P(G)-e}(x) > \frac{|G|}{p}$ then G has a non-trivial normal subgroup N. Set

$$A = \{g \in G \mid \ | \ \langle x\rangle \leq \langle g\rangle \ or \ \langle g\rangle \leq \langle x\rangle\}.$$

By the assumption $|A| > \frac{n}{p}$, we have $r = |G : C_G(x)| < p$. If $r = 1$ then $x \in Z(G)$, as desired. Otherwise $r \neq 1$. If G is simple then G can be embedded into A_r. This implies that $p|r!$, which is impossible. This completes the proof. □

2.4. **A Classification Theorem.** In this subsection a complete classification of groups in which the power graph is a union of complete graphs that share the identity is presented. We need to the following general result for the main result of this section.

Lemma 15. Let G be a finite group. If $P(G)$ is a union of complete graphs that share the identity then the power graph of each Sylow subgroup of G has the same structure.

Proof. Suppose Q is a Sylow $p-$subgroup of G of order p^s. By [6, Proposition 5.4], $P(Q)$ is an induced subgraph of $P(G)$ and so $P(Q) \subseteq \cup X$, where each X is a complete subgraph of $P(G)$. Suppose $\cup_{i=1}^{r} X_i$ is an irredundant cover of $P(Q)$, i.e., no proper subset is also a cover. Hence, each X_i contains an element y_i such that $p|o(y_i)$. We first show that X_i is a subgroup of G. Set $X_i = \{e = x_1, \cdots, x_n\}$. Choose $x_i, x_j \in X_i$. Since x_i and x_j are adjacent, $x_i = x_j^u$ or $x_j = x_i^v$. Therefore, $x_ix_j = x_j^{u+1}$ or $x_ix_j = x_i^{v+1}$ and so $x_ix_j \in X_i$, as desired. Since each X_i has an element of Q, by [6, Theorem 2.12] that states that the power graph is complete if and only if the group is cyclic of prime power order, X_i has prime power order.

Define $Y_i = \{x \in X_i \mid x \in Q\}$, $1 \leq i \leq r$. Since $P(Q)$ is the induced subgraph of $P(G)$ and X_i is a complete graph, then $Y_i = P(G) \cap X_i$ is always complete. $\square$

The following lemma is crucial in the last result of this paper.

Lemma 16 (See [8, Theorem 12]). $P(G)$ is a union of complete graphs which share the identity element of G if and only if G is an EPPO-group and for every maximal cyclic subgroup A and B with $A \neq B$, $A \cap B = e$.

In [5, Examples], P. J. Cameron proved that if G is a finite group of order n and exponent 3 then $P(G)$ consists of $\frac{n-1}{2}$ triangles sharing a common vertex. In the following theorem we classify such $p-$groups.

Theorem 17. Let G be a finite p-group. Then the power graph $P(G)$ is a union of complete subgraphs which share the identity element of G if and only if G is isomorphic to a cyclic group, p-group of exponent p or a dihedral group.

Proof. Let $|G| = p^n$ with $\exp(G) \neq p$. Suppose that $P(G) \subseteq \cup_{i=1}^{r} \Gamma_i$ is an irredundant cover of complete subgraphs of $P(G)$. The vertex set $X_i = V(\Gamma_i)$ is a subgroup of G. Since $\exp(G) \neq p$, there exists a cyclic subgroup $T = X_j$ of order grater than p. Assume that $T = \langle t \rangle$ and $g \in G \backslash T$ such that $[t, g] = 1$, then $o(tg) \neq p$. Apply Lemma 16 to prove that $\langle t \rangle \cap \langle tg \rangle = e$. If g is a central element of order p of G then $\langle t \rangle \cap \langle tg \rangle \neq e$ and hence $g \in T$. Therefore, $\Omega_1(Z(G)) \leq T$ and so $Z(G)$ is cyclic. Also, T is a unique subgroup of order grater than p, which deduces that $T \lhd G$ and $Z(G) \leq T$. Suppose $Z(G) = \langle z \rangle$. If $o(z) > p$ then for all $g \in G$, we have $o(zg) > p$, therefore $zg \in T$ and so $g \in G$. This implieas that $G = T$ is cyclic.

Let $|Z(G)| = p$ and G is not cyclic. We also assume that $|T| = \exp(G) = p^{\ell}$ and $g \in G \backslash T$. Consider the subgroup $H = T\langle g \rangle$. Clearly, H is non-abelian. Let p is odd. Since g is of order p and $\mathrm{Aut}(T)$ has just one subgroup of order p, $t^g = t^s$ where $s = 1 + p^{\ell-1}$. Set $s_i = 1 + ip^{\ell-1}$, then $t^{s^i} = t^{s_i}$ and hence

$$(tg)^p = \prod_{i=0}^{p-1} t^{s^i} = \prod_{i=0}^{p-1} t^{s_i} = t^{\sum_{i=0}^{p-1} s_i} = t^{p+p^{\ell-1}[p(p-1)/2]} = t^p,$$

which is impossible. So $p = 2$ and $T\langle g \rangle$ is isomorphic to a dihedral, semidihedral or quasidihedral group. Since semidihedral and quasidihedral groups have more than one subgroup of order greater than 2, $T\langle g\rangle \cong D_{2^{\ell+2}}$. Now, if $x \in G\backslash H$, then $T\langle x\rangle$ must be dihedral and so for both x and g we have $t^x = t^g = t^{-1}$. Therefore, $[t, xg] = 1$ and $T\langle xg\rangle$ is abelian, which is impossible. We conclude that $G = T\langle g\rangle$ is of dihedral type. □

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