OSCILLATION OF SECOND-ORDER NONLINEAR NEUTRAL DYNAMIC EQUATIONS WITH NONCANONICAL OPERATORS

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Abstract. The study of half-linear differential equations has become an important area of research due to the fact that such equations occur in a variety of real world problems such as in the study of p-Laplace equations, non-Newtonian fluid theory, and the turbulent flow of a polytrophic gas in a porous medium. On the basis of these background details, we study oscillatory behavior of a class of second-order neutral functional dynamic equations on a time scale. New criteria improve and complement related results reported in the literature. Some examples are included to illustrate the results obtained. In particular, an example regarding the second-order neutral differential equation is also provided to show that these theorems improve those in the continuous case.

1. Introduction

In this paper, we study oscillation of a class of second-order nonlinear neutral functional dynamic equations

$$\Big(r(t)\left(\left(x(t)+p(t)x(\eta(t))\right)^{\Delta}\right)^{\gamma}\Big)^{\Delta}+f\big(t,x(g(t))\big)=0 \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$ with $\sup\mathbb{T}=\infty$. We assume $t_0\in\mathbb{T}$ and it is convenient to assume $t_0>0$, and define the time scale interval of the form $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$. Throughout, we assume the following assumptions hold.

(A_1) γ is a quotient of odd positive integers, r and p are real-valued positive rd-continuous functions defined on $\mathbb{T}$;

(A_2) $\eta:[t_0,\infty)_{\mathbb{T}}\to\mathbb{T}$ is rd-continuous, $g:[t_0,\infty)_{\mathbb{T}}\to\mathbb{T}$ is rd-continuous, and $\lim_{t\to\infty}\eta(t)=\lim_{t\to\infty}g(t)=\infty$;

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(A_3) $f(t,u) : [t_0,\infty)_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $uf(t,u) > 0$ for all $u \neq 0$ and there exists a positive rd-continuous function q defined on $\mathbb{T}$ such that $|f(t,u)| \geq q(t)|u|^\gamma$.

By a solution of equation (1.1) we mean a nontrivial real-valued function $x \in \mathrm{C}^1_{\mathrm{rd}}[T_x,\infty)_{\mathbb{T}}$, $T_x \in [t_0,\infty)_{\mathbb{T}}$ which has the properties that $x + px \circ \eta$ and $r\left((x + px\circ\eta)^\Delta\right)^\gamma$ are defined and Δ-differentiable for $t \in [T_x,\infty)_{\mathbb{T}}$, and satisfies (1.1) on $[T_x,\infty)_{\mathbb{T}}$. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Equation (1.1) is called oscillatory if all its solutions oscillate.

The theory of time scales is initiated by Hilger [19] in order to unify continuous and discrete analysis. Several authors have expounded on various aspects of the theory of dynamic equations on time scales; see the survey paper by Agarwal et al. [1] and the references cited therein. The books on the subject of time scales, by Bohner and Peterson [6,7], summarize and organize much of time scale calculus. There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, combinatorics, etc. A cover story article in New Scientist [30] discusses several possible applications. Recently, an increasing interest in obtaining sufficient conditions for oscillatory or nonoscillatory behavior of different classes of dynamic equations has been manifested, we refer the reader to [2–5,8–14,16–18,20–29,33–36,38–40] and the references cited therein. Agarwal et al. [4], Candan [9], Chen [10], Erbe et al. [12], Şahiner [24], Saker [25, 26], Saker et al. [27, 28], Saker and O'Regan [29], Tripathy [34], Wu et al. [36], Yang and Xu [38], and Zhang and Wang [40] investigated (1.1) and obtained some oscillation results in the canonical case

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{1/\gamma}(t)} = \infty, \tag{1.2}$$

some of which we present below for the convenience of the reader.

Theorem 1.1 (See [4, Theorem 3.4]). *Let* (1.2) *and* (A_1)–(A_3) *hold,* $\eta(t) = t - \tau < t$, $g(t) = t - \delta < t$, *and* $0 \leq p(t) < 1$. *Assume that* $\gamma \geq 1$, $r^\Delta(t) \geq 0$, *and there exists a positive rd-continuous Δ-differentiable function α such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\alpha(s)q(s)(1-p(s-\delta))^\gamma - \frac{((\alpha^\Delta(s))_+)^2 r(s-\delta)}{4\gamma\left(\frac{s-\delta}{2}\right)^{\gamma-1}\alpha(s)}\right]\Delta s = \infty,$$

where $(\alpha^\Delta(t))_+ := \max\{0, \alpha^\Delta(t)\}$. *Then* (1.1) *is oscillatory.*

In order to improve Theorem 1.1, Saker [25] established the following new result.

Theorem 1.2 (See [25, Corollary 3.1]). *Let* (1.2) *and* (A_1)–(A_3) *hold,* $\eta(t) = t - \tau < t$, $g(t) = t - \delta < t$, *and* $0 \le p(t) < 1$. *Assume that* $\gamma \ge 1$ *and there exists a positive rd-continuous* Δ*-differentiable function* α *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\alpha(s)q(s)(1 - p(s-\delta))^{\gamma} - \frac{((\alpha^{\Delta}(s))_+)^{\gamma+1} r(s-\delta)}{(\gamma+1)^{\gamma+1}\alpha^{\gamma}(s)} \right] \Delta s = \infty,$$

where $(\alpha^{\Delta}(t))_+ := \max\{0, \alpha^{\Delta}(t)\}$. *Then* (1.1) *is oscillatory.*

Note that Theorem 1.1 and Theorem 1.2 can only be applied to the case when $\eta(t) = t - \tau < t$ and $g(t) = t - \delta < t$. Later, Erbe et al. [12] obtained the following new results. For the sake of simplification, we use the notation

$$(\delta^{\Delta}(t))_+ := \max\{0, \delta^{\Delta}(t)\} \quad \text{and} \quad \theta(t,u) := \frac{\int_u^{g(t)} \Delta s / r^{1/\gamma}(s)}{\int_u^{t} \Delta s / r^{1/\gamma}(s)}.$$

Theorem 1.3 (See [12, Theorem 2.1]). *Let* (1.2) *and* (A_1)–(A_3) *hold,* $0 \le p(t) < 1$, $\eta(t) \le t$, *and* $g(t) \ge t$. *Assume that there exists a positive real-valued* Δ*-differentiable function* δ *such that for all sufficiently large* T,

$$\limsup_{t\to\infty} \int_{T}^{t} \Big[\delta(s)q(s)\big(1 - p(g(s))\big)^{\gamma} - \frac{r(s)((\delta^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)} \Big] \Delta s = \infty. \tag{1.3}$$

Then (1.1) *is oscillatory.*

Theorem 1.4 (See [12, Theorem 2.1]). *Let* (1.2) *and* (A_1)–(A_3) *hold,* $0 \le p(t) < 1$, $\eta(t) \le t$, *and* $g(t) \le t$. *Assume that there exists a positive real-valued* Δ*-differentiable function* δ *such that for all sufficiently large* T_* *and for* $g(T) > T_*$,

$$\limsup_{t\to\infty} \int_{T}^{t} \Big[\delta(s)\theta^{\gamma}(s,T_*)q(s)\big(1 - p(g(s))\big)^{\gamma} - \frac{r(s)((\delta^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)} \Big] \Delta s = \infty. \tag{1.4}$$

Then (1.1) *is oscillatory.*

As yet, there are few results for oscillation of (1.1) in the noncanonical case

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{1/\gamma}(t)} < \infty. \tag{1.5}$$

Saker [26] obtained some criteria for (1.1) provided that

$$p^{\Delta}(t) \geq 0, \quad g(t) \leq \eta(t) \leq t, \quad \eta^{\Delta}(t) \geq 0, \tag{1.6}$$

and

$$\int_T^{\infty} \left(\frac{1}{r(s)} \int_T^s q(u)(1-p(u))^{\gamma} \left(\int_u^{\infty} \frac{\Delta t}{r^{1/\gamma}(t)} \right)^{\gamma} \Delta u \right)^{\frac{1}{\gamma}} \Delta s = \infty \tag{1.7}$$

for some $T \in [t_0, \infty)_{\mathbb{T}}$. Tripathy [35] established several new results for (1.1) in the case where $0 \leq p(t) \leq p_0 < \infty$, $\eta(t) \leq t$, $g(t) \leq t$,

$$0 < \gamma \leq 1 \quad \text{and} \quad \eta \circ g = g \circ \eta. \tag{1.8}$$

As a special case when $\mathbb{T} = \mathbb{R}$, equation (1.1) reduces to a second-order neutral differential equation

$$\big(r(t)\left((x(t) + p(t)x(\eta(t)))'\right)^{\gamma}\big)' + f\big(t, x(g(t))\big) = 0 \tag{1.9}$$

and condition (1.5) reduces to

$$\int_{t_0}^{\infty} \frac{\mathrm{d}t}{r^{1/\gamma}(t)} < \infty. \tag{1.10}$$

Most oscillation results given in the literature for the neutral differential equation (1.9) and its particular cases have been obtained under the condition $\int_{t_0}^{\infty} r^{-1/\gamma}(t)\mathrm{d}t = \infty$ which significantly simplifies the analysis of the behavior of $z(t) = x(t) + p(t)x(\eta(t))$ for a nonoscillatory solution x of (1.9). We note that the investigation of oscillation of equation (1.9) in the case (1.10) brings additional difficulties. In fact, if x is an eventually positive solution of (1.9), then the inequality

$$x(t) \geq (1 - p(t))z(t)$$

does not hold when (1.10), $\eta(t) \leq t$, and $0 \leq p(t) < 1$ are satisfied. Han et al. [15] derived [15, Theorem 2.1 and 2.2] under the assumptions that $\eta(t) = t - \tau$, $p'(t) \geq 0$, $g(t) \leq t - \tau$, and (1.10) hold. Xu and Meng [37] established [37, Theorem 2.3] for oscillation and asymptotic behavior of (1.9) provided that $\eta(t) = t - \tau$, $p'(t) \geq 0$, $\lim_{t\to\infty} p(t) = A$, and (1.10) hold. Sun et al. [31, 32] studied (1.9) in the case where

- (A_4) $r, p \in \mathrm{C}([t_0, \infty), \mathbb{R})$, $r(t) > 0$, and $0 \leq p(t) \leq p_0 < \infty$;
- (A_5) $g \in \mathrm{C}^1([t_0, \infty), \mathbb{R})$, $g'(t) > 0$, and $\lim_{t\to\infty} g(t) = \infty$;
- (A_6) $\eta \in \mathrm{C}^1([t_0, \infty), \mathbb{R})$, $\eta'(t) \geq \eta_0 > 0$, and $\eta \circ g = g \circ \eta$,

and obtained several new results, one of which we present below for the convenience of the reader.

Theorem 1.5 (See [31, Theorem 4.1]). *Assume* (1.10), (A_3) *with* $\mathbb{T}=\mathbb{R}$, (A_4)–(A_6), *and let* $\gamma \geq 1$ *and* $g(t) \leq \eta(t) \leq t$ *for all* $t \geq t_0$. *Suppose further that there exists a function* $\delta \in \mathrm{C}^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{\delta(s)Q(s)}{2^{\gamma-1}} - \frac{\left(1+\frac{p_0{}^{\gamma}}{\eta_0}\right) r(g(s))(\delta'_+(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\delta(s)g'(s))^{\gamma}} \right] \mathrm{d}s = \infty.$$

If there exists a function $\tau \in \mathrm{C}^1([t_0,\infty),\mathbb{R})$ *such that* $\tau(t) \geq t$, $\tau'(t) > 0$, *and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{\pi^{\gamma}(s)Q(s)}{2^{\gamma-1}} - \frac{\gamma^{\gamma+1}\left(1+\frac{p_0{}^{\gamma}}{\eta_0}\right)\tau'(s)}{(\gamma+1)^{\gamma+1}\pi(s)r^{1/\gamma}(\tau(s))} \right] \mathrm{d}s = \infty,$$

where $Q(t) := \min\{q(t), q(\eta(t))\}$, $\delta'_+(t) := \max\{0,\delta'(t)\}$, *and* $\pi(t) := \int_{\tau(t)}^{\infty} r^{-1/\gamma}(s)\mathrm{d}s$, *then* (1.9) *is oscillatory.*

Hence the natural question now is: Can one establish some new oscillation results for (1.1) *in the case where* (1.5) *holds and without conditions* (1.6), (1.7), *and* (1.8)? The purpose of this paper is to derive some new oscillation criteria for (1.1) and reply this question. This paper is organized as follows: In Section 2, we suggest some oscillation criteria for (1.1). In Section 3, three examples are provided to show applications of the results obtained in Section 2. In Section 4, some remarks are given to summarize the contents of this paper.

2. Oscillation results

In what follows, all functional inequalities are assumed to hold eventually, that is, they are satisfied for all t large enough. For the sake of convenience, we use the notation $z(t) := x(t) + p(t)x(\eta(t))$ and $R(t) := \int_t^{\infty} \frac{\Delta s}{r^{1/\gamma}(s)}$. To prove the main theorems, we will use the formula

$$(x^{\gamma})^{\Delta}(t) = \gamma x^{\Delta}(t) \int_0^1 \left[hx^{\sigma}(t) + (1-h)x(t)\right]^{\gamma-1}\mathrm{d}h, \tag{2.1}$$

which is a simple consequence of Keller's chain rule [6, Theorem 1.90].

Theorem 2.1. *Let* (1.5) *and* (A_1)–(A_3) *hold,* $0 \leq p(t) < 1$, $\eta(t) \leq t$, $g(t) \geq \sigma(t)$, *and* $\gamma \leq 1$. *Assume that there exists a positive real-valued* Δ*-differentiable function* δ *such that* (1.3) *holds for all sufficiently large*

T. If there exists a positive real-valued Δ-differentiable function m such that

$$\frac{m(t)}{r^{1/\gamma}(t)R(t)}+m^{\Delta}(t)\le 0,\quad 1-p(t)\frac{m(\eta(t))}{m(t)}>0, \tag{2.2}$$

and

$$\limsup_{t\to\infty}\int_T^t\left[q(s)\left(1-p(g(s))\frac{m(\eta(g(s)))}{m(g(s))}\right)^{\gamma}\left(\frac{m(g(s))}{m(\sigma(s))}\right)^{\gamma}R^{\gamma\sigma}(s)\right.$$
$$\left.-\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{1}{R^{\sigma}(s)r^{1/\gamma}(s)}\right]\Delta s=\infty, \tag{2.3}$$

then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality we assume $x(t)>0$, $x(\eta(t))>0$, and $x(g(t))>0$ for $t\in[t_0,\infty)_{\mathbb{T}}$. In view of (1.1), we obtain

$$(r(z^{\Delta})^{\gamma})^{\Delta}(t)\le -q(t)x^{\gamma}(g(t))<0\quad\text{for}\quad t\in[t_0,\infty)_{\mathbb{T}}. \tag{2.4}$$

Hence $r(z^{\Delta})^{\gamma}$ is eventually strictly decreasing and there exists a $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $z^{\Delta}(t)>0$, or $z^{\Delta}(t)<0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. Assume first that $z^{\Delta}(t)>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. From the proof of [12, Theorem 2.1], we can obtain a contradiction to (1.3). Assume now that $z^{\Delta}(t)<0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. Define the function ω by

$$\omega(t):=\frac{r(t)(z^{\Delta}(t))^{\gamma}}{z^{\gamma}(t)}. \tag{2.5}$$

Then $\omega(t)<0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. By (2.4), we get

$$z^{\Delta}(s)\le\frac{r^{1/\gamma}(t)}{r^{1/\gamma}(s)}z^{\Delta}(t)\quad\text{for}\quad s\in[t,\infty)_{\mathbb{T}}.$$

Integrating this from t to l, we have

$$z(l)\le z(t)+r^{1/\gamma}(t)z^{\Delta}(t)\int_t^l\frac{\Delta s}{r^{1/\gamma}(s)}\quad\text{for}\quad l\in[t,\infty)_{\mathbb{T}}.$$

Letting $l\to\infty$ in the latter inequality yields

$$z(t)+r^{1/\gamma}(t)z^{\Delta}(t)R(t)\ge 0\quad\text{for}\quad t\in[t_1,\infty)_{\mathbb{T}}.$$

Thus, we obtain

$$R(t)r^{1/\gamma}(t)\frac{z^{\Delta}(t)}{z(t)}\ge -1. \tag{2.6}$$

By virtue of (2.5) and (2.6), we have

$$-1 \leq R^{\gamma}(t)\omega(t) \leq 0. \tag{2.7}$$

On the other hand, it follows from (2.6) that

$$\frac{z^{\Delta}(t)}{z(t)} \geq -\frac{1}{r^{1/\gamma}(t)R(t)}.$$

Then, we have

$$\begin{aligned}\left(\frac{z}{m}\right)^{\Delta}(t) &= \frac{z^{\Delta}(t)m(t)-z(t)m^{\Delta}(t)}{m(t)m^{\sigma}(t)} \\ &\geq -\frac{z(t)}{m(t)m^{\sigma}(t)}\left[\frac{m(t)}{r^{1/\gamma}(t)R(t)}+m^{\Delta}(t)\right] \geq 0,\end{aligned}$$

and thus z/m is nondecreasing. Hence we obtain

$$\begin{aligned}x(t) &= z(t)-p(t)x(\eta(t)) \geq z(t)-p(t)z(\eta(t)) \\ &\geq z(t)-p(t)\frac{m(\eta(t))}{m(t)}z(t) = \left(1-p(t)\frac{m(\eta(t))}{m(t)}\right)z(t)\end{aligned}$$

and

$$\frac{z(g(t))}{z(\sigma(t))} \geq \frac{m(g(t))}{m(\sigma(t))} \quad \text{since} \quad g(t) \geq \sigma(t).$$

Differentiating (2.5) and using (2.4), we obtain

$$\begin{aligned}\omega^{\Delta}(t) \leq &- q(t)\left(1-p(g(t))\frac{m(\eta(g(t)))}{m(g(t))}\right)^{\gamma}\left(\frac{m(g(t))}{m(\sigma(t))}\right)^{\gamma} \\ &- \frac{r(t)(z^{\Delta}(t))^{\gamma}(z^{\gamma})^{\Delta}(t)}{z^{\gamma}(t)z^{\gamma}(\sigma(t))}.\end{aligned} \tag{2.8}$$

In view of (2.1), we see that

$$(z^{\gamma})^{\Delta}(t) \leq \gamma z^{\gamma-1}(t)z^{\Delta}(t) \quad \text{since} \quad \gamma \leq 1.$$

Thus, (2.8) yields

$$\begin{aligned}\omega^{\Delta}(t) \leq &- q(t)\left(1-p(g(t))\frac{m(\eta(g(t)))}{m(g(t))}\right)^{\gamma}\left(\frac{m(g(t))}{m(\sigma(t))}\right)^{\gamma} \\ &- \gamma\frac{r(t)(z^{\Delta}(t))^{\gamma+1}}{z(t)z^{\gamma}(\sigma(t))}.\end{aligned} \tag{2.9}$$

On the other hand, we have by $z^{\Delta}(t) < 0$ that $z(t) \geq z^{\sigma}(t)$ and

$$-\gamma\frac{r(t)(z^{\Delta}(t))^{\gamma+1}}{z(t)z^{\gamma}(\sigma(t))} \leq -\gamma\left(\frac{1}{r(t)}\right)^{1/\gamma}\omega^{(\gamma+1)/\gamma}(t).$$

Hence by (2.9), we get

$$\begin{aligned} \omega^{\Delta}(t) &+ q(t)\left(1-p(g(t))\frac{m(\eta(g(t)))}{m(g(t))}\right)^{\gamma}\left(\frac{m(g(t))}{m(\sigma(t))}\right)^{\gamma} \\ &+ \gamma r^{-1/\gamma}(t)\omega^{(\gamma+1)/\gamma}(t)\leq 0 \quad \text{for} \quad t\in[t_1,\infty)_{\mathbb{T}}. \end{aligned} \tag{2.10}$$

Multiplying (2.10) by $R^{\gamma\sigma}(t)$ implies that

$$\begin{aligned} R^{\gamma\sigma}(t)\omega^{\Delta}(t) &+ q(t)\left(1-p(g(t))\frac{m(\eta(g(t)))}{m(g(t))}\right)^{\gamma}\left(\frac{m(g(t))}{m(\sigma(t))}\right)^{\gamma}R^{\gamma\sigma}(t) \\ &+ \gamma R^{\gamma\sigma}(t)r^{-1/\gamma}(t)\omega^{(\gamma+1)/\gamma}(t)\leq 0 \quad \text{for} \quad t\in[t_1,\infty)_{\mathbb{T}}. \end{aligned}$$

Integrating this from t_1 to t, we get

$$\begin{aligned} &\int_{t_1}^{t} q(s)\left(1-p(g(s))\frac{m(\eta(g(s)))}{m(g(s))}\right)^{\gamma}\left(\frac{m(g(s))}{m(\sigma(s))}\right)^{\gamma}R^{\gamma\sigma}(s)\Delta s \\ &+\int_{t_1}^{t} R^{\gamma\sigma}(s)\omega^{\Delta}(s)\Delta s+\gamma\int_{t_1}^{t} R^{\gamma\sigma}(s)r^{-1/\gamma}(s)\omega^{(\gamma+1)/\gamma}(s)\Delta s\leq 0. \end{aligned} \tag{2.11}$$

Integrating by parts, we have

$$\begin{aligned} \int_{t_1}^{t} R^{\gamma\sigma}(s)\omega^{\Delta}(s)\Delta s = R^{\gamma}(t)\omega(t) &- R^{\gamma}(t_1)\omega(t_1) \\ &- \int_{t_1}^{t}(R^{\gamma}(s))^{\Delta}\omega(s)\Delta s. \end{aligned} \tag{2.12}$$

From (2.1), we obtain

$$(R^{\gamma})^{\Delta}(t)=\gamma R^{\Delta}(t)\int_0^1\left[hR^{\sigma}(t)+(1-h)R(t)\right]^{\gamma-1}\mathrm{d}h. \tag{2.13}$$

Noting that $R^{\Delta}(t)=-(1/r(t))^{1/\gamma}<0$, we get by (2.13) and $\gamma\leq 1$ that

$$-\int_{t_1}^{t}(R^{\gamma})^{\Delta}(s)\omega(s)\Delta s\geq\gamma\int_{t_1}^{t}\left(\frac{1}{r(s)}\right)^{1/\gamma}(R^{\sigma}(s))^{\gamma-1}\omega(s)\Delta s. \tag{2.14}$$

By virtue of (2.11), (2.12), and (2.14), we see that

$$\begin{aligned} &\int_{t_1}^{t} q(s)\left(1-p(g(s))\frac{m(\eta(g(s)))}{m(g(s))}\right)^{\gamma}\left(\frac{m(g(s))}{m(\sigma(s))}\right)^{\gamma}R^{\gamma\sigma}(s)\Delta s \\ &+R^{\gamma}(t)\omega(t)-R^{\gamma}(t_1)\omega(t_1)+\gamma\int_{t_1}^{t}\left(\frac{1}{r(s)}\right)^{1/\gamma}(R^{\sigma}(s))^{\gamma-1}\omega(s)\Delta s \\ &+\gamma\int_{t_1}^{t} R^{\gamma\sigma}(s)r^{-1/\gamma}(s)\omega^{(\gamma+1)/\gamma}(s)\Delta s\leq 0. \end{aligned} \tag{2.15}$$

Set $p := (\gamma+1)/\gamma$, $q := \gamma+1$,

$$A := -(\gamma+1)^{\gamma/(\gamma+1)} \left(\frac{R^{\gamma\sigma}(t)}{r^{1/\gamma}(t)}\right)^{\gamma/(\gamma+1)} \omega(t),$$

and

$$B := \frac{\gamma}{\gamma+1}(\gamma+1)^{1/(\gamma+1)} \left(\frac{1}{r^{1/\gamma}(t)}\right)^{1/(\gamma+1)} \frac{1}{(R^\sigma(t))^{1/(\gamma+1)}}.$$

Using the inequality

$$\frac{A^p}{p} + \frac{B^q}{q} \geq AB, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{2.16}$$

we have

$$\begin{aligned} &\gamma R^{\gamma\sigma}(t) r^{-1/\gamma}(t) \omega^{(\gamma+1)/\gamma}(t) + \frac{\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}}{R^\sigma(t) r^{1/\gamma}(t)} \\ &\geq -\gamma \left(\frac{1}{r(t)}\right)^{1/\gamma} (R^\sigma(t))^{\gamma-1} \omega(t). \end{aligned} \tag{2.17}$$

Thus, we get by (2.15) and (2.17) that

$$\begin{aligned} &\int_{t_1}^{t} \Bigg[q(s) \left(1 - p(g(s)) \frac{m(\eta(g(s)))}{m(g(s))}\right)^\gamma \left(\frac{m(g(s))}{m(\sigma(s))}\right)^\gamma R^{\gamma\sigma}(s) \\ &\quad - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1} \frac{1}{R^\sigma(s) r^{1/\gamma}(s)} \Bigg] \Delta s \leq R^\gamma(t_1)\omega(t_1) - R^\gamma(t)\omega(t). \end{aligned}$$

Therefore by (2.7), we get a contradiction to (2.3). The proof is complete. □

Regarding the case where $g(t) \leq t$, similar as in the proof of Theorem 2.1, we obtain the following result.

Theorem 2.2. *Let* (1.5) *and* (A_1)–(A_3) *hold,* $0 \leq p(t) < 1$, $\eta(t) \leq t$, $g(t) \leq t$, *and* $\gamma \leq 1$. *Assume that there exists a positive real-valued* Δ-*differentiable function* δ *such that* (1.4) *holds for all sufficiently large* T_* *and for* $g(T) > T_*$. *If there exists a positive real-valued* Δ-*differentiable function* m *such that* (2.2) *holds and*

$$\begin{aligned} \limsup_{t\to\infty} &\int_T^t \Bigg[q(s) \left(1 - p(g(s)) \frac{m(\eta(g(s)))}{m(g(s))}\right)^\gamma R^{\gamma\sigma}(s) \\ &\quad - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1} \frac{1}{R^\sigma(s) r^{1/\gamma}(s)} \Bigg] \Delta s = \infty, \end{aligned} \tag{2.18}$$

then (1.1) *is oscillatory.*

Theorem 2.3. *Let* (1.5) *and* (A_1)–(A_3) *hold,* $0 \le p(t) < 1$*,* $\eta(t) \le t$*,* $g(t) \ge \sigma(t)$*, and* $\gamma \ge 1$*. Assume that there exists a positive real-valued* Δ*-differentiable function* δ *such that* (1.3) *holds for all sufficiently large* T*. If there exists a positive real-valued* Δ*-differentiable function* m *such that* (2.2) *holds and*

(2.19)
$$\limsup_{t\to\infty} \int_T^t \left[q(s)\left(1 - p(g(s))\frac{m(\eta(g(s)))}{m(g(s))}\right)^\gamma \left(\frac{m(g(s))}{m(\sigma(s))}\right)^\gamma R^{\gamma\sigma}(s) - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1} \frac{R^{\gamma^2-1}(s)}{(R^\sigma(s))^{\gamma^2} r^{1/\gamma}(s)}\right]\Delta s = \infty,$$

then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality we assume $x(t) > 0$, $x(\eta(t)) > 0$, and $x(g(t)) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. We obtain (2.4) by (1.1). Therefore, $r(z^\Delta)^\gamma$ is eventually strictly decreasing and there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $z^\Delta(t) > 0$, or $z^\Delta(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Assume first that $z^\Delta(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. From the proof of [12, Theorem 2.1], we can obtain a contradiction to (1.3). Consider now the case where $z^\Delta(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Defining ω as in (2.5), we have (2.7). Differentiating (2.5) and using (2.4), we obtain (2.8). In view of (2.1), we have

$$(z^\gamma)^\Delta(t) \le \gamma z^{\gamma-1}(\sigma(t)) z^\Delta(t) \quad \text{since} \quad \gamma \ge 1.$$

Thus, we get

$$\begin{aligned} \omega^\Delta(t) \le \;&- q(t)\left(1 - p(g(t))\frac{m(\eta(g(t)))}{m(g(t))}\right)^\gamma \left(\frac{m(g(t))}{m(\sigma(t))}\right)^\gamma \\ &- \gamma\frac{r(t)(z^\Delta(t))^{\gamma+1}}{z^\gamma(t) z(\sigma(t))}. \end{aligned} \tag{2.20}$$

On the other hand, we have by $z^\Delta(t) < 0$ that $z(t) \ge z^\sigma(t)$ and

$$-\gamma\frac{r(t)(z^\Delta(t))^{\gamma+1}}{z^\gamma(t) z(\sigma(t))} \le -\gamma\left(\frac{1}{r(t)}\right)^{1/\gamma} \omega^{(\gamma+1)/\gamma}(t).$$

Hence by (2.20), we get (2.10). Then we obtain that (2.11) and (2.12) hold. By virtue of (2.1), we have (2.13). From (2.13), $\gamma \ge 1$, and $R^\Delta(t) = -(1/r(t))^{1/\gamma} < 0$, we see that

$$-\int_{t_1}^t (R^\gamma)^\Delta(s)\omega(s)\Delta s \ge \gamma \int_{t_1}^t \left(\frac{1}{r(s)}\right)^{1/\gamma} R^{\gamma-1}(s)\omega(s)\Delta s. \tag{2.21}$$

It follows from (2.11), (2.12), and (2.21) that

$$\int_{t_1}^{t} q(s)\left(1-p(g(s))\frac{m(\eta(g(s)))}{m(g(s))}\right)^{\gamma}\left(\frac{m(g(s))}{m(\sigma(s))}\right)^{\gamma} R^{\gamma\sigma}(s)\Delta s$$
$$+R^{\gamma}(t)\omega(t)-R^{\gamma}(t_1)\omega(t_1)+\gamma\int_{t_1}^{t}\left(\frac{1}{r(s)}\right)^{1/\gamma} R^{\gamma-1}(s)\omega(s)\Delta s$$
$$+\gamma\int_{t_1}^{t} R^{\gamma\sigma}(s)r^{-1/\gamma}(s)\omega^{(\gamma+1)/\gamma}(s)\Delta s\leq 0. \tag{2.22}$$

Set $p:=(\gamma+1)/\gamma$, $q:=\gamma+1$,

$$A:=-(\gamma+1)^{\gamma/(\gamma+1)}\left(\frac{R^{\gamma\sigma}(t)}{r^{1/\gamma}(t)}\right)^{\gamma/(\gamma+1)}\omega(t),$$

and

$$B:=\frac{\gamma}{\gamma+1}(\gamma+1)^{1/(\gamma+1)}\left(\frac{1}{r^{1/\gamma}(t)}\right)^{1/(\gamma+1)}\frac{R^{\gamma-1}(t)}{(R^{\sigma}(t))^{\gamma^2/(\gamma+1)}}.$$

Using inequality (2.16), we have

$$\gamma R^{\gamma\sigma}(t)r^{-1/\gamma}(t)\omega^{(\gamma+1)/\gamma}(t)+\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{R^{\gamma^2-1}(t)}{(R^{\sigma}(t))^{\gamma^2}r^{1/\gamma}(t)}$$
$$\geq -\gamma\left(\frac{1}{r(t)}\right)^{1/\gamma}R^{\gamma-1}(t)\omega(t). \tag{2.23}$$

Thus, we obtain by (2.22) and (2.23) that

$$\int_{t_1}^{t}\left[q(s)\left(1-p(g(s))\frac{m(\eta(g(s)))}{m(g(s))}\right)^{\gamma}\left(\frac{m(g(s))}{m(\sigma(s))}\right)^{\gamma}R^{\gamma\sigma}(s)\right.$$
$$\left.-\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{R^{\gamma^2-1}(s)}{(R^{\sigma}(s))^{\gamma^2}r^{1/\gamma}(s)}\right]\Delta s\leq R^{\gamma}(t_1)\omega(t_1)-R^{\gamma}(t)\omega(t).$$

Hence by (2.7), we obtain a contradiction to (2.19). This completes the proof. □

Similar as in the proof of Theorem 2.3, we establish the following result for the case where $g(t)\leq t$.

Theorem 2.4. *Let* (1.5) *and* (A_1)–(A_3) *hold,* $0\leq p(t)<1$, $\eta(t)\leq t$, $g(t)\leq t$, *and* $\gamma\geq 1$. *Assume that there exists a positive real-valued* Δ-*differentiable function* δ *such that* (1.4) *holds for all sufficiently large* T_*

and for $g(T) > T_*$. *If there exists a positive real-valued* Δ*-differentiable function* m *such that* (2.2) *holds and*

$$\limsup_{t\to\infty} \int_T^t \left[q(s)\left(1 - p(g(s))\frac{m(\eta(g(s)))}{m(g(s))}\right)^{\gamma} R^{\gamma\sigma}(s) - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1} \frac{R^{\gamma^2-1}(s)}{(R^{\sigma}(s))^{\gamma^2} r^{1/\gamma}(s)} \right] \Delta s = \infty, \tag{2.24}$$

then (1.1) *is oscillatory.*

Theorem 2.5. *Let* (1.2) *and* (A_1)*–*(A_3) *hold,* $0 \le p(t) < 1$, $\eta(t) \ge t$, *and* $g(t) \ge t$. *Assume that there exists a positive real-valued* Δ*-differentiable function* m *such that for all sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$,

$$\frac{m(t)}{r^{1/\gamma}(t)\int_{t_1}^t \frac{\Delta s}{r^{1/\gamma}(s)}} - m^{\Delta}(t) \le 0 \quad \text{and} \quad 1 - p(t)\frac{m(\eta(t))}{m(t)} > 0. \tag{2.25}$$

If there exists a positive real-valued Δ*-differentiable function* δ *such that for all sufficiently large* T,

$$\limsup_{t\to\infty} \int_T^t \left[\delta(s)q(s)\left(1 - p(g(s))\frac{m(\eta(g(s)))}{m(g(s))}\right)^{\gamma} - \frac{r(s)((\delta^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)} \right] \Delta s = \infty, \tag{2.26}$$

then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality we assume $x(t) > 0$, $x(\eta(t)) > 0$, and $x(g(t)) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. It follows from (1.2) that there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $z^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. From (1.1), we see that

$$z(t) = z(t_1) + \int_{t_1}^t \frac{(r(z^{\Delta})^{\gamma})^{1/\gamma}(s)}{r^{1/\gamma}(s)} \Delta s \ge r^{1/\gamma}(t) z^{\Delta}(t) \int_{t_1}^t \frac{\Delta s}{r^{1/\gamma}(s)}.$$

Since

$$\begin{aligned} \left(\frac{z}{m}\right)^{\Delta}(t) &= \frac{z^{\Delta}(t)m(t) - z(t)m^{\Delta}(t)}{m(t)m^{\sigma}(t)} \\ &\le \frac{z(t)}{m(t)m^{\sigma}(t)} \left[\frac{m(t)}{r^{1/\gamma}(t)\int_{t_1}^t \frac{\Delta s}{r^{1/\gamma}(s)}} - m^{\Delta}(t) \right] \le 0, \end{aligned}$$

we find that z/m is nonincreasing. Hence we have

$$x(t) = z(t) - p(t)x(\eta(t)) \ge z(t) - p(t)z(\eta(t)) \ge \left(1 - p(t)\frac{m(\eta(t))}{m(t)}\right) z(t).$$

Define the function u by

$$u(t) := \delta(t)\frac{r(t)(z^{\Delta}(t))^{\gamma}}{z^{\gamma}(t)}.$$

Then $u(t) > 0$. The rest of the proof is similar to that of [12, Theorem 2.1], and so is omitted. □

Theorem 2.6. *Let* (1.2) *and* (A_1)–(A_3) *hold,* $0 \le p(t) < 1$, $\eta(t) \ge t$, *and* $g(t) \le t$. *Assume that there exists a positive real-valued* Δ-*differentiable function* m *such that* (2.25) *holds for all sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$. *If there exists a positive real-valued* Δ-*differentiable function* δ *such that for all sufficiently large* T_* *and for* $g(T) > T_*$,

$$\limsup_{t\to\infty}\int_T^t \left[\delta(s)\theta^{\gamma}(s,T_*)q(s)\left(1-p(g(s))\frac{m(\eta(g(s)))}{m(g(s))}\right)^{\gamma} - \frac{r(s)((\delta^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right]\Delta s = \infty, \tag{2.27}$$

where θ *is as in Section 1, then* (1.1) *is oscillatory.*

Proof. The proof is similar to those of Theorem 2.5 and [12, Theorem 2.1], and hence is omitted. □

Theorem 2.7. *Let* (1.5) *and* (A_1)–(A_3) *hold,* $0 \le p(t) < 1$, $\eta(t) \ge t$, $g(t) \ge \sigma(t)$, *and* $\gamma \le 1$. *Assume that there exists a positive real-valued* Δ-*differentiable function* m *such that* (2.25) *holds for all sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$. *Suppose further that there exists a positive real-valued* Δ-*differentiable function* δ *such that* (2.26) *holds for all sufficiently large* T. *If there exists a positive real-valued* Δ-*differentiable function* h *such that*

$$\frac{h(t)}{r^{1/\gamma}(t)R(t)} + h^{\Delta}(t) \le 0 \tag{2.28}$$

and

$$\limsup_{t\to\infty}\int_T^t \left[q(s)\left(1-p(g(s))\right)^{\gamma}\left(\frac{h(g(s))}{h(\sigma(s))}\right)^{\gamma}R^{\gamma\sigma}(s) - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\frac{1}{R^{\sigma}(s)r^{1/\gamma}(s)}\right]\Delta s = \infty, \tag{2.29}$$

then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality we assume $x(t) > 0$, $x(\eta(t)) > 0$, and $x(g(t)) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. In view of (1.1), we obtain (2.4). Therefore, $r(z^{\Delta})^{\gamma}$ is eventually strictly decreasing and there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $z^{\Delta}(t) > 0$, or $z^{\Delta}(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Assume first that $z^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. From the proof of Theorem 2.5, we can obtain a contradiction to (2.26). Assume now that $z^{\Delta}(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Define ω as in (2.5). Note that

$$x(t) = z(t) - p(t)x(\eta(t)) \geq z(t) - p(t)z(\eta(t)) \geq \big(1 - p(t)\big)z(t).$$

Using the similar proof with that of Theorem 2.1 completes the proof. □

Theorem 2.8. *Let* (1.5) *and* (A_1)–(A_3) *hold,* $0 \leq p(t) < 1$, $\eta(t) \geq t$, $g(t) \leq t$, *and* $\gamma \leq 1$. *Assume that there exists a positive real-valued* Δ-*differentiable function* m *such that* (2.25) *holds for all sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$. *Suppose further that there exists a positive real-valued* Δ-*differentiable function* δ *such that* (2.27) *holds for all sufficiently large* T_* *and for* $g(T) > T_*$. *If*

$$\limsup_{t\to\infty} \int_T^t \left[q(s)\left(1 - p(g(s))\right)^{\gamma} R^{\gamma\sigma}(s) - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1} \frac{1}{R^{\sigma}(s) r^{1/\gamma}(s)}\right] \Delta s = \infty, \tag{2.30}$$

then (1.1) *is oscillatory.*

Proof. The proof is similar to those of Theorem 2.1 and Theorem 2.7, and so is omitted. □

Theorem 2.9. *Let* (1.5) *and* (A_1)–(A_3) *hold,* $0 \leq p(t) < 1$, $\eta(t) \geq t$, $g(t) \geq \sigma(t)$, *and* $\gamma \geq 1$. *Assume that there exists a positive real-valued* Δ-*differentiable function* m *such that* (2.25) *holds for all sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$. *Suppose further that there exists a positive real-valued* Δ-*differentiable function* δ *such that* (2.26) *holds for all sufficiently large* T. *If there exists a positive real-valued* Δ-*differentiable function* h *such that* (2.28) *holds and*

$$\limsup_{t\to\infty} \int_T^t \left[q(s)\left(1 - p(g(s))\right)^{\gamma} \left(\frac{h(g(s))}{h(\sigma(s))}\right)^{\gamma} R^{\gamma\sigma}(s) - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1} \frac{R^{\gamma^2-1}(s)}{(R^{\sigma}(s))^{\gamma^2} r^{1/\gamma}(s)}\right] \Delta s = \infty, \tag{2.31}$$

then (1.1) *is oscillatory.*

Proof. The proof is similar to those of Theorem 2.3 and Theorem 2.7, and hence is omitted. □

Theorem 2.10. *Let* (1.5) *and* (A_1)–(A_3) *hold,* $0 \leq p(t) < 1$*,* $\eta(t) \geq t$*,* $g(t) \leq t$*, and* $\gamma \geq 1$*. Assume that there exists a positive real-valued* Δ*-differentiable function* m *such that* (2.25) *holds for all sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$*. Suppose further that there exists a positive real-valued* Δ*-differentiable function* δ *such that* (2.27) *holds for all sufficiently large* T_* *and for* $g(T) > T_*$*. If*

$$\limsup_{t\to\infty} \int_T^t \left[q(s)\,(1-p(g(s)))^\gamma R^{\gamma\sigma}(s) - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1} \frac{R^{\gamma^2-1}(s)}{(R^\sigma(s))^{\gamma^2} r^{1/\gamma}(s)} \right] \Delta s = \infty, \tag{2.32}$$

then (1.1) *is oscillatory.*

Proof. The proof is similar to those of Theorem 2.3 and Theorem 2.7, and so is omitted. □

3. Examples

Example 3.1. For $t \in [1, \infty)_{\mathbb{T}}$, consider a neutral dynamic equation

$$\left(t\sigma(t) \left(x(t) + p_0 \frac{\eta(t)}{t} x(\eta(t)) \right)^\Delta \right)^\Delta + g(t)x(g(t)) = 0, \tag{3.1}$$

where $\eta(t) \leq t$, $g(t) \geq \sigma(t)$, and $p_0 \in (0,1)$ is a constant. Let $\delta(t) = 1$ and $m(t) = t^{-1}$. Using Theorem 2.1 and [7, Theorem 5.68], we obtain that (3.1) is oscillatory.

Example 3.2. For $t \in [1, \infty)_{\mathbb{T}}$, consider a neutral dynamic equation

$$\left(t\sigma(t) \left(x(t) + p_0 \frac{t}{\eta(t)} x(\eta(t)) \right)^\Delta \right)^\Delta + g(t)x(g(t)) = 0, \tag{3.2}$$

where $\eta(t) \geq t$, $g(t) \geq \sigma(t)$, and $p_0 \in (0,1)$ is a constant. Let $\delta(t) = 1$, $m(t) = t$, and $h(t) = t^{-1}$. Using Theorem 2.7 and [7, Theorem 5.68], we get that (3.2) is oscillatory.

Example 3.3. For $t \geq 1$, consider a neutral differential equation

$$\left(t^3 \left(x(t) + \frac{1}{8} x\left(\frac{t}{2}\right) \right)' \right)' + q_0 t x\left(\frac{t}{3}\right) = 0. \tag{3.3}$$

Set $r(t) = t^3$, $\eta(t) = t/2$, $q(t) = q_0 t$, $g(t) = t/3$, $m(t) = t^{-2}/2$, and $\delta(t) = 1$. It is not difficult to verify that (3.3) is oscillatory if $q_0 > 2$ when using Theorem 2.2. *However, Theorem 1.5 implies that* (3.3) *is oscillatory when* $q_0 > 5/2$.

4. Discussions

Remark 4.1. Results of [26, 35] cannot be applied to equations (3.1), (3.2), and (3.3) due to assumptions (1.6), (1.7), and $g(t) \geq \sigma(t)$.

Remark 4.2. In this paper, we suggest some new oscillation results for a second-order nonlinear neutral dynamic equation (1.1). From Theorem 2.1, Theorem 2.2, Theorem 2.3, and Theorem 2.4, one can derive various classes of oscillation criteria in the case when $\eta(t) \leq t$, e.g., by letting

$$m(t) = R(t).$$

Based on Theorem 2.5, Theorem 2.6, Theorem 2.7, Theorem 2.8, Theorem 2.9, and Theorem 2.10, one can obtain various oscillation criteria in the case where $\eta(t) \geq t$, e.g., by letting

$$m(t) = \int_{t_1}^{t} \frac{\Delta s}{r^{1/\gamma}(s)} \quad \text{and} \quad h(t) = R(t).$$

Remark 4.3. Grace et al. [13, 14] and Hassan [18] studied equations

$$(rx^{\Delta})^{\Delta}(t) + q(t)x(t) = 0 \quad \text{and} \quad (rx^{\Delta})^{\Delta}(t) + q(t)x^{\sigma}(t) = 0$$

in the case

$$\int_{t_0}^{\infty} \frac{1}{r(t)} \int_{t_0}^{t} q(s) \int_{s}^{\infty} \frac{\Delta u}{r(u)} \Delta s \Delta t = \infty. \tag{4.1}$$

It is well known that second-order differential equation

$$(t^2 x'(t))' + q_0 x(t) = 0$$

is oscillatory if $q_0 > 1/4$. By Theorem 2.1, we also obtain this conclusion. But results of [13, 14, 18, 26, 35] cannot give this conclusion due to conditions (1.7), (4.1), and [35, Theorem 3.2, Theorem 3.3, and Theorem 3.4].

Remark 4.4. To the best of our knowledge, there are two classes of ideas in the study of oscillatory properties of (1.1) under the assumption (1.5) holds; see [26, 35]. Saker [26] obtained some criteria under conditions (1.6) and (1.7). Tripathy [35] established several related results in the case

$$0 < \gamma \leq 1, \quad 0 \leq p(t) \leq p_0 < \infty, \quad \eta(t) \leq t, \quad g(t) \leq t,$$

and

$$\eta^{\Delta} \geq \eta_0 > 0, \quad \eta([t_0, \infty)_{\mathbb{T}}) = [\eta(t_0), \infty)_{\mathbb{T}}, \quad \eta \circ g = g \circ \eta.$$

Saker et al. [27] showed that the latter assumptions may be restrictive in some applications. To achieve new results, we are forced to require that

$$1 - p(t)m(\eta(t))/m(t) > 0.$$

Combining the methods given in this paper and those reported in [21], one can easily derive some new oscillation theorems for (1.1) in the case where $p(t) > 1$. The details are left to the reader.

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