THE BEST CONSTANTS IN MULTIDIMENSIONAL HILBERT-TYPE INEQUALITIES INVOLVING SOME WEIGHTED MEANS OPERATORS

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Abstract. In this paper we establish several multidimensional Hilbert-type inequalities with a homogeneous kernel, involving the weighted geometric and harmonic mean operators in the integral case. The general results are derived for the case of non-conjugate parameters. A special emphasis is dedicated to determining conditions under which the obtained inequalities include the best possible constants on their right-hand sides, which can be established after reduction to the conjugate case. As an application, we consider some particular examples and compare our results with the previously known from the literature.

1. Introduction

At the beginning of the 20th century, the following inequality has been established:

$$\int_{\mathbb{R}_+^2} \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin\frac{\pi}{p}}\|f\|_p\|g\|_q, \tag{1}$$

that holds for non-negative functions $f \in L^p(\mathbb{R}_+)$ and $g \in L^q(\mathbb{R}_+)$, such that $0 < \|f\|_p, \|g\|_q < \infty$. Here, and throughout this paper $||\cdot||_r$ denotes the usual norm in $L^r(\mathbb{R}_+)$, i.e. $||f||_r = \left(\int_{\mathbb{R}_+} |f(x)|^r dx\right)^{1/r}$, $r \geq 1$. The parameters p and q appearing in (1) are mutually conjugate, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, where $p > 1$. In addition, the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ is the best possible in the sense that it can not be replaced with a smaller constant so that (1) still holds.

The above inequality was first studied by D. Hilbert at the end of the 19th century, hence, in his honor, it is referred to as the Hilbert inequality.

After its discovery, the Hilbert inequality was studied by numerous authors, who either reproved it using various techniques, or applied and generalized it in many different ways. Such generalizations included inequalities with more general kernels, weight functions and integration sets, extension to a multidimensional case, equivalent forms, and so forth. The resulting relations are usually referred to as the Hilbert-type inequalities. For a comprehensive inspection of the initial development of the Hilbert inequality, the reader is referred to classical monographs [10] and [16].

Although classical, the Hilbert inequality is still of interest to numerous mathematicians. Nowadays, more than a century after its discovery, this problem area offers diverse possibilities for generalizations and extensions (see, e.g. [5], [9], [11],

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[12], [17], [19], [20], and references therein). The most important recent results regarding Hilbert-type inequalities are collected in a monograph [14], that provides a unified treatment of such inequalities.

The starting point in this article is a recent result of Adiyasuren and Batbold [1], who derived a pair of Hilbert-type inequalities with a homogeneous kernel, involving an arithmetic mean integral operator $\mathcal{A} : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$, $p > 1$, defined by $(\mathcal{A}f)(x) = \frac{1}{x}\int_0^x f(t)dt$. Namely, they obtained inequalities

$$\int_{\mathbb{R}_+^2} K_\lambda(x,y)x^{r-\frac{1}{q}}y^{s-\frac{1}{p}}(\mathcal{A}f)(x)(\mathcal{A}g)(y)dxdy < pqc_\lambda(s)\|f\|_p\|g\|_q \tag{2}$$

and

$$\left[\int_{\mathbb{R}_+} y^{ps-1}\left(\int_{\mathbb{R}_+} K_\lambda(x,y)x^{r-\frac{1}{q}}(\mathcal{A}f)(x)dx\right)^p dy\right]^{\frac{1}{p}} < qc_\lambda(s)\|f\|_p, \tag{3}$$

where $\frac{1}{p}+\frac{1}{q} = 1$, $p > 1$, $r, s > 0$, $\lambda = r+s$, $0 < \|f\|_p, \|g\|_q < \infty$, and $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a homogeneous function of degree $-\lambda$ such that $0 < c_\lambda(s) = \int_0^\infty K_\lambda(1,t)t^{s-1}dt < \infty$. Observe that the operator $\mathcal{A}$ is well-defined due to the famous Hardy inequality $\|\mathcal{A}f\|_{L^p(\mathbb{R}_+)} < q\|f\|_{L^p(\mathbb{R}_+)}$, that holds for $0 < \|f\|_p < \infty$. In fact, this operator is known in the literature as the Hardy operator (for more details see [15]).

Inequalities (2) and (3) may be deduced throughout a unified treatment of Hilbert-type inequalities (see [11] or [14], Corollary 1.1, p. 12) and with the help of the Hardy inequality. The most interesting fact in connection with inequalities (2) and (3) is that the constants $pqc_\lambda(s)$ and $qc_\lambda(s)$ remain the best possible (for more details, see [1]). In addition, the above inequalities (2) and (3) were recently extended to a multidimensional case in [13]. That result will be cited in the next section.

The main objective of this paper is a study of multidimensional Hilbert-type inequalities with a homogeneous kernel and with some other means operators, i.e. the weighted geometric and harmonic mean operators. A special emphasis is placed on establishing conditions under which such inequalities include the best possible constants on their right-hand sides.

The paper is divided into six sections as follows: After this Introduction, in Section 2 we introduce the concept of non-conjugate parameters in the multidimensional case and cite the corresponding general Hilbert-type inequalities which will be the base of our study. Further, in Section 3 we define the weighted geometric and harmonic mean operators, and also cite some well-known inequalities closely connected to these operators. In Section 4 we derive Hilbert-type inequalities with a homogeneous kernel, including the weighted geometric and harmonic mean operators. After reducing to the conjugate case, in Section 5 we establish conditions under which the derived inequalities include the best possible constants. Finally, in the last section we deal with some particular examples, and compare them with the previously known, from the literature.

2. A unified treatment of Hilbert-type inequalities with non-conjugate exponents

After the discovery, one of the most interesting problems in connection with the Hilbert inequality was the question whether it is possible to establish the corresponding inequalities where the exponents are not conjugate. The answer to that question appeared to be true. This problem was dealt by some famous mathematicians such as F.F. Bonsall, G.H. Hardy, V. Levin, J. Littlewood, G. Pólya, in the first half of the twentieth century, and later, by E.K. Godunova. This bring us to the concept of non-conjugate parameters.

Let p_i be the real parameters satisfying

$$\sum_{i=1}^{n} \frac{1}{p_i} > 1, \quad p_i > 1, \ i = 1, 2, \dots, n. \tag{4}$$

The parameters p_i' are defined as associated conjugates, that is,

$$\frac{1}{p_i} + \frac{1}{p_i'} = 1, \quad i = 1, 2, \dots, n. \tag{5}$$

Since $p_i > 1$, it follows that $p_i' > 1$, $i = 1, 2, \dots, n$. In addition, we define

$$\lambda_n = \frac{1}{n-1} \sum_{i=1}^{n} \frac{1}{p_i'}. \tag{6}$$

Clearly, relations (4) and (5) imply that $0 < \lambda_n < 1$. Finally, let q_i be defined by

$$\frac{1}{q_i} = \lambda_n - \frac{1}{p_i'}, \quad i = 1, 2, \dots, n, \tag{7}$$

provided that $q_i > 0$, $i = 1, 2, \dots, n$. The above conditions $(4) - (7)$ provide the n-tuple of non-conjugate exponents and were given by Bonsall [2], more than half a century ago. Note also that $\lambda_n = \sum_{i=1}^{n} 1/q_i$ and $1/q_i + 1 - \lambda_n = 1/p_i$, $i = 1, 2, \dots, n$. Of course, if $\lambda_n = 1$, then $\sum_{i=1}^{n} 1/p_i = 1$, which represents the setting with conjugate parameters.

In 2005, Brnetić *et.al.* [3], provided a unified treatment of multidimensional Hilbert-type inequalities with non-conjugate exponents, with a main result including a general measurable kernel and weight functions. Moreover, Perić and Vuković [17], studied the latter inequalities for the case of a homogeneous kernel. Before we state the corresponding result, we need some conventions.

Recall that the function $K : \mathbb{R}_+^n \to \mathbb{R}$ is said to be homogeneous of degree $-s$, $s > 0$, if $K(t\mathbf{x}) = t^{-s} K(\mathbf{x})$ for all $t > 0$ and $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$. Furthermore, if $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, we define

$$k_i(\mathbf{a}) = \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^i) \prod_{j=1, j \neq i}^{n} u_j^{a_j} \hat{d}^i \mathbf{u}, \quad i = 1, 2, \dots, n, \tag{8}$$

where $\hat{\mathbf{u}}^i = (u_1, \dots, u_{i-1}, 1, u_{i+1}, \dots, u_n)$, $\hat{d}^i \mathbf{u} = du_1 \dots du_{i-1} du_{i+1} \dots du_n$, and provided that the above integral converges. Further, in the sequel $d\mathbf{u}$ is an abbreviation for $du_1 du_2 \dots du_n$.

The following pair of multidimensional Hilbert-type inequalities, in a slightly altered notation, can be found in [17] (see also [3]):

Theorem 1. *Let p_i, p_i', q_i, $i = 1, 2, \dots, n$, and λ_n be as in (4)–(7), and let A_{ij}, $i, j = 1, 2, \dots, n$, be real parameters such that $\sum_{i=1}^{n} A_{ij} = 0$. If $K : \mathbb{R}_+^n \to \mathbb{R}$ is a non-negative measurable homogeneous function of degree $-s$, $s > 0$, and $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \dots, n$, are non-negative measurable functions, then the following two* *inequalities hold and are equivalent:*

$$\int_{\mathbb{R}_+^n} K^{\lambda_n}(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i) d\mathbf{x} \leq \prod_{i=1}^{n} k_i^{1/q_i}(q_i \mathbf{A_i}) \prod_{i=1}^{n} ||x_i^{(n-1-s)/q_i+\alpha_i} f_i||_{p_i}, \tag{9}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(1-\lambda_n p_n')(n-1-s)-p_n'\alpha_n} \left(\int_{\mathbb{R}_+^{n-1}} K^{\lambda_n}(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i) \hat{d}^n\mathbf{x} \right)^{p_n'} dx_n \right]^{1/p_n'}$$

$$\leq \prod_{i=1}^{n} k_i^{1/q_i}(q_i \mathbf{A_i}) \prod_{i=1}^{n-1} ||x_i^{(n-1-s)/q_i+\alpha_i} f_i||_{p_i}, \tag{10}$$

where $\alpha_i = \sum_{j=1}^{n} A_{ij}$, $\mathbf{A_i} = (A_{i1}, A_{i2}, \ldots, A_{in})$ *and* $k_i(q_i \mathbf{A_i}) < \infty$, $i = 1, 2, \ldots, n$.

Remark 1. It should be noticed here that the problem of the best constants in Hilbert-type inequalities with non-conjugate exponents seems to be very hard and remains open. That problem can be solved after reducing to the conjugate case.

In [13], Krnić considered inequalities (9) and (10) with the Hardy operator $(\mathcal{A}f)(x) = \frac{1}{x}\int_0^x f(t)dt$. In this setting, the above inequalities read respectively

$$\int_{\mathbb{R}_+^n} K^{\lambda_n}(\mathbf{x}) \prod_{i=1}^{n} x_i^{\frac{s+1-n}{q_i}-\alpha_i} (\mathcal{A}f_i)^{\mu_i}(x_i) d\mathbf{x} \leq h_n^s(\mathbf{p}, \mathbf{q}, \mathbf{A}, \boldsymbol{\mu}) \prod_{i=1}^{n} ||f_i^{\mu_i}||_{p_i}, \tag{11}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(1-\lambda_n p_n')(n-1-s)-p_n'\alpha_n} \times \right.$$

$$\left. \times \left(\int_{\mathbb{R}_+^{n-1}} K^{\lambda_n}(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\frac{s+1-n}{q_i}-\alpha_i} (\mathcal{A}f_i)^{\mu_i}(x_i) \hat{d}^n\mathbf{x} \right)^{p_n'} dx_n \right]^{1/p_n'}$$

$$\leq h_{n-1}^s(\mathbf{p}, \mathbf{q}, \mathbf{A}, \boldsymbol{\mu}) \prod_{i=1}^{n-1} ||f_i^{\mu_i}||_{p_i}, \tag{12}$$

where μ_i are real parameters such that $p_i\mu_i > 1$, $i = 1, 2, \ldots, n$, and

$$h_n^s(\mathbf{p}, \mathbf{q}, \mathbf{A}, \boldsymbol{\mu}) = \prod_{i=1}^{n} k_i^{1/q_i}(q_i \mathbf{A_i}) \prod_{i=1}^{n} \left(\frac{p_i\mu_i}{p_i\mu_i - 1} \right)^{\mu_i},$$

$$h_{n-1}^s(\mathbf{p}, \mathbf{q}, \mathbf{A}, \boldsymbol{\mu}) = \prod_{i=1}^{n} k_i^{1/q_i}(q_i \mathbf{A_i}) \prod_{i=1}^{n-1} \left(\frac{p_i\mu_i}{p_i\mu_i - 1} \right)^{\mu_i}.$$

Observe that inequalities (11) and (12) may be regarded as a multidimensional extension of relations (2) and (3).

Remark 2. According to Remark 1, we can not decide whether or not the constants $h_n^s(\mathbf{p}, \mathbf{q}, \mathbf{A}, \boldsymbol{\mu})$ and $h_{n-1}^s(\mathbf{p}, \mathbf{q}, \mathbf{A}, \boldsymbol{\mu})$ are the best possible in (11) and (12). Roughly speaking, these constants appear to be the best possible for the conjugate setting, in the case when the part of the constant regarding homogeneous kernel contain no exponents (for more details see [13]).

Our further step is to derive analogues of (11) and (12), with the weighted geometric and harmonic mean operators, instead of the Hardy operator $\mathcal{A}$. Therefore, we need to define the corresponding operators and derive some auxiliary results.

3. The weighted geometric and harmonic mean operators

We have already seen that the arithmetic mean (Hardy) operator $\mathcal{A}$ appears in the Hardy inequality. Similarly, the weighted geometric and harmonic mean operators are closely connected to some well-known inequalities.

In 1984, Cochran and Lee [4], obtained the following inequality

$$\int_0^\infty x^{\gamma-1} \exp\left[\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\log f(t)dt\right]dx \le e^{\gamma/\alpha}\int_0^\infty x^{\gamma-1}f(x)dx, \tag{13}$$

with the best constant $e^{\gamma/\alpha}$, where $\alpha,\gamma\in\mathbb{R},\alpha>0$, and $\int_0^\infty x^{\gamma-1}f(x)dx<\infty$. Inequality (13) is known in the literature as the Levin-Cochran-Lee inequality and it includes the weighted geometric mean operator $\mathcal{G}$ defined by

$$(\mathcal{G}f)(x) = \exp\left[\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\log f(t)dt\right]. \tag{14}$$

Clearly, if $\gamma=1$, the above inequality may be rewritten as $\|\mathcal{G}f\|_p \le e^{1/\alpha p}\|f\|_p$, $p>1$, which means that the norm of operator $\mathcal{G}: L^p(\mathbb{R}_+)\to L^p(\mathbb{R}_+)$ is equal to $e^{1/\alpha p}$. It should be noticed here that for $\alpha=\gamma=1$, inequality (13) reduces to the well-known Knopp inequality (see [16]).

In order to define the weighted harmonic mean operator, we first cite the following inequality from [6]: Let $a,b,r,s\in\mathbb{R}, a<b, r<s, r,s\neq 0$, and f be a non-negative measurable function. Then,

$$\begin{aligned}&\left\{\frac{1}{(b-a)^\gamma}\int_a^b (x-a)^{\gamma-1}\left[\frac{1}{(x-a)^\alpha}\int_a^x (t-a)^{\alpha-1}f^r(t)dt\right]^{\frac{s}{r}}dx\right\}^{\frac{1}{s}}\\ &\le \left\{\frac{1}{(b-a)^\alpha}\int_a^b (x-a)^{\alpha-1}\left[\frac{1}{(x-a)^\gamma}\int_a^x (t-a)^{\gamma-1}f^s(t)dt\right]^{\frac{r}{s}}dx\right\}^{\frac{1}{r}},\end{aligned} \tag{15}$$

where $\alpha,\gamma\in\mathbb{R}$. The above inequality is crucial in establishing the mixed means inequality (for more details see [6]).

Lemma 1. *Let α,γ, and $r>0$ be real numbers such that $\alpha+\gamma r>0$ and f be a non-negative measurable function. If $\int_0^\infty x^{\gamma-1}f(x)dx<\infty$, then*

$$\int_0^\infty x^{\gamma-1}\left[\frac{x^\alpha}{\int_0^x t^{\alpha-1}f^{-r}(t)dt}\right]^{\frac{1}{r}}dx \le (\alpha+\gamma r)^{\frac{1}{r}}\int_0^\infty x^{\gamma-1}f(x)dx, \tag{16}$$

where the constant $(\alpha+\gamma r)^{\frac{1}{r}}$ is the best possible.

Proof. Setting $a=0, s=1$, and $r=-r$, inequality (15) reduces to

$$\int_0^b x^{\gamma-1}\left[\frac{x^\alpha}{\int_0^x t^{\alpha-1}f^{-r}(t)dt}\right]^{\frac{1}{r}}dx \le b^{\frac{\alpha}{r}+\gamma}\left[\int_0^b \frac{x^{\alpha-1+\gamma r}}{\left(\int_0^x t^{\gamma-1}f(t)dt\right)^r}dx\right]^{-\frac{1}{r}}. \tag{17}$$

Further, since $\int_0^x t^{\gamma-1}f(t)dt \le \int_0^b t^{\gamma-1}f(t)dt,\ 0 \le x \le b$, the right-hand side of (17) does not exceed

$$b^{\frac{\alpha}{r}+\gamma}\left(\int_0^b x^{\alpha-1+\gamma r}dx\right)^{-\frac{1}{r}}\left(\int_0^b x^{\gamma-1}f(x)dx\right) = (\alpha+\gamma r)^{\frac{1}{r}}\int_0^b x^{\gamma-1}f(x)dx.$$

Therefore we have

$$\int_0^b x^{\gamma-1}\left[\frac{x^\alpha}{\int_0^x t^{\alpha-1}f^{-r}(t)dt}\right]^{\frac{1}{r}}dx \le (\alpha+\gamma r)^{\frac{1}{r}}\int_0^b x^{\gamma-1}f(x)dx,$$

so (16) follows by letting b to infinity.

In order to prove that (16) includes the best possible constant, we suppose that there exists a positive L, smaller than $(\alpha+\gamma r)^{\frac{1}{r}}$, such that the inequality

$$\int_0^\infty x^{\gamma-1}\left[\frac{x^\alpha}{\int_0^x t^{\alpha-1}f^{-r}(t)dt}\right]^{\frac{1}{r}}dx \le L\int_0^\infty x^{\gamma-1}f(x)dx$$

holds for all non-negative functions $f:\mathbb{R}_+\to\mathbb{R}$, provided $\int_0^\infty x^{\gamma-1}f(x)dx<\infty$. Considering the function

$$\widetilde{f}(x) = \begin{cases} x^{\varepsilon-\gamma}, & 0<x\le 1 \\ 0, & x>1 \end{cases},$$

where $\varepsilon>0$ is sufficiently small number, we have

$$\int_0^1 x^{\gamma-1}\left[\frac{x^\alpha}{\int_0^x t^{\alpha-1-r(\varepsilon-\gamma)}dt}\right]^{\frac{1}{r}}dx \le L\int_0^1 x^{\varepsilon-1}dx = \frac{L}{\varepsilon}.$$

The above relation yields $(\alpha-r\varepsilon+\gamma r)^{\frac{1}{r}} \le L$, and for $\varepsilon\to 0^+$, it follows that $(\alpha+\gamma r)^{\frac{1}{r}}\le L$. This contradicts with $L<(\alpha+\gamma r)^{\frac{1}{r}}$, which means that $(\alpha+\gamma r)^{\frac{1}{r}}$ is the best possible constant in (16). □

Motivated by Lemma 1, we define the weighted harmonic mean operator $\mathcal{H}$ by

$$(\mathcal{H}f)(x) = \frac{x^\alpha}{\int_0^x t^{\alpha-1}f^{-1}(t)dt}. \tag{18}$$

If $\gamma=1$, the inequality (16) may be rewritten as $\|\mathcal{H}f\|_p \le (\alpha+1/p)\|f\|_p$, $p>1$, so Lemma 1 implies that the norm of operator $\mathcal{H}: L^p(\mathbb{R}_+)\to L^p(\mathbb{R}_+)$ is equal to $\alpha+1/p$.

4. Multidimensional Hilbert-type inequalities with geometric and harmonic mean operators

In this section we derive analogues of relations (11) and (12), where the Hardy operator $\mathcal{A}$ is replaced by geometric and harmonic mean operators, defined in the previous section. Obviously, the starting point is Theorem 1, which provides a unified treatment of Hilbert-type inequalities with a homogeneous kernel in the non-conjugate case. The first result refers to the weighted geometric operator $\mathcal{G}$ defined by (14).

Theorem 2. *Suppose* p_i, p_i', q_i, $i = 1, 2, \dots, n$, *and* λ_n *are as in* (4)–(7), *and* A_{ij}, $i, j = 1, 2, \dots, n$, *are real parameters satisfying* $\sum_{i=1}^n A_{ij} = 0$. *Further, let* $\alpha_i = \sum_{j=1}^n A_{ij}$, $i = 1, 2, \dots, n$, *and let* ν_i, μ_i, *and* $\alpha > 0$ *be real parameters. If* $K : \mathbb{R}_+^n \to \mathbb{R}$ *is a non-negative measurable homogeneous function of degree* $-s$, $s > 0$, *and* $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \dots, n$, *non-negative measurable functions, then*

$$\int_{\mathbb{R}_+^n} K^{\lambda_n}(\mathbf{x}) \prod_{i=1}^n x_i^{\nu_i} \left(\mathcal{G} f_i\right)^{\mu_i}(x_i) d\mathbf{x} \le k_n^s(\mathbf{p}, \mathbf{q}, \mathbf{A}, \boldsymbol{\nu}) \prod_{i=1}^n ||x_i^{\frac{(n-1-s)}{q_i} + \alpha_i + \nu_i} f_i^{\mu_i}||_{p_i}, \tag{19}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(1-\lambda_n p_n')(n-1-s) - p_n' \alpha_n} \left(\int_{\mathbb{R}_+^{n-1}} K^{\lambda_n}(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\nu_i} \left(\mathcal{G} f_i\right)^{\mu_i}(x_i) \hat{d}^n \mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le k_{n-1}^s(\mathbf{p}, \mathbf{q}, \mathbf{A}, \boldsymbol{\nu}) \prod_{i=1}^{n-1} ||x_i^{\frac{(n-1-s)}{q_i} + \alpha_i + \nu_i} f_i^{\mu_i}||_{p_i}, \tag{20}$$

where

$$k_n^s(\mathbf{p}, \mathbf{q}, \mathbf{A}, \boldsymbol{\nu}) = e^{\frac{1}{\alpha}\left[-\lambda_n s + n + \sum_{i=1}^n \nu_i\right]} \prod_{i=1}^n k_i^{1/q_i}(q_i \mathbf{A_i}),$$
$$k_{n-1}^s(\mathbf{p}, \mathbf{q}, \mathbf{A}, \boldsymbol{\nu}) = e^{\frac{1}{\alpha}\left[\lambda_n(1-s) + n - 1 - \alpha_n - \frac{n-s}{q_n} + \sum_{i=1}^{n-1} \nu_i\right]} \prod_{i=1}^n k_i^{1/q_i}(q_i \mathbf{A_i}),$$

$\mathbf{A_i} = (A_{i1}, A_{i2}, \dots, A_{in})$, *and* $k_i(q_i \mathbf{A_i}) < \infty$, $i = 1, 2, \dots, n$.

Proof. The result is an immediate consequence of general Hilbert-type inequalities (9) and (10) equipped with the functions $x_i^{\nu_i} \left(\mathcal{G} f_i\right)^{\mu_i}(x_i)$ instead of $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \dots, n$, and the Levin-Cochran-Lee inequality (13). Namely, applying (13) to the right-hand sides of (9) and (10) yields

$$||x_i^{(n-1-s)/q_i + \alpha_i + \nu_i} (\mathcal{G} f_i)^{\mu_i}(x_i)||_{p_i} \le e^{\frac{1}{\alpha}\left[\frac{n-s}{q_i} + \alpha_i + \nu_i - \lambda_n + 1\right]} ||x_i^{\frac{(n-1-s)}{q_i} + \alpha_i + \nu_i} f_i^{\mu_i}||_{p_i},$$

which completes the proof. □

The following pair of Hilbert-type inequalities deals with the weighted harmonic mean operator $\mathcal{H}$, defined by (18).

Theorem 3. *Suppose* p_i, p_i', q_i, $i = 1, 2, \dots, n$, *and* λ_n *are as in* (4)–(7), *and* A_{ij}, $i, j = 1, 2, \dots, n$, *are real parameters such that* $\sum_{i=1}^n A_{ij} = 0$. *Further, let* $\alpha_i = \sum_{j=1}^n A_{ij}$, $i = 1, 2, \dots, n$, *and let* α, ν_i *and* $\mu_i > 0$ *be real parameters such that* $\alpha + \frac{1}{\mu_i}\left(1 - \lambda_n + \alpha_i + \nu_i + \frac{n-s}{q_i}\right) > 0$. *If* $K : \mathbb{R}_+^n \to \mathbb{R}$ *is a non-negative measurable homogeneous function of degree* $-s$, $s > 0$, *and* $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \dots, n$, *non-negative measurable functions, then*

$$\int_{\mathbb{R}_+^n} K^{\lambda_n}(\mathbf{x}) \prod_{i=1}^n x_i^{\nu_i} \left(\mathcal{H} f_i\right)^{\mu_i}(x_i) d\mathbf{x} \le l_n^s(\mathbf{p}, \mathbf{q}, \mathbf{A}, \boldsymbol{\nu}, \boldsymbol{\mu}) \prod_{i=1}^n ||x_i^{\frac{(n-1-s)}{q_i} + \alpha_i + \nu_i} f_i^{\mu_i}||_{p_i}, \tag{21}$$

and

$$\left[\int_{\mathbb{R}_+} x_n{}^{(1-\lambda_n p_n')(n-1-s)-p_n'\alpha_n}\left(\int_{\mathbb{R}_+^{n-1}} K^{\lambda_n}(\mathbf{x})\prod_{i=1}^{n-1} x_i^{\nu_i}\left(\mathcal{H}f_i\right)^{\mu_i}(x_i)\hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$

$$\leq l_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A},\boldsymbol{\nu},\boldsymbol{\mu})\prod_{i=1}^{n-1}||x_i^{\frac{(n-1-s)}{q_i}+\alpha_i+\nu_i} f_i^{\mu_i}||_{p_i}, \tag{22}$$

where

$$l_n^s(\mathbf{p},\mathbf{q},\mathbf{A},\boldsymbol{\nu},\boldsymbol{\mu}) = \prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^n\left[\alpha+\frac{1}{\mu_i}\left(1-\lambda_n+\alpha_i+\nu_i+\frac{n-s}{q_i}\right)\right]^{\mu_i},$$

$$l_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A},\boldsymbol{\nu},\boldsymbol{\mu}) = \prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^{n-1}\left[\alpha+\frac{1}{\mu_i}\left(1-\lambda_n+\alpha_i+\nu_i+\frac{n-s}{q_i}\right)\right]^{\mu_i},$$

$\mathbf{A_i} = (A_{i1}, A_{i2}, \dots, A_{in})$, $k_i(q_i\mathbf{A_i}) < \infty$, *and* $i = 1, 2, \dots, n$.

Proof. We follow the same procedure as in the proof of the previous theorem, except that we use inequality (16) instead of the Levin-Cochran-Lee inequality.

More precisely, considering (9) and (10) with the functions $x_i^{\nu_i}\left(\mathcal{H}f_i\right)^{\mu_i}(x_i)$, $i = 1, 2, \dots, n$, it follows that

$$\begin{aligned}&||x_i^{(n-1-s)/q_i+\alpha_i+\nu_i}(\mathcal{H}f_i)^{\mu_i}(x_i)||_{p_i}\\ &\leq\left[\alpha+\frac{1}{\mu_i}\left(1-\lambda_n+\alpha_i+\nu_i+\frac{n-s}{q_i}\right)\right]^{\mu_i}||x_i^{\frac{(n-1-s)}{q_i}+\alpha_i+\nu_i} f_i^{\mu_i}||_{p_i},\end{aligned}$$

and the proof is completed. □

Our next step is to determine conditions under which the constants $k_n^s(\mathbf{p},\mathbf{q},\mathbf{A},\boldsymbol{\nu})$, $k_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A},\boldsymbol{\nu})$, $l_n^s(\mathbf{p},\mathbf{q},\mathbf{A},\boldsymbol{\nu},\boldsymbol{\mu})$, and $l_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A},\boldsymbol{\nu},\boldsymbol{\mu})$ are the best possible in the corresponding inequalities. This happens in the case of conjugate exponents.

5. Reduction to the conjugate case and the best constants

We have already mentioned that the problem of the best possible constants in Hilbert-type inequalities, for the case of non-conjugate exponents, seems to be very hard and remains unsolved.

Hence, in order to obtain the best possible constants in inequalities (19), (20), (21), and (22), we consider here their conjugate forms. Namely, if $p_i > 1$, $i = 1, 2, \dots, n$, is the set of conjugate exponents, then inequalities (19) and (20) become respectively

$$\int_{\mathbb{R}_+^n} K(\mathbf{x})\prod_{i=1}^n x_i^{\nu_i}\left(\mathcal{G}f_i\right)^{\mu_i}(x_i)d\mathbf{x} \leq \overline{k}_n^s(\mathbf{p},\mathbf{A},\boldsymbol{\nu})\prod_{i=1}^n||x_i^{\frac{(n-1-s)}{p_i}+\alpha_i+\nu_i} f_i^{\mu_i}||_{p_i}, \tag{23}$$

$$\left[\int_{\mathbb{R}_+} x_n{}^{(1-p_n')(n-1-s)-p_n'\alpha_n}\left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x})\prod_{i=1}^{n-1} x_i^{\nu_i}\left(\mathcal{G}f_i\right)^{\mu_i}(x_i)\hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$

$$\leq \overline{k}_{n-1}^s(\mathbf{p},\mathbf{A},\boldsymbol{\nu})\prod_{i=1}^{n-1}||x_i^{\frac{(n-1-s)}{p_i}+\alpha_i+\nu_i} f_i^{\mu_i}||_{p_i}, \tag{24}$$

where

$$\overline{k}_n^s(\mathbf{p},\mathbf{A},\boldsymbol{\nu}) = e^{\frac{1}{\alpha}\left[-s+n+\sum_{i=1}^n \nu_i\right]}\prod_{i=1}^n k_i^{1/p_i}(p_i\mathbf{A_i}),$$

$$\overline{k}_{n-1}^s(\mathbf{p},\mathbf{A},\boldsymbol{\nu}) = e^{\frac{1}{\alpha}\left[-s+n-\alpha_n-\frac{n-s}{p_n}+\sum_{i=1}^{n-1} \nu_i\right]}\prod_{i=1}^n k_i^{1/p_i}(p_i\mathbf{A_i}).$$

Similarly, the conjugate forms of inequalities (21) and (22) read respectively

$$\int_{\mathbb{R}_+^n} K(\mathbf{x})\prod_{i=1}^n x_i^{\nu_i}\left(\mathcal{H}f_i\right)^{\mu_i}(x_i)d\mathbf{x} \le \overline{l}_n^s(\mathbf{p},\mathbf{A},\boldsymbol{\nu},\boldsymbol{\mu})\prod_{i=1}^n ||x_i^{\frac{(n-1-s)}{p_i}+\alpha_i+\nu_i} f_i^{\mu_i}||_{p_i}, \tag{25}$$

$$\left[\int_{\mathbb{R}_+} x_n^{(1-p_n')(n-1-s)-p_n'\alpha_n}\left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x})\prod_{i=1}^{n-1} x_i^{\nu_i}\left(\mathcal{H}f_i\right)^{\mu_i}(x_i)\hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$

$$\le \overline{l}_{n-1}^s(\mathbf{p},\mathbf{A},\boldsymbol{\nu},\boldsymbol{\mu})\prod_{i=1}^{n-1} ||x_i^{\frac{(n-1-s)}{p_i}+\alpha_i+\nu_i} f_i^{\mu_i}||_{p_i}, \tag{26}$$

where

$$\overline{l}_n^s(\mathbf{p},\mathbf{A},\boldsymbol{\nu},\boldsymbol{\mu}) = \prod_{i=1}^n k_i^{1/p_i}(p_i\mathbf{A_i})\prod_{i=1}^n\left[\alpha+\frac{1}{\mu_i}\left(\alpha_i+\nu_i+\frac{n-s}{p_i}\right)\right]^{\mu_i},$$

$$\overline{l}_{n-1}^s(\mathbf{p},\mathbf{A},\boldsymbol{\nu},\boldsymbol{\mu}) = \prod_{i=1}^n k_i^{1/p_i}(p_i\mathbf{A_i})\prod_{i=1}^{n-1}\left[\alpha+\frac{1}{\mu_i}\left(\alpha_i+\nu_i+\frac{n-s}{p_i}\right)\right]^{\mu_i}.$$

In the sequel we determine the conditions under which the inequalities (23), (24), (25), and (26) include the best possible constants on their right-hand sides. To do this, we establish some more specific conditions about the convergence of the integral $k_1(\mathbf{a})$, $\mathbf{a}=(a_1,a_2,\dots,a_n)$, defined by (8). More precisely, we assume that

$$k_1(\mathbf{a})<\infty \text{ for } a_2,\dots,a_n>-1, \sum_{i=2}^n a_i < s-n+1, n\in\mathbb{N}, n\ge 2. \tag{27}$$

By the similar reasoning as in some recent results known from the literature (see [17], [18], [20], [19]), the best possible constants can be obtained if their parts regarding homogeneous kernel contain no exponents. For that sake, assume that

$$k_1(p_1\mathbf{A_1}) = k_2(p_2\mathbf{A_2}) = \cdots = k_n(p_n\mathbf{A_n}). \tag{28}$$

Utilizing the change of variables $u_1 = 1/t_2, u_3 = t_3/t_2, u_4 = t_4/t_2, \dots, u_n = t_n/t_2$, which provides the Jacobian of the transformation

$$\left|\frac{\partial(u_1,u_3,\dots,u_n)}{\partial(t_2,t_3,\dots,t_n)}\right| = t_2^{-n},$$

we have

$$\begin{aligned} k_2(p_2\mathbf{A_2}) &= \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{t}}^1)t_2^{s-n-p_2(\alpha_2-A_{22})}\prod_{j=3}^n t_j^{p_2A_{2j}}\hat{d}^1\mathbf{t} \\ &= k_1(p_1A_{11}, s-n-p_2(\alpha_2-A_{22}), p_2A_{23},\dots,p_2A_{2n}). \end{aligned}$$

According to (28), we have $p_1A_{12} = s - n - p_2(\alpha_2 - A_{22})$, $p_1A_{13} = p_2A_{23}$, ..., $p_1A_{1n} = p_2A_{2n}$. In a similar manner we express $k_i(p_i\mathbf{A_i})$, $i = 3, \dots, n$, in terms of $k_1(\cdot)$. In such a way we see that (28) is fulfilled if

$$p_jA_{ji} = s - n - p_i(\alpha_i - A_{ii}), i, j = 1, 2, \dots, n,\ i \neq j. \tag{29}$$

The above set of conditions also implies that $p_iA_{ik} = p_jA_{jk}$, when $k \neq i, j$. Hence, we use abbreviations $\widetilde{A}_1 = p_nA_{n1}$ and $\widetilde{A}_i = p_1A_{1i}$, $i \neq 1$. Since $\sum_{i=1}^n A_{ij} = 0$, one easily obtains that $p_jA_{jj} = \widetilde{A}_j(1 - p_j)$. Moreover, $\sum_{i=1}^n \widetilde{A}_i = s - n$ (see also [18]).

Now, if the set of conditions (29) is fulfilled, then, with the above abbreviations, inequalities (23) and (24) become respectively

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\nu_i} (\mathcal{G}f_i)^{\mu_i} (x_i) d\mathbf{x} \leq m_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}) \prod_{i=1}^n ||x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} f_i^{\mu_i}||_{p_i}, \tag{30}$$

$$\left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\widetilde{A}_n)} \left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\nu_i} (\mathcal{G}f_i)^{\mu_i} (x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$

$$\leq m_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}) \prod_{i=1}^{n-1} ||x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} f_i^{\mu_i}||_{p_i}, \tag{31}$$

where

$$m_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}) = k_1(\widetilde{\mathbf{A}}) e^{\frac{1}{\alpha}\left[-s+n+\sum_{i=1}^n \nu_i\right]},$$

$$m_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}) = k_1(\widetilde{\mathbf{A}}) e^{\frac{1}{\alpha}\left[-s+n+\widetilde{A}_n+\sum_{i=1}^{n-1} \nu_i\right]}.$$

and $\widetilde{\mathbf{A}} = (\widetilde{A}_1, \widetilde{A}_2, \dots, \widetilde{A}_n)$.

In the same way, inequalities (25) and (26) read respectively

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\nu_i} (\mathcal{H}f_i)^{\mu_i} (x_i) d\mathbf{x} \leq \overline{m}_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu}) \prod_{i=1}^n ||x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} f_i^{\mu_i}||_{p_i}, \tag{32}$$

$$\left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\widetilde{A}_n)} \left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\nu_i} (\mathcal{H}f_i)^{\mu_i} (x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$

$$\leq \overline{m}_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu}) \prod_{i=1}^{n-1} ||x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} f_i^{\mu_i}||_{p_i}, \tag{33}$$

where

$$\overline{m}_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu}) = k_1(\widetilde{\mathbf{A}}) \prod_{i=1}^n \left[\alpha + \frac{1}{\mu_i}\left(\nu_i - \widetilde{A}_i\right)\right]^{\mu_i},$$

$$\overline{m}_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu}) = k_1(\widetilde{\mathbf{A}}) \prod_{i=1}^{n-1} \left[\alpha + \frac{1}{\mu_i}\left(\nu_i - \widetilde{A}_i\right)\right]^{\mu_i}.$$

Finally, we show that the constants $m_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu})$, $m_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu})$, $\overline{m}_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu})$, and $\overline{m}_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu})$ are the best possible in the corresponding inequalities.

Theorem 4. *Let $\alpha > 0$, $\widetilde{A}_i \leq \nu_i \leq \frac{\alpha}{p_i} + \widetilde{A}_i$, $i = 1, 2, \dots, n$, and let the parameters $\widetilde{A}_i$, $i = 2, \dots, n$, fulfill conditions as in* (27). *Then, the constant $m_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu})$ is the best possible in the inequality* (30).

Proof. Suppose that there exists a positive constant C_n, $0 < C_n < m_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu})$, such that inequality

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\nu_i} (\mathcal{G}f_i)^{\mu_i} (x_i) d\mathbf{x} \leq C_n \prod_{i=1}^n ||x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} f_i^{\mu_i}||_{p_i} \tag{34}$$

holds for all non-negative measurable functions $f_i : \mathbb{R}_+ \to \mathbb{R}$. Considering this inequality with the functions

$$f_i^\varepsilon(x_i) = \begin{cases} 1, & 0 < x_i < 1, \\ e^{-\frac{1}{p_i\mu_i}} x_i^{\frac{\widetilde{A}_i - \nu_i}{\mu_i} - \frac{\varepsilon}{p_i\mu_i}}, & x \geq 1, \end{cases}$$

where ε is sufficiently small number, its right-hand side becomes

$$C_n \prod_{i=1}^n ||x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} (f_i^\varepsilon)^{\mu_i}||_{p_i} = \frac{C_n}{\varepsilon} \prod_{i=1}^n \left(\frac{1}{e} - \frac{\varepsilon}{p_i(\nu_i - \widetilde{A}_i)} \right)^{\frac{1}{p_i}}. \tag{35}$$

On the other hand, since

$$(\mathcal{G}f_i^\varepsilon)(x_i) = \begin{cases} 0, & 0 < x_i < 1, \\ e^{-\frac{1}{\mu_i p_i} - \frac{\widetilde{A}_i - \nu_i}{\alpha\mu_i}} x_i^{\frac{\widetilde{A}_i - \nu_i}{\mu_i} - \frac{\varepsilon}{\mu_i p_i}} e^{\frac{\varepsilon}{\mu_i p_i \alpha} + \frac{1}{x_i^\alpha}\left(\frac{1}{\mu_i p_i} + \frac{\widetilde{A}_i - \nu_i}{\alpha\mu_i} - \frac{\varepsilon}{\mu_i p_i \alpha}\right)}, & x \geq 1, \end{cases}$$

the left-hand side of (34), denoted here by L, can be estimated as

$$\begin{aligned} L &= \int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\nu_i} (\mathcal{G}f_i^\varepsilon)^{\mu_i} (x_i) d\mathbf{x} \\ &> e^{-1+\frac{1}{\alpha}\left(n-s+\sum_{i=1}^n \nu_i\right)} \int_{[1,\infty\rangle^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} e^{\frac{\varepsilon}{p_i\alpha} + \frac{1}{x_i^\alpha}\left(\frac{1}{p_i} + \frac{\widetilde{A}_i - \nu_i}{\alpha} - \frac{\varepsilon}{p_i\alpha}\right)} d\mathbf{x} \\ &\geq e^{-1+\frac{1}{\alpha}\left(n-s+\sum_{i=1}^n \nu_i\right)} \cdot I, \end{aligned}$$

where $I = \int_{[1,\infty\rangle^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} d\mathbf{x}$. Obviously, the integral I can be rewritten as

$$I = \int_1^\infty x_1^{-1-\varepsilon} \left[\int_{[1/x_1,\infty\rangle^{n-1}} K(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\widetilde{A}_i - \varepsilon/p_i} \hat{d}^1\mathbf{u} \right] dx_1,$$

providing the inequality

$$\begin{aligned} I &\geq \int_1^\infty x_1^{-1-\varepsilon} \left[\int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\widetilde{A}_i - \varepsilon/p_i} \hat{d}^1\mathbf{u} \right] dx_1 \\ &\quad - \int_1^\infty x_1^{-1-\varepsilon} \left[\sum_{i=2}^n \int_{\mathbb{D}_i} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \hat{d}^1\mathbf{u} \right] dx_1 \\ &\geq \frac{1}{\varepsilon} \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\widetilde{A}_i - \varepsilon/p_i} \hat{d}^1\mathbf{u} \\ &\quad - \int_1^\infty x_1^{-1} \left[\sum_{i=2}^n \int_{\mathbb{D}_i} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \hat{d}^1\mathbf{u} \right] dx_1, \end{aligned} \tag{36}$$

where $\mathbb{D}_i = \{(u_2, u_3, \ldots, u_n); 0 < u_i \leq 1/x_1, u_j > 0, j \neq i\}$, $\mathbf{1}/\mathbf{p} = (1/p_1, \ldots, 1/p_n)$.

Without loss of generality, it suffices to find the appropriate estimate for the integral $\int_{\mathbb{D}_2} K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\hat{d}^1\mathbf{u}$. In fact, setting $\alpha>0$ such that $\widetilde{A}_2+1>\varepsilon/p_2+\alpha$, since $-u_2^\alpha\log u_2\to 0$ $(u_2\to 0^+)$, there exists $M\geq 0$ such that $-u_2^\alpha\log u_2\leq M$ $(u_2\in(0,1])$. On the other hand, it follows easily that the parameters $a_2=\widetilde{A}_2-(\varepsilon/p_2+\alpha)$ and $a_i=\widetilde{A}_i-\varepsilon/p_i,\ i=3,\dots,n$ satisfy conditions as in (27). Then, by virtue of the Fubini theorem, we have

$$\begin{aligned}
0 &\leq \int_1^\infty x_1^{-1}\int_{\mathbb{D}_2} K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\hat{d}^1\mathbf{u}dx_1\\
&= \int_1^\infty x_1^{-1}\left[\int_{\mathbb{R}_+^{n-2}}\int_0^{1/x_1} K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\hat{d}^1\mathbf{u}\right]dx_1\\
&= \int_{\mathbb{R}_+^{n-2}}\int_0^1 K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\left(\int_1^{1/u_2} x_1^{-1}dx_1\right)\hat{d}^1\mathbf{u}\\
&= \int_{\mathbb{R}_+^{n-2}}\int_0^1 K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}(-\log u_2)\hat{d}^1\mathbf{u}\\
&\leq M\int_{\mathbb{R}_+^{n-2}}\int_0^1 K(\hat{\mathbf{u}}^1)u_2^{\widetilde{A}_2-(\varepsilon/p_2+\alpha)}\prod_{j=3}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\hat{d}^1\mathbf{u}\\
&\leq M\int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1)u_2^{\widetilde{A}_2-(\varepsilon/p_2+\alpha)}\prod_{j=3}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\hat{d}^1\mathbf{u}\\
&= M\cdot k_1(\widetilde{A}_2-(\varepsilon/p_2+\alpha),\widetilde{A}_3-\varepsilon/p_3,\dots,\widetilde{A}_n-\varepsilon/p_n)<\infty.
\end{aligned} \tag{37}$$

Hence, taking into account (36), we obtain

$$L\geq e^{-1+\frac{1}{\alpha}\left(n-s+\sum_{i=1}^n\nu_i\right)}\left(\frac{1}{\varepsilon}k_1\left(\widetilde{\mathbf{A}}-\varepsilon\mathbf{1}/\mathbf{p}\right)-O(1)\right).$$

Moreover, the relation (35) implies that

$$\frac{C_n}{\varepsilon}\prod_{i=1}^n\left(\frac{1}{e}-\frac{\varepsilon}{p_i(\nu_i-\widetilde{A}_i)}\right)^{\frac{1}{p_i}}\geq e^{-1+\frac{1}{\alpha}\left(n-s+\sum_{i=1}^n\nu_i\right)}\left(\frac{1}{\varepsilon}k_1\left(\widetilde{\mathbf{A}}-\varepsilon\mathbf{1}/\mathbf{p}\right)-O(1)\right),$$

that is,

$$C_n\prod_{i=1}^n\left(\frac{1}{e}-\frac{\varepsilon}{p_i(\nu_i-\widetilde{A}_i)}\right)^{\frac{1}{p_i}}\geq e^{-1+\frac{1}{\alpha}\left(n-s+\sum_{i=1}^n\nu_i\right)}\left(k_1\left(\widetilde{\mathbf{A}}-\varepsilon\mathbf{1}/\mathbf{p}\right)-\varepsilon O(1)\right).$$

Obviously, if $\varepsilon\to 0^+$, then $C_n\geq m_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu})$, which contradicts with our assumption $0<C_n<m_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu})$. Hence, $m_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu})$ is the best possible in (30). □

Theorem 5. *Let $\alpha>0$, $\widetilde{A}_i\leq\nu_i\leq\frac{\alpha}{p_i}+\widetilde{A}_i$, $i=1,2,\dots,n$, and let parameters $\widetilde{A}_i$, $i=2,\dots,n$, fulfill conditions as in* (27)*. Then, the constant $m_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu})$ is the best possible in* (31)*.*

Proof. Assume that there exists a positive constant C_{n-1}, smaller than $m_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu})$, such that the inequality (31) holds when replacing $m_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu})$ by C_{n-1}.

The left-hand side of inequality (30), denoted here by L, can be rewritten in the following form:

$$L=\int_{\mathbb{R}_+}\left(x_n^{\frac{1}{p_n}+\widetilde{A}_n}\int_{\mathbb{R}_+^{n-1}}K(\mathbf{x})\prod_{i=1}^{n-1}x_i^{\nu_i}\left(\mathcal{G}f_i\right)^{\mu_i}(x_i)\hat{d}^n\mathbf{x}\right)x_n^{\nu_n-\frac{1}{p_n}-\widetilde{A}_n}\left(\mathcal{G}f_n\right)^{\mu_n}(x_n)dx_n.$$

Now, applying the Hölder inequality with conjugate exponents p_n and p_n' to the above expression yields inequality

$$L\leq L'||x_n^{\nu_n-\frac{1}{p_n}-\widetilde{A}_n}\left(\mathcal{G}f_n\right)^{\mu_n}||_{p_n}, \tag{38}$$

where L' denotes the left-hand side of (31).

Moreover, $L'\leq C_{n-1}\prod_{i=1}^{n-1}||x_i^{\nu_i-\frac{1}{p_i}-\widetilde{A}_i}f_i^{\mu_i}||_{p_i}$, while the Levin-Cochran-Lee inequality (13) yields

$$||x_n^{\nu_n-\frac{1}{p_n}-\widetilde{A}_n}\left(\mathcal{G}f_n\right)^{\mu_n}||_{p_n}\leq e^{\frac{\nu_n-\tilde{A}_n}{\alpha}}\cdot||x_n^{\nu_n-\frac{1}{p_n}-\widetilde{A}_n}f_n^{\mu_n}||_{p_n}.$$

Therefore relation (38) yields the inequality

$$L\leq C_{n-1}e^{\frac{\nu_n-\tilde{A}_n}{\alpha}}\cdot\prod_{i=1}^{n}||x_i^{\nu_i-\frac{1}{p_i}-\widetilde{A}_i}f_i^{\mu_i}||_{p_i}. \tag{39}$$

Finally, taking into account our assumption $0<C_{n-1}<m_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu})$, we have

$$0<C_{n-1}e^{\frac{\nu_n-\tilde{A}_n}{\alpha}}<m_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu})e^{\frac{\nu_n-\tilde{A}_n}{\alpha}}=m_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu}).$$

Hence, relation (39) contradicts with the fact that $m_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu})$ is the best possible constant in inequality (30). Thus, the assumption that $m_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu})$ is not the best possible is false. The proof is now completed. □

Theorem 6. *Let α,ν_i, and $\mu_i>0$ be real parameters such that $\alpha+\frac{1}{\mu_i}(\nu_i-\widetilde{A}_i)>0$, $i=1,2,\dots,n$, and let parameters $\widetilde{A}_i$, $i=2,\dots,n$, fulfill conditions as in* (27). *Then, the constant $\overline{m}_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu},\boldsymbol{\mu})$ is the best possible in* (32).

Proof. We follow the same procedure as in the proof of Theorem 4, that is, we suppose that the inequality

$$\int_{\mathbb{R}_+^n}K(\mathbf{x})\prod_{i=1}^{n}x_i^{\nu_i}\left(\mathcal{H}f_i\right)^{\mu_i}(x_i)d\mathbf{x}\leq C_n\prod_{i=1}^{n}||x_i^{\nu_i-\frac{1}{p_i}-\widetilde{A}_i}f_i^{\mu_i}||_{p_i}, \tag{40}$$

holds with a positive constant C_n, smaller than $\overline{m}_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu},\boldsymbol{\mu})$. Considering this inequality with the functions

$$f_i^\varepsilon(x_i)=\begin{cases}x_i^{\frac{\tilde{A}_i-\nu_i}{\mu_i}+\frac{\varepsilon}{p_i\mu_i}}, & 0<x_i\leq 1,\\ 0, & x>1,\end{cases}$$

where ε is sufficiently small number, its right-hand side reduces to

$$C_n\prod_{i=1}^{n}||x_i^{\nu_i-\frac{1}{p_i}-\widetilde{A}_i}(f_i^\varepsilon)^{\mu_i}||_{p_i}=\frac{C_n}{\varepsilon}. \tag{41}$$

Moreover, since

$$(\mathcal{H}f_i^\varepsilon)(x_i)=\begin{cases}\left[\alpha+\frac{\nu_i-\widetilde{A}_i}{\mu_i}-\frac{\varepsilon}{\mu_ip_i}\right]x_i^{\frac{\tilde{A}_i-\nu_i}{\mu_i}+\frac{\varepsilon}{\mu_ip_i}}, & 0<x_i\leq 1,\\ 0, & x_i>1,\end{cases}$$

the left-hand side of (40), denoted here by L, reads

$$\begin{aligned} L &= \int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\nu_i} \left(\mathcal{H} f_i^{\varepsilon}\right)^{\mu_i} (x_i) d\mathbf{x} \\ &= \varphi(\varepsilon) \cdot I, \end{aligned}$$

where

$$\varphi(\varepsilon) = \prod_{i=1}^n \left[\alpha + \frac{\nu_i - \widetilde{A}_i}{\mu_i} - \frac{\varepsilon}{\mu_i p_i}\right]^{\mu_i} \quad \text{and} \quad I = \int_{\langle 0,1]^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\widetilde{A}_i + \frac{\varepsilon}{p_i}} d\mathbf{x}.$$

Obviously, the integral I can be rewritten as

$$I = \int_0^1 x_1^{\varepsilon - 1} \left[\int_{\langle 0, 1/x_1]^{n-1}} K(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\widetilde{A}_i + \varepsilon/p_i} \hat{d}^1 \mathbf{u}\right] dx_1,$$

providing the estimate

$$\begin{aligned} I &\geq \int_0^1 x_1^{\varepsilon-1} \left[\int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\widetilde{A}_i + \varepsilon/p_i} \hat{d}^1 \mathbf{u}\right] dx_1 \\ &\quad - \int_0^1 x_1^{\varepsilon - 1} \left[\sum_{i=2}^n \int_{\mathbb{E}_i} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j + \varepsilon/p_j} \hat{d}^1 \mathbf{u}\right] dx_1 \\ &\geq \frac{1}{\varepsilon} \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\widetilde{A}_i + \varepsilon/p_i} \hat{d}^1 \mathbf{u} \\ &\quad - \int_0^1 x_1^{-1} \left[\sum_{i=2}^n \int_{\mathbb{E}_i} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j + \varepsilon/p_j} \hat{d}^1 \mathbf{u}\right] dx_1, \end{aligned} \tag{42}$$

where $\mathbb{E}_i = \{(u_2, u_3, \ldots, u_n); 1/x_1 \leq u_i < \infty, u_j > 0, j \neq i\}$, $\mathbf{1/p} = (1/p_1, \ldots, 1/p_n)$.

Clearly, it is enough to estimate the integral $\int_{\mathbb{E}_2} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j + \varepsilon/p_j} \hat{d}^1 \mathbf{u}$. Namely, choosing $\alpha > 0$ such that $\widetilde{A}_2 + 1 > -\varepsilon/p_2 - \alpha$, since $-u_2^{-\alpha} \log \frac{1}{u_2} \to 0$ $(u_2 \to \infty)$, there exists $M \geq 0$ such that $-u_2^{-\alpha} \log \frac{1}{u_2} \leq M$ $(u_2 \in [1, \infty))$. Further, the parameters $a_2 = \widetilde{A}_2 + (\varepsilon/p_2 + \alpha)$ and $a_i = \widetilde{A}_i + \varepsilon/p_i,\ i = 3, \ldots, n$, fulfill conditions as in (27). Then, similarly to (37), we have

$$\begin{aligned} &\int_0^1 x_1^{-1} \int_{\mathbb{E}_2} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j + \varepsilon/p_j} \hat{d}^1 \mathbf{u} dx_1 \\ &\quad \leq M \cdot k_1(\widetilde{A}_2 + (\varepsilon/p_2 + \alpha), \widetilde{A}_3 + \varepsilon/p_3, \ldots, \widetilde{A}_n + \varepsilon/p_n) < \infty, \end{aligned}$$

and utilizing (42), it follows that

$$L \geq \varphi(\varepsilon) \cdot \left(\frac{1}{\varepsilon} k_1\left(\widetilde{\mathbf{A}} + \varepsilon \mathbf{1/p}\right) - O(1)\right). \tag{43}$$

Finally, taking into account (41) and (43), we have that $\overline{m}_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu}) \leq C_n$ when $\varepsilon \to 0^+$, which is an obvious contradiction. This means that the constant $\overline{m}_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu})$ is the best possible in (32). □

Theorem 7. *Let α, ν_i, and $\mu_i > 0$ be real parameters such that $\alpha + \frac{1}{\mu_i}(\nu_i - \widetilde{A}_i) > 0$, $i = 1, 2, \ldots, n$, and let parameters $\widetilde{A}_i$, $i = 2, \ldots, n$, fulfill conditions as in (27). Then, the constant $\overline{m}^s_{n-1}(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu})$ is the best possible in (33).*

Proof. Suppose, on the contrary, that there exists a positive constant C_{n-1}, $0 < C_{n-1} < \overline{m}^s_{n-1}(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu})$, such that the inequality (33) holds with the constant C_{n-1} instead of $\overline{m}^s_{n-1}(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu})$.

Now, rewriting the left-hand side of inequality (32) in the form

$$\int_{\mathbb{R}_+} \left(x_n^{\frac{1}{p_n} + \widetilde{A}_n} \int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\nu_i} (\mathcal{H} f_i)^{\mu_i} (x_i) \hat{d}^n \mathbf{x} \right) x_n^{\nu_n - \frac{1}{p_n} - \widetilde{A}_n} (\mathcal{H} f_n)^{\mu_n} (x_n) dx_n,$$

and applying the Hölder inequality with conjugate exponents p_n and p_n', we have

$$L \leq L' || x_n^{\nu_n - \frac{1}{p_n} - \widetilde{A}_n} (\mathcal{H} f_n)^{\mu_n} ||_{p_n}, \tag{44}$$

where L and L' respectively denote the left-hand sides of inequalities (32) and (33). In addition, $L' \leq C_{n-1} \prod_{i=1}^{n-1} || x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} f_i^{\mu_i} ||_{p_i}$, while (16) yields the inequality

$$|| x_n^{\nu_n - \frac{1}{p_n} - \widetilde{A}_n} (\mathcal{H} f_n)^{\mu_n} ||_{p_n} \leq \left(\alpha + \frac{\nu_n - \widetilde{A}_n}{\mu_n} \right)^{\mu_n} \cdot || x_n^{\nu_n - \frac{1}{p_n} - \widetilde{A}_n} f_n^{\mu_n} ||_{p_n}.$$

Hence, relation (44) provides the inequality

$$L \leq C_{n-1} \left(\alpha + \frac{\nu_n - \widetilde{A}_n}{\mu_n} \right)^{\mu_n} \cdot \prod_{i=1}^{n} || x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} f_i^{\mu_i} ||_{p_i}. \tag{45}$$

Finally, with our assumption $0 < C_{n-1} < m^s_{n-1}(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\mu})$, we have

$$C_{n-1} \left(\alpha + \frac{\nu_n - \widetilde{A}_n}{\mu_n} \right)^{\mu_n} < \overline{m}^s_{n-1}(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu}) \left(\alpha + \frac{\nu_n - \widetilde{A}_n}{\mu_n} \right)^{\mu_n} = \overline{m}^s_n(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu}).$$

Therefore, inequality (45) contradicts with the fact that $\overline{m}^s_n(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu})$ is the best possible constant in (32). The proof is now completed. □

6. Some examples and remarks

We conclude the paper with some consequences of the general results established in Section 4 and Section 5. More precisely, we derive here several new Hilbert-type inequalities with geometric and harmonic mean operators and with some particular homogeneous kernels. In this section we deal with the case of conjugate exponents and the inequalities that follow include the best possible constants on their right-hand sides.

6.1. **First example.** A typical example of a homogeneous kernel with the negative degree of homogeneity is the function $K_1 : \mathbb{R}_+^n \to \mathbb{R}$, defined by

$$K_1(\mathbf{x}) = \frac{1}{\left(\sum_{i=1}^n x_i\right)^s}, \ s > 0.$$

Clearly, K_1 is a homogeneous function of degree $-s$, and the constant factor $k_1(\widetilde{\mathbf{A}})$, appearing in inequalities (30), (31), (32), and (33), can be expressed in terms of the usual Gamma function Γ. Namely, utilizing the well-known formula

$$\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} u_i^{a_i-1}}{\left(1+\sum_{i=1}^{n-1} u_i\right)^{\sum_{i=1}^n a_i}} \hat{d}^n \mathbf{u} = \frac{\prod_{i=1}^n \Gamma(a_i)}{\Gamma\left(\sum_{i=1}^n a_i\right)}, \tag{46}$$

which holds for $a_i > 0$, $i = 1, 2, \dots n$, it follows that

$$k_1(\widetilde{\mathbf{A}}) = \frac{1}{\Gamma(s)} \prod_{i=1}^n \Gamma(1 + \widetilde{A}_i), \quad i = 1, 2, \dots, n,$$

provided that $\widetilde{A}_i > -1$, $i = 1, 2, \dots, n$, and $\sum_{i=1}^n \widetilde{A}_i = s - n$. In addition, considering the parameters $\widetilde{A}_i = r_i - 1$, $\mu_i = 1$, $\nu_i = r_i - 1/p_i'$, $i = 1, 2, \dots, n$, where $r_i > 0$ and $\sum_{i=1}^n r_i = s$, inequalities (30), (31), (32), and (33) reduce respectively to

$$\int_{\mathbb{R}_+^n} \frac{1}{\left(\sum_{i=1}^n x_i\right)^s} \prod_{i=1}^n x_i^{r_i - \frac{1}{p_i'}} \left(\mathcal{G} f_i\right)(x_i) d\mathbf{x} \le \frac{e^{1/\alpha}}{\Gamma(s)} \prod_{i=1}^n \Gamma(r_i) \prod_{i=1}^n ||f_i||_{p_i},$$

$$\left[\int_{\mathbb{R}_+} {x_n}^{r_n p_n' - 1} \left(\int_{\mathbb{R}_+^{n-1}} \frac{1}{\left(\sum_{i=1}^n x_i\right)^s} \prod_{i=1}^{n-1} x_i^{r_i - \frac{1}{p_i'}} \left(\mathcal{G} f_i\right)(x_i) \hat{d}^n \mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le \frac{e^{1/(\alpha p_n')}}{\Gamma(s)} \prod_{i=1}^n \Gamma(r_i) \prod_{i=1}^{n-1} ||f_i||_{p_i},$$

$$\int_{\mathbb{R}_+^n} \frac{1}{\left(\sum_{i=1}^n x_i\right)^s} \prod_{i=1}^n x_i^{r_i - \frac{1}{p_i'}} \left(\mathcal{H} f_i\right)(x_i) d\mathbf{x} \le \prod_{i=1}^n \left(\alpha + \frac{1}{p_i}\right) \frac{\prod_{i=1}^n \Gamma(r_i)}{\Gamma(s)} \prod_{i=1}^n ||f_i||_{p_i},$$

and

$$\left[\int_{\mathbb{R}_+} {x_n}^{r_n p_n' - 1} \left(\int_{\mathbb{R}_+^{n-1}} \frac{1}{\left(\sum_{i=1}^n x_i\right)^s} \prod_{i=1}^{n-1} x_i^{r_i - \frac{1}{p_i'}} \left(\mathcal{H} f_i\right)(x_i) \hat{d}^n \mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le \prod_{i=1}^{n-1} \left(\alpha + \frac{1}{p_i}\right) \frac{\prod_{i=1}^n \Gamma(r_i)}{\Gamma(s)} \prod_{i=1}^{n-1} ||f_i||_{p_i}.$$

Clearly, the constants appearing on their right-hand sides are the best possible.

6.2. **Second example.** Another example of a homogeneous kernel with degree $-s$, is the function

$$K_2(\mathbf{x}) = \frac{1}{\max\{x_1^s, \dots, x_n^s\}}, \quad s > 0.$$

In order to derive analogues of the inequalities from the previous example, we utilize the integral formula

$$\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} u_i^{a_i}}{\max\{1, x_1^s, \dots, x_{n-1}^s\}} \hat{d}^n \mathbf{u} = \frac{s}{\prod_{i=1}^n (1 + a_i)},$$

where $a_i > -1$ and $\sum_{i=1}^n a_i = s - n$ (for more details see [13]). Hence, with this kernel and parameters $\widetilde{A}_i = r_i - 1$, $\mu_i = 1$, $\nu_i = r_i - 1/p_i'$, $i = 1, 2, \ldots, n$, where $r_i > 0$ and $\sum_{i=1}^n r_i = s$, inequalities (30), (31), (32), and (33) become respectively

$$\int_{\mathbb{R}_+^n} \frac{1}{\max\{x_1^s, \ldots, x_n^s\}} \prod_{i=1}^n x_i^{r_i - \frac{1}{p_i'}} (\mathcal{G} f_i)(x_i) d\mathbf{x} \le \frac{s e^{1/\alpha}}{\prod_{i=1}^n r_i} \prod_{i=1}^n ||f_i||_{p_i},$$

$$\left[\int_{\mathbb{R}_+} {x_n}^{r_n p_n' - 1} \left(\int_{\mathbb{R}_+^{n-1}} \frac{1}{\max\{x_1^s, \ldots, x_n^s\}} \prod_{i=1}^{n-1} x_i^{r_i - \frac{1}{p_i'}} (\mathcal{G} f_i)(x_i) \hat{d}^n \mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le \frac{s e^{1/(\alpha p_n')}}{\prod_{i=1}^n r_i} \prod_{i=1}^{n-1} ||f_i||_{p_i},$$

$$\int_{\mathbb{R}_+^n} \frac{1}{\max\{x_1^s, \ldots, x_n^s\}} \prod_{i=1}^n x_i^{r_i - \frac{1}{p_i'}} (\mathcal{H} f_i)(x_i) d\mathbf{x} \le s \prod_{i=1}^n \frac{\alpha + 1/p_i}{r_i} \prod_{i=1}^n ||f_i||_{p_i},$$

and

$$\left[\int_{\mathbb{R}_+} {x_n}^{r_n p_n' - 1} \left(\int_{\mathbb{R}_+^{n-1}} \frac{1}{\max\{x_1^s, \ldots, x_n^s\}} \prod_{i=1}^{n-1} x_i^{r_i - \frac{1}{p_i'}} (\mathcal{H} f_i)(x_i) \hat{d}^n \mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le \frac{s}{\alpha + 1/p_n} \prod_{i=1}^n \frac{\alpha + 1/p_i}{r_i} \prod_{i=1}^{n-1} ||f_i||_{p_i},$$

where the constants appearing on their right-hand sides are the best possible.

Remark 3. It should be noticed here that the analogues of inequalities from this section, including the arithmetic mean (Hardy) operator, were derived in [13], while the two-dimensional case was studied in [7] and [8].

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