On S-propermutable subgroups of finite groups

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Abstract

Let H be a subgroup of a finite group G. Then we say that H is S-propermutable in G provided G has a subgroup B such that $G = N_G(H)B$ and H permutes with all Sylow subgroups of B. In this paper we analyze the influence of S-propermutable subgroups on the structure of G.

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover p is always supposed to be a prime and π is a non-empty subset of the set $\mathbb{P}$ of all primes. We use $\mathcal{M}_\phi(G)$ to denote a set of maximal subgroups of G such that $\Phi(G)$ coincides with the intersection of all subgroups in $\mathcal{M}_\phi(G)$. If for subgroups A and B of G we have $AB = BA$, then A is said to *permute* with B. If $G = AB$, then B is said to be a *supplement* of A to G.

Recall that a subgroup H of G is said to be *S-permutable*, *S-quasinormal*, or *π-quasinormal* Kegel [11] in G provided $HP = PH$ for all Sylow subgroups P of G. The S-permutable subgroups possess many interesting properties (see [11, 3, 15] or Chapter 1 in [1]), and such subgroups are used for the analysis of many questions of the group theory (see Section 5 in [20]). This circumstance was the main motivation for the introduction and study of various generalizations of the S-permutability. One of the most interesting generalizations of S-permutability was found by Shirong Li, Zhencai Shen, Jianjun Liu and Xiaochun Liu: A subgroup H of G is called *SS-quasinormal* [18] in G if H

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permutes with all Sylow subgroups of some supplement of H to G. Nice results obtained in the papers [18, 19, 22] were based on applications of this concept.

In this paper we consider another generalization of S-permutable subgroups.

Definition 1.1. *Let H be a subgroup of G. Then we say that H is S-*propermutable *in G provided there is a subgroup B of G such that $G = N_G(H)B$ and H permutes with all Sylow subgroups of B.*

In fact, we meet S-propermutable subgroups quite often.

Example 1.1. *1. Every maximal subgroup of a soluble group G and every its Hall subgroup E with $|G : N_G(E)| = p^a$ are S-propermutable in G. Indeed, since G is soluble, there is a Sylow p-subgroup P of G such that $EP = PE$. On the other hand, since $|G : N_G(E)| = p^a|$ we have $G = N_G(E)P$. Hence E is S-propermutable in G.*

2. If $|H| = p^a$ and $H \leq Z_\infty(G)$, then $H \leq P$, where P is the Sylow p-subgroup of $Z_\infty(G)$. Therefore, since $G/C_G(P)$ is a p-group (see Lemma 2.9 below), $G = N_G(H)G_p$ and $H \leq P \leq G_p$, where G_p is a Sylow p-subgroup of G. Hence H is S-propermutable in G.

3. If G is metanilpotent, that is $G/F(G)$ is nilpotent, then for every Sylow subgroup P of G we have $G = N_G(P)F(G)$. Therefore, in this case, every characteristic subgroup of every Sylow subgroup of G is S-propermutable in G. In particular, every Sylow subgroup of a supersoluble group is S-propermutable.

It is clear that every SS-quasinormal subgroup is S-propermutable. The following elementary example shows that in general the set of all S-propermutable subgroups of G is wider than the set of all its SS-quasinormal subgroups.

Example 1.2. *Let $p > q > r$ be primes such that qr divides $p - 1$. Let P be a group of order p and $QR \leq Aut(P)$, where Q and R are groups with order q and r, respectively. Let $G = P \rtimes (QR)$. Then R is S-propermutable in G. Suppose that R is SS-quasinormal in G. Then $Q^xR = RQ^x$ for all $x \in G$ (see Lemma 1.4 below). But $Q^xR \simeq G/P$ is cyclic, so $Q^G = PQ \leq N_G(R)$. Hence R is normal in G, which implies that $R \leq C_G(P) = P$. This contradiction shows that R is not SS-quasinormal in G.*

The results of the above-mentioned papers [18, 19, 22] are motivations for the following our theorem.

Theorem A. *Let E be a normal subgroup of G and P a Sylow p-subgroup of E. Suppose that $|P| > p$.*

(I) *If every number V of some fixed $\mathcal{M}_\phi(P)$ is S-propermutable in G, then E is p-supersoluble.*

(II) *If every maximal subgroup of P is S-propermutable in G, then every chief factor of G between E and $O_{p'}(E)$ is cyclic.*

As a first application of Theorem A, we prove also the following result.

Theorem B. *Let $X \leq E$ be normal subgroups of G. Suppose that every maximal subgroup of every non-cyclic Sylow subgroup of X is S-propermutable in G. If either $X = E$ or $X = F^*(E)$, then every chief factor of G below E is cyclic.*

Let $\mathfrak{F}$ be a class of groups. If $1 \in \mathfrak{F}$, then we write $G^{\mathfrak{F}}$ to denote the intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$. The class $\mathfrak{F}$ is said to be a *formation* if either $\mathfrak{F} = \varnothing$ or $1 \in \mathfrak{F}$ and every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$ for any group G. The formation $\mathfrak{F}$ is said to be *solubly saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(N) \in \mathfrak{F}$ for some soluble normal subgroup N of G.

Note that if $\mathfrak{F}$ is a solubly saturated formation and $G/E \in \mathfrak{F}$, where every chief factor of G below E is cyclic, then $G \in \mathfrak{F}$ (see Lemma 2.13 below). Therefore from Theorem B we get

Corollary 1.1. *Let $\mathfrak{F}$ be a solubly saturated formation containing all supersoluble groups and $X \leq E$ normal subgroups of G such that $G/E \in \mathfrak{F}$. Suppose that every maximal subgroup of every non-cyclic Sylow subgroup of X is S-propermutable in G. If either $X = E$ or $X = F^*(E)$, then $G \in \mathfrak{F}$.*

Note Theorem A and Corollary 1.4 cover results of many papers and, in particular, some main results in [14, 18, 19] (see Section 4).

The proof of Theorem A consists of many steps, and the following useful result is one of them.

Theorem C. *Let E be a normal subgroup of G and P is a Sylow p-subgroups of E. If P is S-propermutable in G, then E is p-soluble.*

All unexplained notation and terminology are standard. The reader is referred to [17], [4], [6] or [2] if necessary.

2 Preliminaries

Lemma 2.1 (See [9]). *Let A and B be subgroups of G with $G = AB$.*

(1) *If G is π-soluble, then there are Hall π-subgroups A_π, B_π and G_π of A, B and G, respectively, such that $G_\pi = A_\pi B_\pi$*

(2) *For any prime p dividing $|G|$, there are Sylow p-subgroups A_p, B_p and G_p of A, B and G, respectively, such that $G_p = A_p B_p$.*

Lemma 2.2 (See Lemma 1.6 in [4]). *Let H, K and N be subgroups of G. If $HK = KH$ and $HN = NH$, then $H\langle K, N\rangle = \langle K, N\rangle H$.*

We say that H is *propermutable* in G provided there is a subgroup B of G such that $G = N_G(H)B$ and H permutes with all subgroups of B.

Lemma 2.3. *Let $H \leq G$ and N be a normal subgroup of G. Suppose that H is S-propermutable (propermutable) in G.*

(1) *HN/N is S-propermutable (propermutable, respectively) in G/N.*

(2) *H permutes with some Sylow p-subgroup of G for any prime p dividing $|G|$.*

(3) *If G is π-soluble, then H permutes with some Hall π-subgroup of G.*

(4) *$|G : N_G(H \cap N)|$ is a π-number, where $\pi = \pi(N) \cup \pi(H)$.*

Proof. (1) First suppose that H is S-propermutable in G. By hypothesis there is a subgroup B of G such that $G = N_G(H)B$ and H permutes with all Sylow p-subgroups of B for all primes p dividing $|B|$. Then

$$G/N = (N_G(H)N/N)(BN/N) = N_{G/N}(HN/N)(BN/N).$$

Suppose that p divides $|BN/N|$ and let K/N be any Sylow p-subgroup of BN/N. Then $K = (K \cap B)N$, so by Lemma 2.1, there are Sylow p-subgroups K_p, P and N_p of K, $K \cap B$ and N, respectively, such that $K_p = PN_p$. Let $P \leq B_p$, where B_p is a Sylow p-subgroup of B. Then $K/N \leq B_pN/N$, which implies that $K/N = B_pN/N$. But H permutes with B_p, so that HN/N permutes with K/N. Therefore HN/N is S-propermutable in G/N. The second assertion of (1) is proved similarly.

(2) By Lemma 2.1 there are Sylow p-subgroups P_1, P_2 and P of $N_G(H)$, B and G, respectively, such that $P = P_1P_2$. Then

$$HP = H(P_1P_2) = (HP_1)P_2 = (P_1H)P_2 =$$

$$P_1(HP_2) = P_1(P_2H) = (P_1P_2)H = PH.$$

(3) See the proof of (2) and use Lemma 2.2.

(4) Let p be a prime such that $p \notin \pi$. Then by (3) there is a Sylow p-subgroup P of G such that $HP = PH$ is a subgroup of G. Hence $HP \cap N = H \cap N$ is a normal subgroup of HP. Thus p does not divide $|G : N_G(H \cap N)|$. □

Lemma 2.4. *Let H and B be subgroups of G. If $G = N_G(H)B$ and $HV^b = V^bH$ for some subgroup V of B and for all $b \in B$, then $HV^x = V^xH$ for all $x \in G$.*

Proof. Since $G = N_G(H)B$ we have $x = bn$ for some $b \in B$ and $n \in N_G(H)$. Hence $HV^x = HV^{bn} = Hn(V^b)n^{-1} = n(V^b)n^{-1}H = V^xH$. □

Lemma 2.5. *Suppose that for subgroups A an B of G we have $AB = BA$ and $G = N_G(A)B$. Then*

(1) *$A^G = A(A^G \cap B)$.*

(2) *If A permutes with all Sylow p-subgroups of B, then A permutes with all Sylow p-subgroups of $A^G \cap B$.*

Proof. (1) Since $AB = BA$, AB is a subgroup of G and so $A^G = A^{N_G(A)B} = A^B \leq \langle A, B\rangle = AB$. Hence $A^G = A^G \cap AB = A(A^G \cap B)$.

(2) By (1) we have $A^G = A(A^G \cap B)$. Let P be any Sylow p-subgroup of $A^G \cap B$ and $P \leq B_p$, where B_p is a Sylow of B. Then $AB_p = B_pA$ and $P = A^G \cap B \cap B_p = A^G \cap B_p$. Hence $AB_p \cap A^G = A(B_p \cap A^G) = AP = PA$. □

Lemma 2.6 (See Kegel [12])**.** *Let A and B be subgroups of G such that $G \neq AB$ and $AB^x = B^xA$, for all $x \in G$. Then G has a proper normal subgroup N such that either $A \leq N$ or $B \leq N$.*

In our proofs we shall need the following well-known properties of supersoluble and p-supersoluble groups.

Lemma 2.7. *Let N and R be normal subgroups of G.*

(1) *If $N \leq \Phi(G) \cap R$ and R/N is p-supersoluble, then R is p-supersoluble.*

(2) *If G is p-supersoluble and $O_{p'}(G) = 1$, then p is the largest prime dividing $|G|$, G is supersoluble and $F(G) = O_p(G)$ is a normal Sylow p-subgroup of G.*

(3) *If G is supersoluble, then $G' \leq F(G)$.*

Lemma 2.8 (See Knyagina and Monakhov [13])**.** *Let H, K and N be subgroups of G. If N is normal in G, H permutes with K and H is a Hall subgroup of G, then*

$$N \cap HK = (N \cap H)(N \cap K).$$

We use $\mathcal{A}(p-1)$ to denote the class of all abelian groups of exponent dividing $p-1$. The symbol $Z_{\mathfrak{U}}(G)$ denotes the product of all normal subgroups N of G such that every chief factor of G below N is cyclic.

Lemma 2.9 (See Lemma 2.2 in [21])**.** *Let E be a normal p-subgroup of a group G. If $E \leq Z_{\mathfrak{U}}(G)$ (if $E \leq Z_\infty(G)$), then*

$$(G/C_G(E))^{\mathcal{A}(p-1)} \leq O_p(G/C_G(E))$$

($G/C_G(E)$ is a p-group, respectively).

Proof. See the proof of Lemma 2.2 in [21]. □

Lemma 2.10. *Suppose that G is p-soluble and $O_{p'}(G) = 1$. Then $F^*(G) = O_p(G)$.*

Proof. It is clear that $F(G) = O_p(G) \leq F^*(G)$. Suppose that $O_p(G) \neq F^*(G)$ and let $H/O_p(G)$ be a chief factor of G below $F^*(G)$. Then, since G is p-soluble, $H/O_p(G)$ is a non-abelian p'-group and $O_p(G) \leq Z_\infty(H)$ by [10, Chapter X, Theorems 13.6 and 13.7]. Hence $H/C_H(O_p(G))$ is a p-group by Lemma 2.9. On the other hand, by the Schur-Zassenhaus theorem, $O_p(G)$ has a complement E in H. Then $E \leq C_H(O_p(G))$, which implies that E is normal in H. Thus E is a characteristic subgroup of E, so $E \leq O_{p'}(G) = 1$, a contradiction. □

Lemma 2.11 (See Lemma 2.15 in [7]). *Let E be a normal non-identity quasinilpotent subgroup of G. If $\Phi(G) \cap E = 1$, then E is the direct product of some minimal normal subgroups of G.*

Let $\mathfrak{F}$ be a class of groups. A chief factor H/K of G is called *$\mathfrak{F}$-central* in G provided $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$.

Lemma 2.12 (See Theorem B in [21]). *Let $\mathfrak{F}$ be any formation and E a normal subgroup of G. If each chief factor of G below $F^*(E)$ is $\mathfrak{F}$-central in G, then each chief factor of G below E is $\mathfrak{F}$-central in G as well.*

Lemma 2.13 (See Lemma 3.3 in [7]). *Let $\mathfrak{F}$ be a solubly saturated formation containing all supersolble groups and E a normal subgroups of G with $G/E \in \mathfrak{F}$. If every chief factor of G below E is cyclic, then $G \in \mathfrak{F}$.*

Recall that G is called a *Schmidt group* provided G is not nilpotent but every proper subgroup of G is nilpotent. We shall need in our proofs the following facts on Schmidt groups.

Lemma 2.14 (See Theorem 25.4 in [16]). *Let G be a Schmidt group Then*

(a) *$G = P \rtimes Q$, where P is a Sylow p-subgroup of G of exponent p or exponent* 4 *(if P is a non-abelian* 2*-group), Q is a Sylow q-subgroup of G for some primes $p \neq q$.*

(b) *$P/\Phi(P)$ is a chief factor of G and $C_G(P/\Phi(P)) \neq G$.*

Lemma 2.15. *Let E be a normal subgroup of G and P a Sylow p-subgroup of E such that $(p-1, |G|) = 1$. If either P is cyclic or G is p-supersoluble, then E is p-nilpotent and $E/O_{p'}(E) \leq Z_\infty(G/O_{p'}(E))$.*

Proof. Let H/K be any chief factor of G such that $O_{p'}(E) \leq K < H \leq E$. Then $|H/K| = p$, so $G/C_G(H/K)$ divides $p-1$. But by hypothesis, $(p-1, |G|) = 1$. Hence $C_G(H/K) = G$. Thus $E/O_{p'}(E) \leq Z_\infty(G/O_{p'}(E))$. □

Lemma 2.16. *Let P be a normal p-subgroup of G. If $P/\Phi(P) \leq Z_{\mathfrak{U}}(G/\Phi(P))$, then $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. Let $C = C_G(P)$, H/K any chief factor of G below P. Then $O_p(G/C_G(H/K)) = 1$ by [23, Appendix C, Corollary 6.4]. Suppose that $P/\Phi(P) \leq Z_{\mathfrak{U}}(G/\Phi(P))$. Then by Lemma 2.9, $(G/C_G(P/\Phi(P)))^{\mathcal{A}(p-1)}$ is a p-group. Hence $(G/C)^{\mathcal{A}(p-1)}$ is a p-group by [5, Chapter 5, Theorem 1.4]. Thus $G/C_G(H/K) \in \mathcal{A}(p-1)$ and so $|H/K| = p$ by [23, Chapter 1, Theorem 1.4]. This implies that $P \leq Z_{\mathfrak{U}}(G)$. □

Lemma 2.17 (See Corollary 1.11 in [7]). *Let N be a normal soluble subgroup of G. Then $F^*(G/\Phi(N)) = F^*(G)/\Phi(N)$.*

Lemma 2.18 (See Theorem A* in [8]). *Let H be a Hall π-subgroup of G. Let $G = HT$ for some subgroup T of G, and q a prime. If H permutes with every Sylow p-subgroup of T for all primes $p \neq q$, then T contains a complement of H in G and any two complements of H in G are conjugate.*

Lemma 2.19. *Let A and B be subgroups of G. If $A^xB = BA^x$ for all $x \in G$, then $AB^x = B^xA$ for all $x \in G$.*

Proof. Indeed, from $A^{x^{-1}}B = BA^{x^{-1}}$ we get $AB^x = (A^{x^{-1}}B)^x = (BA^{x^{-1}})^x = B^xA$. □

A group G is said to be π-closed (p-closed) provided G has a normal Hall π-subgroup (a normal Sylow p-subgroup, respectively).

Lemma 2.20 (See Corollary 1.7 in [7]). *Let N and R be normal subgroups of G. If $N \leq \Phi(G) \cap R$ and R/N is π-closed, then R is π-closed*

3 Proofs of Theorems A, B and C

Proof of Theorem C. Suppose that this theorem is false and let G be a counterexample with $|G| + |E|$ minimal. Suppose that there is a non-identity p-soluble normal subgroup N of G such that $N \leq E$. If $P \leq N$, then G/N is a p'-group and so the p-solubility of N implies the p-solubility of E. On the other hand, if $P \nleq N$, then the hypothesis holds for G/N by Lemma 2.3 (1). Hence E/N is p-soluble by the choice of (G, E) since $|G/N| < |G|$. Therefore E is p-soluble. But this contradicts the choice of (G, E). Hence every non-identity normal subgroup N of G contained in E is not p-soluble.

By hypothesis there is a subgroup B of G such that $G = N_G(P)B$ and P permutes with all Sylow subgroups of B. We shall show that $E = P^G = G = PB$. Indeed, by Lemma 2.5, $P^G = P(P^G \cap B)$ and P permutes with all Sylow subgroups of $P^G \cap B$. Hence P is S-propermutable in P^G. If $P^G \neq G$, then P^G is p-soluble by the choice of (G, E) since $P^G \leq E$. Therefore G has a non-identity p-soluble normal subgroup, a contradiction. Thus $E = P^G = G = PB$.

Let Q be any Sylow q-subgroup of B such that $q \neq p$. Then p divides $|Q^G|$ and $P_0 = P \cap Q^G$ is a Sylow p-subgroup of Q^G. We show that the hypothesis holds for (Q^G, P_0). Indeed, let R be a Sylow r-subgroup of $Q^G \cap B$, where $r \neq p$. Then for some Sylow r-subgroup B_r of B we have

$$R = B_r \cap (Q^G \cap B) = B_r \cap Q^G.$$

By Lemma 2.8 we also know that

$$PB_r \cap Q^G = (P \cap Q^G)(B_r \cap Q^G) = P_0R = RP_0.$$

Therefore P_0 is S-propermutable in Q^G. But since G has no non-identity p-soluble normal subgroups, the choice of (G, E) implies that $Q^G = G$. Note that by Burnside's p^aq^b-theorem we have $PQ \neq G$.

On the other hand, by Lemma 2.4, $PQ^x = Q^xP$ for all $x \in G$ and so by Lemma 2.6, $P^G \neq G$. This contradiction completes the proof of the result.

Proof of Theorem A. (I) Suppose that this assertion is false and let G be a counterexample with $|G|+|E|$ minimal. Let $V \in \mathcal{M}_\phi(P)$. By hypothesis there is a subgroup B of G that $G = N_G(V)B$ and V permutes with all Sylow q-subgroups of B.

(1) $V^G = V(V^G \cap B)$ *and* V *permutes with every Sylow* q*-subgroup of* $V^G \cap B$ *for all primes* q *dividing* $|V^G \cap B|$ (This directly follows from Lemma 2.5).

(2) $O_{p'}(N) = 1$ *for every normal subgroup* N *of* G *contained in* E.

Suppose that for some normal subgroup N of G contained in E we have $O_{p'}(N) \neq 1$. Since $O_{p'}(N)$ is a characteristic subgroup of N, it is normal in G. On the other hand, by Lemma 2.3 (1), the hypothesis holds for $(G/O_{p'}(N), E/O_{p'}(N))$. Hence $E/O_{p'}(N)$ is p-supersoluble by the choice of (G, E). Thus E is p-supersoluble, a contradiction.

(3) *If* L *is a minimal normal subgroup of* G, *then* $L \nleq \Phi(P)$.

Indeed, in the case, where $L \leq \Phi(P)$, we have $L \leq \Phi(E)$ and the hypothesis holds for $(G/L, E/L)$ by Lemma 2.3 (1). Hence E/L is p-supersoluble by the choice of (G, E). Therefore E is p-supersoluble by Lemma 2.7 (1), which contradicts to our assumption on E.

(4) *If* D *is a normal* p*-soluble subgroup of* G *contained in* E, *then* D *is supersoluble and* p*-closed.*

By (2), $O_{p'}(D) = 1$. Therefore $O_p = O_p(D) \neq 1$. Let N be a minimal normal subgroup of G contained in O_p. In view of (3) we have $N \nleq \Phi(P)$. Hence for some subgroup $W \in \mathcal{M}_\phi(P)$ we have $P = NW$. Let $S = N \cap W$. Then S is normal in P. On the other hand, by Lemma 2.3 (4), $|G : N_G(S)|$ is a power of p. Hence $|E : N_E(S)| = |E : N_G(S) \cap E| = |EN_G(S) : N_G(S)|$ is a power of p. Thus S is normal in E. By Proposition 4.13 (c) in [4, Chapter A], $N = N_1 \times \ldots \times N_t$, where $N_1, \ldots, N_t$ are minimal normal subgroups of E, and from the proof of this proposition we know also that $|N_i| = |N_j|$ for all i, j. Therefore there is a minimal normal subgroup L of E such that $N = SL$ and $S \cap L = 1$. Hence $P = L \rtimes W$, which implies by Gaschütz's theorem [9, Chapter I, Satz 17.4] that L has a complement M in E. Thus $N \nleq \Phi(E)$ and $N_1, \ldots, N_t$ are groups of order p. It is clear that $\Phi(E) \cap O_p$ is normal in G. Therefore $\Phi(E) \cap O_p = 1$. Hence $O_p = L_1 \times \ldots \times L_t$, where $L_1, \ldots, L_t$ are minimal normal subgroups of E by Lemma 2.11. If for some i we have $L_i \nleq \Phi(P)$, then, as above, one can show that $|L_i| = p$. Therefore there are normal subgroups F and M of E such that $O_p = FM$, every chief factor of E below M is cyclic and $F \leq \Phi(P) \leq \Phi(E)$. Now consider D/F. It is clear $O_p(D/F) = O_p/F = MF/F$. On the other hand, by Lemma 2.20, $O_{p'}(D/F) = 1$ since $O_{p'}(D) = 1$. Therefore by Lemma 2.10, $F^*(D/F) = O_p/F$, where every chief factor of D/F below $F^*(D/F)$ is cyclic. Hence D/F is supersoluble, so D is supersoluble by Lemma 2.7 (1). But $O_{p'}(D) = 1$, so O_p is a Sylow p-subgroup of D by Lemma 2.7 (2).

(5) E *is* p*-soluble.*

Assume that E is not p-soluble.

(a) *If* $O_p(E) \neq 1$*, then* P *is not cyclic.*

Suppose that P is cyclic. Let L be a minimal normal subgroup of G contained in $O_p(E) \leq P$. Suppose that $C_E(L) = E$, so $L \leq Z(E)$. Let $N = N_E(P)$. If $P \leq Z(N)$, then E is p-nilpotent by Burnside's theorem [9, Chapter IV, Satz 2.6], which contradicts the choice of (G, E). Hence $N \neq C_E(P)$. Let $x \in N \backslash C_E(P)$ with $(|x|, |P|) = 1$ and $K = P \rtimes \langle x \rangle$. By [9, Chapter III, Satz 13.4], $P = [K, P] \times (P \cap Z(K))$. Since $L \leq P \cap Z(K)$ and P is cyclic, it follows that $P = P \cap Z(K)$ and so $x \in C_K(P)$. This contradiction shows that $C_E(L) \neq E$.

Since P is cyclic, $|L| = p$. Hence $G/C_G(L)$ is a cyclic group of order dividing $p-1$. If $|P/L| > p$, then the hypothesis holds for $(G/L, E/L)$ by Lemma 2.3 (1). Hence E/L is p-supersoluble by the choice of (G, E) and so E is p-soluble, a contradiction. Thus $|P/L| = p$ and hence $V = L$ is normal in G. Therefore the hypothesis holds for $(G, C_E(L))$, so $C_E(L)$ is p-supersoluble since $C_E(L) \neq E$. But then E is p-soluble since $E/C_E(L) = E/E \cap C_G(L) \simeq EC_G(L)/C_G(L)$ is cyclic. This contradiction shows that we have (a).

(b) *If* $P \nleq V^G$*, then* V *is normal in* G*.*

Indeed, since $P \nleq V^G \leq E$, V is a Sylow p-subgroup of V^G. On the other hand, by (1) we have $V^G = V(V^G \cap B)$ and V is S-propermutable in V^G. Therefore V^G is p-soluble by Theorem C. Thus V is normal in V^G by (4). Since V is a Sylow p-subgroup of V^G, V is characteristic in V^G. Hence $V = V^G$ is normal in G.

(c) P *is not cyclic.*

Suppose that P is cyclic. Then $\mathcal{M}_\phi(P) = \{V\}$, and by (1), (a) and (b) we have $P \leq V^G = V(V^G \cap B)$ and V permutes with every Sylow q-subgroup of $V^G \cap B$ for all primes q dividing $|V^G \cap B|$. Hence the hypothesis holds for (V^G, V^G). Assume that $V^G \neq G$. Then V^G is p-supersoluble by the choice of (G, E). Hence by (4), P is normal in G, which contradicts (a). Therefore $V^G = G$, which implies that $G = VB$ by (1). Hence $P = P \cap VB = V(P \cap B)$, so $P \leq B$ since P is cyclic. Therefore $B = G$, so V is S-permutable in G. Hence $V \leq P_E \leq O_p(E)$, which contradicts (a). Hence P is not cyclic.

(d) P *permutes with every Sylow* q*-subgroup* Q *of* P^G *for all primes* $q \neq p$ *dividing* $|P^G|$*.*

Let $D = P^G$. In view (c), there is a subgroup $W \in \mathcal{M}_\phi(P)$ such that $V \neq W$. Then $P = VW$. Hence in view of Lemma 2.2 we have only to show that V and W permute with Q. In view of (b) we may suppose that $P \leq V^G$ and $P \leq W^G$. Then $D = P^G \leq V^G$ and so by (1), $D = V(D \cap B)$ and V permutes with every Sylow q-subgroup Q_1 of $D \cap B$. It is also clear that Q_1 is a Sylow q-subgroup of D. Therefore for some $x \in D$ we have $Q_1 = Q^x$. Hence V permutes with Q by Lemma 2.4. Similarly, it may be proved that W permutes with Q.

Final contradiction for (5). By (d) and Lemma 2.18, P^G has a Hall p'-subgroup. Hence by (d), P is S-propermutable in P^G. Therefore by Theorem C, P^G is p-soluble. Hence by (4), P is normal in G. Therefore E is p-soluble. This contradiction completes the proof of (5).

By (5), E is p-soluble. Hence E is supersoluble by (4). This contradiction completes the proof of (I).

(II) Suppose that this assertion is false and let G be a counterexample with $|G| + |E|$ minimal. Let $Z = Z_{\mathfrak{U}}(G)$. First we show that $O_{p'}(E) = 1$. Indeed, suppose that $O_{p'}(E) \neq 1$. It is clear that $O_{p'}(E)$ is normal in G. Moreover, the hypothesis holds for $(G/O_{p'}(E), E/O_{p'}(E))$ by Lemma 2.3 (1). Therefore every chief factor of $G/O_{p'}(E)$ below $E/O_{p'}(E)$ is cyclic by the choice of (G, E). Hence every chief factor of G between E and $O_{p'}(E)$ is cyclic, a contradiction. Thus $O_{p'}(E) = 1$.

By (I), E is p-supersoluble. Hence by Lemma 2.7 (2), E is supersoluble and $P = F(E)$. Hence the hypothesis is true for (G, P). If $P \neq E$, then every chief factor of G below P is cyclic by the choice of (G, E). Hence every chief factor of G below E is cyclic by Lemma 2.12, contrary to the choice of (G, E). Hence $P = E$.

Let N be any minimal normal subgroup of G contained in P. Then the hypothesis holds for $(G/N, P/N)$, so every chief factor of G/N below P/N is cyclic by the choice of (G, E). Thus $|N| > p$. Moreover, $N \nleq \Phi(P)$, otherwise every chief factor of G below P is cyclic by Lemma 2.16. Thus $\Phi(P) = 1$ and so P is elementary abelian p-group. Let W be a maximal subgroup of N such that W is normal in a Sylow p-subgroup G_p of G. Let $V = WS$, where S is a complement of N in P. Then $W = V \cap N$ and V is S-propermutable in G by hypothesis. Hence by Lemma 2.3 (4), $G = G_pN_G(W)$. Therefore W is normal in G, so $W = 1$. This contradiction completes the proof of Assertion (II).

Theorem is proved.

Proof of Theorem B. First we assume that $X = E$. Suppose that in this case the theorem is false and consider a counterexample (G, E) for which $|G| + |E|$ is minimal. Let p be the smallest prime dividing $|E|$ and P a Sylow p-subgroup of E. Then E is p-nilpotent by Lemma 2.15 and Theorems A. Let V be the normal Hall p'-subgroup of E. Since $V char E \lhd G$, V is normal in G. Moreover, the hypothesis holds for (G, V) and for $(G/V, E/V)$ by Lemma 2.3 (1). Hence in the case when $V \neq 1$ we have $V \leq Z_{\mathfrak{U}}(G)$ and $E/V \leq Z_{\mathfrak{U}}(G/V)$ by the choice of (G, E). This induces that $E \leq Z_{\mathfrak{U}}(G)$, a contradiction. Therefore $E = P$ and consequently $E \leq Z_{\mathfrak{U}}(G)$ by Theorem A.

Finally, if $X = F^*(E)$, then as above we have $F^*(E) \leq Z_{\mathfrak{U}}(G)$. Therefore $E \leq Z_{\mathfrak{U}}(G)$ by Lemma 2.12.

4 Some applications of Theorem A and Corollary 1.4

In the literature one can find many special cases of Theorem A and Corollary 1.4. Here we discuss only some of them.

From Theorem A and Lemma 2.15 we get

Corollary 4.1 (See Theorem 1.1 in [18])**.** *Let P be a Sylow subgroup of G, where p is the smallest* *prime dividing $|G|$. If every number V of some fixed $\mathcal{M}_\phi(P)$ is SS-quasinormal in G, then G is p-nilpotent.*

Corollary 4.2. *Let P be a Sylow subgroup of G. If $N_G(P)$ is p-nilpotent and every number V of some fixed $\mathcal{M}_\phi(P)$ is S-propermutable in G, then G is p-nilpotent.*

Proof. If $|P| = p$, then G is p-nilpotent by Burnside's theorem [9, IV, 2.6]. Otherwise, G is p-supersoluble by Theorem A. The hypothesis holds for $G/O_{p'}(G)$ by Lemma 2.3(1), so in the case, where $O_{p'}(G) \neq 1$, $G/O_{p'}(G)$ is p-nilpotent by induction. Hence G is p-nilpotent. Therefore we may assume that $O_{p'}(G) = 1$. But then, by Lemma 2.7(2), P is normal in G. Hence G is p-nilpotent by hypothesis. □

From Corollary 4.2 we get

Corollary 4.3 (See Theorem 1.2 in [18])**.** *Let P be a Sylow subgroup of G. If $N_G(P)$ is p-nilpotent and every number V of some fixed $\mathcal{M}_\phi(P)$ is SS-quasinormal in G, then G is p-nilpotent.*

Corollary 4.4. *Let P be a Sylow subgroup of G. If G is p-soluble and every number V of some fixed $\mathcal{M}_d(P)$ is S-propermutable in G, then G is p-supersoluble.*

Proof. In the case, when $|P| = p$, this directly follows from the p-solubility of G. If $|P| > p$, this corollary follows from Theorem A. □

The next fact follows from Corollary 4.4.

Corollary 4.5 (See Theorem 1.3 in [18])**.** *Let P be a Sylow subgroup of G. If G is p-soluble and every number V of some fixed $\mathcal{M}_\phi(P)$ is SS-quasinormal in G, then G is p-supersoluble.*

Corollary 4.6. *If, for every prime p dividing $|G|$ and $P \in Syl_p(G)$, every number V of some fixed $\mathcal{M}_\phi(P)$ is S-propermutable in G, then G is supersoluble.*

Proof. Let p be the smallest prime dividing $|G|$. Then G is p-nilpotent by Corollary 4.1, so G is soluble by Fait-Thompson's theorem. Hence G is supersoluble by Corollary 4.4. □

From Corollary 4.6 we get

Corollary 4.7 (See Theorem 1.4 in [18])**.** *If, for every prime p dividing $|G|$ and $P \in Syl_p(G)$, every number V of some fixed $\mathcal{M}_\phi(P)$ is SS-quasinormal in G, then G is supersoluble.*

The formation $\mathcal{F}$ is said to be *saturated* if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$. It is clear that every saturated formation is soluble saturated. Hence from Corollary 1.4 we get

Corollary 4.8. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and $X \leq E$ normal subgroups of G such that $G/E \in \mathfrak{F}$. Suppose that every maximal subgroup of any non-cyclic Sylow subgroup of X is S-propermutable in G. If either $X = E$ or $X = F^*(E)$, then $G \in \mathfrak{F}$.*

The following results are special cases of Corollary 4.8.

Corollary 4.9 (See Theorem 1.5 in [18]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and E a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that for every maximal subgroup of every non-cyclic Sylow subgroup of E is SS-quasinormal in G. Then $G \in \mathfrak{F}$.*

Corollary 4.10 (See Theorem 3.2 in [19]). *Let E a normal subgroup of G such that G/E is supersoluble. Suppose that for every maximal subgroup of every Sylow subgroup of $F^*(E)$ is SS-quasinormal in G. Then G is supersoluble.*

Corollary 4.11 (See Theorem 3.3 in [19]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and E a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that for every maximal subgroup of every Sylow subgroup of $F^*(E)$ is SS-quasinormal in G. Then $G \in \mathfrak{F}$.*

Corollary 4.12 (See Theorem 3.2 in [14]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and E a normal subgroup of G such that $G/E \in \mathfrak{F}$. If all maximal subgroups of $F^*(E)$ are S-permutable in G, then $G \in \mathfrak{F}$.*

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