Existence of periodic solutions for a $2n$th-order difference equation involving p-Laplacian[*]

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Abstract. By using the critical point theory, the existence of periodic solutions for a $2n$th-order nonlinear difference equation containing both advance and retardation involving p-Laplacian is obtained. The main approaches used in our paper are variational techniques and the Saddle Point Theorem. The problem is to solve the existence of periodic solutions for a $2n$th-order p-Laplacian difference equation. The obtained results successfully generalize and complement the existing ones.

Keywords: Periodic solutions; $2n$th-order; Nonlinear difference equation; Discrete variational theory; p-Laplacian

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1. Introduction

Existence of periodic solutions of higher-order differential equations has been the subject of many investigations [9,14,15,26,30,31]. By using various methods and techniques, such as fixed point theory, the Kaplan-Yorke method, critical point theory, coincidence degree theory, bifurcation theory and dynamical system theory etc., a series of existence results for periodic solutions have been obtained in the literature. Difference equations, the discrete analogs of differential equations, occur widely in numerous settings and forms, both in mathematics itself and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology and other fields. For the general background of difference equations, one can refer to monographs [1,3,4,21]. Since the last decade, there has been much progress on the qualitative properties of difference equations, which included results on stability and attractivity [16,21,25,41] and results on oscillation and other topics [1-4,18-20,23,24,37-40]. Only a few papers discuss the periodic solutions of higher-order difference equations. Therefore, it is worthwhile to explore this topic.

Let $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of all natural numbers, integers and real numbers respectively. For a, $b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$, $\mathbf{Z}(a,b) = \{a, a+1, \cdots, b\}$ when $a \leq b$. $*$ denotes the transpose of a vector.

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In this paper, we consider the following $2n$th-order difference equation containing both advance and retardation with p-Laplacian

$$\Delta^n\left(r_{k-n}\varphi_p\left(\Delta^n u_{k-1}\right)\right)=(-1)^n f(k,u_{k+1},u_k,u_{k-1}),\ n\in\mathbf{Z}(1),\ k\in\mathbf{Z}, \tag{1.1}$$

where Δ is the forward difference operator $\Delta u_k=u_{k+1}-u_k$, $\Delta^2 u_k=\Delta(\Delta u_k)$, $r_k>0$ is real valued for each $k\in\mathbf{Z}$, $\varphi_p(s)$ is the p-Laplacian operator $\varphi_p(s)=|s|^{p-2}s(1<p<\infty)$, $f\in C(\mathbf{Z}\times\mathbf{R}^3,\mathbf{R})$, r_k and $f(k,v_1,v_2,v_3)$ are T-periodic in k for a given positive integer T.

We may think of (1.1) as a discrete analogue of the following $2n$th-order functional differential equation

$$d^n/dt^n\left[r(t)\varphi_p\left(d^n u(t)/dt^n\right)\right]=(-1)^n f(t,u(t+1),u(t),u(t-1)),\ t\in\mathbf{R}. \tag{1.2}$$

Equations similar in structure to (1.2) arise in the study of the existence of solitary waves of lattice differential equations, see Smets and Willem [34].

The widely used tools for the existence of periodic solutions of difference equations are the various fixed point theorems in cones [1,3,4,21]. It is well known that critical point theory is an effective approach that deals with the problems of differential equations [9,11,26,32,36]. Only since 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions of difference equations. By using the critical point theory, Guo and Yu [18-20] and Shi *et al.* [33] established sufficient conditions on the existence of periodic solutions of second-order nonlinear difference equations. Compared to one-order or second-order difference equations, the study of higher-order equations has received considerably less attention(see, for example, [1,5,6,12,13,17,21,27,29] and the references contained therein). Ahlbrandt and Peterson [5] in 1994 studied the $2n$th-order difference equation of the form,

$$\sum_{i=0}^{n}\Delta^i\left(r_i(k-i)\Delta^i u(k-i)\right)=0 \tag{1.3}$$

in the context of the discrete calculus of variations, and Peil and Peterson [29] studied the asymptotic behavior of solutions of (1.3) with $r_i(k)\equiv 0$ for $1\le i\le n-1$. In 1998, Anderson [6] considered (1.3) for $k\in\mathbf{Z}(a)$, and obtained a formulation of generalized zeros and (n,n)-disconjugacy for (1.3). Migda [27] in 2004 studied an mth-order linear difference equation.

In 2007, Cai and Yu [10] have obtained some criteria for the existence of periodic solutions of a $2n$th-order difference equation

$$\Delta^n\left(r_{k-n}\Delta^n u_{k-n}\right)+f(k,u_k)=0,\ n\in\mathbf{Z}(3),\ k\in\mathbf{Z}, \tag{1.4}$$

for the case where f grows superlinearly at both 0 and ∞.

If $n=1$ and $r_k\equiv 1$, (1.1) reduces to the following second order p-Laplacian difference equation

$$\Delta\left(\varphi_p\left(\Delta u_{k-1}\right)\right)+f(k,u_{k+1},u_k,u_{k-1})=0,\ k\in\mathbf{Z}. \tag{1.5}$$

Chen and Fang [12] in 2007 have obtained a sufficient condition for the existence of periodic and subharmonic solutions of (1.5).

A great deal of work has also been done in the study of the existence of solutions to discrete boundary value problems with the p-Laplacian operator. Because of their applications in many fields, we refer the reader to the monograph by Agarwal *et al.* and some recent contributions as

[1,2,7,8,22-24,35,38]. However, to the best of our knowledge, the results on periodic solutions of higher-order nonlinear difference equations involving p-Laplacian are very scarce in the literature. Furthermore, since (1.1) contains both advance and retardation, there are very few manuscripts dealing with this subject. Some difficulties lie that the traditional methods [18-20] for difference equations are not applicable to our case. The intention of this paper is to give some sufficient conditions for the existence and multiplicity of periodic and subharmonic solutions for a $2n$th-order nonlinear difference equation containing both advance and retardation with p-Laplacian. The proof is based on the Saddle Point Theorem in combination with variational technique. In particular, our results generalize and complement the results in the literature [10] and [12]. In fact, one can see the following Remark 1.4 for details.

Let

$$\underline{r} = \min_{k\in\mathbf{Z}(1,T)}\{r_k\},\ \bar{r} = \max_{k\in\mathbf{Z}(1,T)}\{r_k\}.$$

Now we state the main results of this paper.

Theorem 1.1. *Assume that the following hypotheses are satisfied:*
(F_1) *there exists a functional* $F(k, v_1, v_2) \in C^1(\boldsymbol{Z}\times\boldsymbol{R}^2, \boldsymbol{R})$ *such that*

$$F(k+T, v_1, v_2) = F(k, v_1, v_2),$$

$$\partial F(k-1, v_2, v_3)/\partial v_2 + \partial F(k, v_1, v_2)/\partial v_2 = f(k, v_1, v_2, v_3);$$

(F_2) *there exists a constant* $M_0 > 0$ *for all* $(k, v_1, v_2) \in \boldsymbol{Z}\times\boldsymbol{R}^2$ *such that*

$$|\partial F(k, v_1, v_2)/\partial v_1| \le M_0,\ |\partial F(k, v_1, v_2)/\partial v_2| \le M_0;$$

(F_3) $F(k, v_1, v_2) \to +\infty$ *uniformly for* $k \in \boldsymbol{Z}$ *as* $\sqrt{v_1^2+v_2^2} \to +\infty$.
Then for any given positive integer $m > 0$, *(1.1) has at least one* mT*-periodic solution.*

Remark 1.1. Assumption (F_2) implies that there exists a constant $M_1 > 0$ such that
(F_2') $|F(k, v_1, v_2)| \le M_1 + M_0(|v_1| + |v_2|),\ \forall (k, v_1, v_2) \in \mathbf{Z}\times\mathbf{R}^2$.

Theorem 1.2. *Assume that* (F_1) *holds; further*
(F_4) *there exist constants* $R_1 > 0$ *and* α, $1 < \alpha < 2$ *such that for* $k \in \boldsymbol{Z}$ *and* $\sqrt{v_1^2+v_2^2} \ge R_1$,

$$0 < \partial F(k, v_1, v_2)/\partial v_1 \cdot v_1 + \partial F(k, v_1, v_2)/\partial v_2 \cdot v_2 \le \alpha/2\cdot pF(k, v_1, v_2);$$

(F_5) *there exist constants* $a_1 > 0$, $a_2 > 0$ *and* γ, $1 < \gamma \le \alpha$ *such that*

$$F(k, v_1, v_2) \ge a_1\left(\sqrt{v_1^2+v_2^2}\right)^{\gamma/2\cdot p} - a_2,\ \forall (k, v_1, v_2) \in \boldsymbol{Z}\times\boldsymbol{R}^2.$$

Then for any given positive integer $m > 0$, *(1.1) has at least one* mT*-periodic solution.*

Remark 1.2. Assumption (F_4) implies that for each $k \in \mathbf{Z}$ there exist constants $a_3 > 0$ and $a_4 > 0$ such that
(F_4') $F(k, v_1, v_2) \le a_3\left(\sqrt{v_1^2+v_2^2}\right)^{\alpha/2\cdot p} + a_4,\ \forall (k, v_1, v_2) \in \mathbf{Z}\times\mathbf{R}^2$.

Remark 1.3. The results of Theorem 1.1 and Theorem 1.2 ensure that (1.1) has at least one mT-periodic solution. However, in some cases, we are interested in the existence of nontrivial periodic solutions for (1.1).

In this case, we have

Theorem 1.3. *Assume that* (F_1) *holds; further*
(F_6) $F(k,0)=0,\ f(k,v_1,v_2,v_3)=0$ *if and only if* $v_2=0$*, for all* $k\in\boldsymbol{Z}$*;*
(F_7) *there exists a constant* α*,* $1<\alpha<2$ *such that for* $k\in\boldsymbol{Z}$*,*

$$0<\partial F(k,v_1,v_2)/\partial v_1\cdot v_1+\partial F(k,v_1,v_2)/\partial v_2\cdot v_2\leq \alpha/2\cdot pF(k,v_1,v_2),\ \forall(v_1,v_2)\neq 0;$$

(F_8) *there exist constants* $a_5>0$ *and* γ*,* $1<\gamma\leq\alpha$ *such that*

$$F(k,v_1,v_2)\geq a_5\left(\sqrt{v_1^2+v_2^2}\right)^{\gamma/2\cdot p},\ \ \forall(k,v_1,v_2)\in\boldsymbol{Z}\times\boldsymbol{R}^2.$$

Then for any given positive integer $m>0$*, (1.1) has at least one nontrivial* mT*-periodic solution.*

Theorem 1.4. *Assume that* $(F_1)-(F_3)$ *and* (F_6) *hold; further*
(F_9) *there exist constants* $a_6>0$ *and* θ*,* $0<\theta<2$ *such that*

$$F(k,v_1,v_2)\geq a_6\left(\sqrt{v_1^2+v_2^2}\right)^{\theta/2\cdot p},\ \ \forall(k,v_1,v_2)\in\boldsymbol{Z}\times\boldsymbol{R}^2.$$

Then for any given positive integer $m>0$*, (1.1) has at least one nontrivial* mT*-periodic solution.*

If $p=2,\ f(k,u_{k+1},u_k,u_{k-1})=(-1)^{n+1}f(k,u_k)$, (1.1) reduces to (1.4). Then, we have the following results.

Theorem 1.5. *Assume that the following hypotheses are satisfied:*
(F_{10}) *there exists a functional* $F(k,v)\in C^1(\boldsymbol{Z}\times\boldsymbol{R},\boldsymbol{R}),\ F(k+T,v)=F(k,v)$ *such that*

$$\partial F(k,v)/\partial v=f(k,v);$$

(F_{11}) $F(k,0)=0$*, for all* $k\in\boldsymbol{Z}$*;*
(F_{12}) *there exists a constant* α*,* $1<\alpha<2$ *such that for* $k\in\boldsymbol{Z}$*,*

$$\alpha F(k,v)\leq vf(k,v)<0,\ \forall|v|\neq 0;$$

(F_{13}) *there exist constants* $a_7>0$ *and* γ*,* $1<\gamma\leq\alpha$ *such that*

$$F(k,v)\leq -a_7|v|^\gamma,\ \forall(k,v)\in\boldsymbol{Z}\times\boldsymbol{R}.$$

Then for any given positive integer $m>0$*, (1.4) has at least one nontrivial* mT*-periodic solution.*

Theorem 1.6. *Assume that* (F_{10}) *holds; further*
(F_{14}) *there exists a constant* $M_0>0$ *for all* $(k,v)\in\boldsymbol{Z}\times\boldsymbol{R}$ *such that* $|f(k,v)|\leq M_0$*;*
(F_{15}) $F(k,v)\to-\infty$ *uniformly for* $k\in\boldsymbol{Z}$ *as* $v\to+\infty$*;*
(F_{16}) $F(k,0)=0,\ f(k,v)=0$ *if and only if* $v=0$*, for all* $k\in\boldsymbol{Z}$*;*
(F_{17}) *there exist constants* $a_8>0$ *and* θ*,* $0<\theta<2$ *such that*

$$F(k,v)\leq -a_8|v|^\theta,\ \forall(k,v)\in\boldsymbol{Z}\times\boldsymbol{R}.$$

Then for any given positive integer $m>0$*, (1.4) has at least one nontrivial* mT*-periodic solution.*

Remark 1.4. When $\beta > 2$, Cai and Yu [10] in Theorem 1.1 have obtained some criteria for the existence of periodic solutions of (1.4) and Chen and Fang [12] in Theorem 3.1 have obtained some criteria for the existence of periodic solutions of (1.5). When $\beta < 2$, we can still find the periodic solutions of (1.4) and (1.5). Hence, Theorems 1.3-1.6 generalize and complement the existing ones.

The rest of the paper is organized as follows. Firstly, in Section 2, we shall establish the variational framework associated with (1.1) and transfer the problem of the existence of periodic solutions of (1.1) into that of the existence of critical points of the corresponding functional. Some related fundamental results will also be recalled. Then, in Section 3, we shall complete the proof of the results by using the critical point method. Finally, in Section 4, we shall give two examples to illustrate the main results.

About the basic knowledge for variational methods, please refer the reader to [26,28,32].

2. Variational structure and some lemmas

In order to apply the critical point theory, we shall establish the corresponding variational framework for (1.1) and give some lemmas which will be of fundamental importance in proving our main results. We start by some basic notations.

Let S be the set of sequences $u = (\cdots, u_{-k}, \cdots, u_{-1}, u_0, u_1, \cdots, u_k, \cdots) = \{u_k\}_{k=-\infty}^{+\infty}$, that is

$$S = \{\{u_k\} | u_k \in \mathbf{R},\ k \in \mathbf{Z}\}.$$

For any $u, v \in S$, $a, b \in \mathbf{R}$, $au + bv$ is defined by

$$au + bv = \{au_k + bv_k\}_{k=-\infty}^{+\infty}.$$

Then S is a vector space.

For any given positive integers m and T, E_{mT} is defined as a subspace of S by

$$E_{mT} = \{u \in S | u_{k+mT} = u_k,\ \forall k \in \mathbf{Z}\}.$$

Clearly, E_{mT} is isomorphic to $\mathbf{R}^{mT}$. E_{mT} can be equipped with the inner product

$$\langle u, v \rangle = \sum_{j=1}^{mT} u_j v_j,\ \forall u, v \in E_{mT}, \tag{2.1}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \left(\sum_{j=1}^{mT} u_j^2 \right)^{1/2},\ \forall u \in E_{mT}. \tag{2.2}$$

It is obvious that E_{mT} with the inner product (2.1) is a finite dimensional Hilbert space and linearly homeomorphic to $\mathbf{R}^{mT}$.

On the other hand, we define the norm $\|\cdot\|_s$ on E_{mT} as follows:

$$\|u\|_s = \left(\sum_{j=1}^{mT} |u_j|^s \right)^{1/s}, \tag{2.3}$$

for all $u \in E_{mT}$ and $s > 1$.

Since $\|u\|_s$ and $\|u\|_2$ are equivalent, there exist constants $c_1,\ c_2$ such that $c_2 \geq c_1 > 0$, and

$$c_1\|u\|_2 \leq \|u\|_s \leq c_2\|u\|_2,\ \forall u \in E_{mT}. \tag{2.4}$$

Clearly, $\|u\| = \|u\|_2$. For all $u \in E_{mT}$, define the functional J on E_{mT} as follows:

$$\begin{aligned} J(u) &= -1/p\sum_{k=1}^{mT} r_{k-1}\left|\Delta^n u_{k-1}\right|^p + \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\ &:= -H(u) + \sum_{k=1}^{mT} F(k, u_{k+1}, u_k), \end{aligned} \tag{2.5}$$

where

$$H(u) = 1/p\sum_{k=1}^{mT} r_{k-1}\left|\Delta^n u_{k-1}\right|^p,\ \partial F(k-1, v_2, v_3)/\partial v_2 + \partial F(k, v_1, v_2)/\partial v_2 = f(k, v_1, v_2, v_3).$$

Clearly, $J \in C^1(E_{mT}, \mathbf{R})$ and for any $u = \{u_k\}_{k\in\mathbf{Z}} \in E_{mT}$, by using $u_0 = u_{mT},\ u_1 = u_{mT+1}$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_k} = -(-1)^n\Delta^n\left(r_{k-n}\varphi_p\left(\Delta^n u_{k-1}\right)\right) + f(k, u_{k+1}, u_k, u_{k-1}).$$

Thus, u is a critical point of J on E_{mT} if and only if

$$\Delta^n\left(r_{k-n}\varphi_p\left(\Delta^n u_{k-1}\right)\right) = f(k, u_{k+1}, u_k, u_{k-1}),\ \forall k \in \mathbf{Z}(1, mT).$$

Due to the periodicity of $u = \{u_k\}_{k\in\mathbf{Z}} \in E_{mT}$ and $f(k, v_1, v_2, v_3)$ in the first variable k, we reduce the existence of periodic solutions of (1.1) to the existence of critical points of J on E_{mT}. That is, the functional J is just the variational framework of (1.1).

Let

$$P = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

be a $mT \times mT$ matrix. By matrix theory, we see that the eigenvalues of P are

$$\lambda_j = 2\left(1 - \cos 2j/(mT)\pi\right), j = 0, 1, 2, \cdots, mT-1. \tag{2.6}$$

Thus, $\lambda_0 = 0, \lambda_1 > 0, \lambda_2 > 0, \cdots, \lambda_{mT-1} > 0$. Therefore,

$$\begin{cases} \lambda_{\min} = \min\{\lambda_1, \lambda_2, \cdots, \lambda_{mT-1}\} = 2\left(1 - \cos 2/(mT)\pi\right), \\ \lambda_{\max} = \max\{\lambda_1, \lambda_2, \cdots, \lambda_{mT-1}\} = \begin{cases} 4, & \text{when mT is even}, \\ 2\left(1 + \cos 1/(mT)\pi\right), & \text{when mT is odd}. \end{cases} \end{cases} \tag{2.7}$$

Let

$$W = \ker P = \{u \in E_{mT} | Pu = 0 \in \mathbf{R}^{mT}\}.$$

Then

$$W = \{u \in E_{mT} | u = \{c\},\ c \in \mathbf{R}\}.$$

Let V be the direct orthogonal complement of E_{mT} to W, i.e., $E_{mT} = V \oplus W$. For convenience, we identify $u \in E_{mT}$ with $u = (u_1, u_2, \cdots, u_{mT})^*$.

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\left\{u^{(i)}\right\} \subset E$ for which $\left\{J\left(u^{(i)}\right)\right\}$ is bounded and $J'\left(u^{(i)}\right) \to 0 (i \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1. *(Saddle Point Theorem [26,32]). Let E be a real Banach space, $E = E_1 \oplus E_2$, where $E_1 \neq \{0\}$ and is finite dimensional. Suppose that $J \in C^1(E, \boldsymbol{R})$ satisfies the P.S. condition and*
(J_1) there exist constants σ, $\rho > 0$ such that $J|_{\partial B_\rho \cap E_1} \leq \sigma$;
(J_2) there exists $e \in B_\rho \cap E_1$ and a constant $\omega \geq \sigma$ such that $J_{e+E_2} \geq \omega$.

Then J possesses a critical value $c \geq \omega$, where

$$c = \inf_{h \in \Gamma} \max_{u \in B_\rho \cap E_1} J(h(u)),\ \Gamma = \{h \in C(\bar{B}_\rho \cap E_1, E) \mid h|_{\partial B_\rho \cap E_1} = id\}$$

and id denotes the identity operator.

Lemma 2.2. *Assume that $(F_1) - (F_3)$ is satisfied. Then J satisfies the P.S. condition.*

Proof. Let $\left\{u^{(i)}\right\} \subset E_{mT}$ be such that $\left\{J\left(u^{(i)}\right)\right\}$ is bounded and $J'\left(u^{(i)}\right) \to 0$ as $i \to \infty$. Then there exists a positive constant M_2 such that $\left|J\left(u^{(i)}\right)\right| \leq M_2$.

Let $u^{(i)} = v^{(i)} + w^{(i)} \in V + W$. For i large enough, since

$$-\|u\|_2 \leq \left\langle J'\left(u^{(i)}\right), u\right\rangle = -\left\langle H'\left(u^{(i)}\right), u\right\rangle + \sum_{k=1}^{mT} f\left(k, u_{k+1}^{(i)}, u_k^{(i)}, u_{k-1}^{(i)}\right) u_k,$$

combining with (F_2) and (F_3), we have

$$\begin{aligned}\left\langle H'\left(u^{(i)}\right), v^{(i)}\right\rangle &\leq \sum_{k=1}^{mT} f\left(k, u_{k+1}^{(i)}, u_k^{(i)}, u_{k-1}^{(i)}\right) v_k^{(i)} + \left\|v^{(i)}\right\|_2 \\ &\leq 2M_0 \sum_{k=1}^{mT} \left|v_k^{(i)}\right| + \left\|v^{(i)}\right\|_2 \\ &\leq \left(2M_0\sqrt{mT} + 1\right)\left\|v^{(i)}\right\|_2.\end{aligned}$$

On the other hand, we know that

$$\left\langle H'\left(u^{(i)}\right), v^{(i)}\right\rangle = \sum_{k=1}^{mT} r_{k-1}\left(\Delta^n v_{k-1}^{(i)}, \Delta^n v_{k-1}^{(i)}\right)^{p/2} = \sum_{k=1}^{mT} r_k \left(\Delta^n v_k^{(i)}, \Delta^n v_k^{(i)}\right)^{p/2} = pH\left(v^{(i)}\right).$$

Since

$$\underline{r}/p \cdot c_1^p \lambda_{\min}^{p/2} \left\|x^{(i)}\right\|_2^p \leq \underline{r}/p \cdot c_1^p \left[\left(x^{(i)}\right)^* P \left(x^{(i)}\right)\right]^{p/2} \leq H\left(v^{(i)}\right),$$

$$H\left(v^{(i)}\right) \leq \bar{r}/p \cdot c_2^p \left[\left(x^{(i)}\right)^* P \left(x^{(i)}\right)\right]^{p/2} \leq \bar{r}/p \cdot c_2^p \lambda_{\max}^{p/2} \left\|x^{(i)}\right\|_2^p,$$

and

$$\lambda_{\min}^{(n-1)p/2} \left\|v^{(i)}\right\|_2^p \leq \left\|x^{(i)}\right\|_2^p = \sum_{k=1}^{mT}\left(\Delta^{n-2} v_{k+1}^{(i)} - \Delta^{n-2} v_k^{(i)}\right)^p \leq \lambda_{\max}^{p/2} \sum_{k=1}^{mT}\left(\Delta^{n-2} v_k^{(i)}\right)^p \leq \lambda_{\max}^{(n-1)p/2} \left\|v^{(i)}\right\|_2^p,$$

where $x^{(i)}=\left(\Delta^{n-1}v_1^{(i)},\Delta^{n-1}v_2^{(i)},\cdots,\Delta^{n-1}v_{mT}^{(i)}\right)^*$, we get

$$\underline{r}/p\cdot c_1^p\lambda_{\min}^{np/2}\left\|v^{(i)}\right\|_2^p\leq H\left(v^{(i)}\right)\leq\bar{r}/p\cdot c_2^p\lambda_{\max}^{np/2}\left\|v^{(i)}\right\|_2^p. \tag{2.3}$$

Thus, we have

$$\underline{r}c_1^p\lambda_{\min}^{np/2}\left\|v^{(i)}\right\|_2^p\leq\left(2M_0\sqrt{mT}+1\right)\left\|v^{(i)}\right\|_2.$$

The above inequality implies that $\left\{v^{(i)}\right\}$ is bounded.

Next, we shall prove that $\left\{w^{(i)}\right\}$ is bounded. Since

$$\begin{aligned}M_2\geq J\left(u^{(i)}\right)&=-H\left(u^{(i)}\right)+\sum_{k=1}^{mT}F\left(k,u_{k+1}^{(i)},u_k^{(i)}\right)\\&=-H\left(v^{(i)}\right)+\sum_{k=1}^{mT}\left[F\left(k,u_{k+1}^{(i)},u_k^{(i)}\right)-F\left(k,w_{k+1}^{(i)},w_k^{(i)}\right)\right]+\sum_{k=1}^{mT}F\left(k,w_{k+1}^{(i)},w_k^{(i)}\right),\end{aligned}$$

combining with (2.8), we get

$$\begin{aligned}&\sum_{k=1}^{mT}F\left(k,w_{k+1}^{(i)},w_k^{(i)}\right)\\&\leq M_2+H\left(v^{(i)}\right)+\sum_{k=1}^{mT}\left|F\left(k,u_{k+1}^{(i)},u_k^{(i)}\right)-F\left(k,w_{k+1}^{(i)},w_k^{(i)}\right)\right|\\&\leq M_2+\bar{r}/p\cdot c_2^p\lambda_{\max}^{np/2}\left\|v^{(i)}\right\|_2^p\\&\quad+\sum_{k=1}^{mT}\left|\partial F\left(k,w_{k+1}^{(i)}+\theta v_{k+1}^{(i)},w_k^{(i)}+\theta v_k^{(i)}\right)/\partial v_1\cdot v_{k+1}^{(i)}+\partial F\left(k,w_{k+1}^{(i)}+\theta v_{k+1}^{(i)},w_k^{(i)}+\theta v_k^{(i)}\right)/\partial v_2\cdot v_k^{(i)}\right|\\&\leq M_2+\bar{r}/pc_2^p\lambda_{\max}^{np/2}\left\|v^{(i)}\right\|_2^p+2M_0\sqrt{mT}\left\|v^{(i)}\right\|_2.\end{aligned}$$

where $\theta\in(0,1)$. It is not difficult to see that $\left\{\sum\limits_{k=1}^{mT}F\left(k,w_{k+1}^{(i)},w_k^{(i)}\right)\right\}$ is bounded.

By (F_3), $\left\{w^{(i)}\right\}$ is bounded. Otherwise, assume that $\left\|w^{(i)}\right\|_2\to+\infty$ as $i\to\infty$. Since there exist $z^{(i)}\in\mathbf{R}$, $i\in\mathbf{N}$, such that $w^{(i)}=\left(z^{(i)},z^{(i)},\cdots,z^{(i)}\right)^*\in E_{mT}$, then

$$\left\|w^{(i)}\right\|_2=\left(\sum_{k=1}^{mT}\left|w_k^{(i)}\right|^2\right)^{1/2}=\left(\sum_{k=1}^{mT}\left|z^{(i)}\right|^2\right)^{1/2}=\sqrt{mT}\left|z^{(i)}\right|\to+\infty$$

as $i\to\infty$. Since $F\left(k,w_{k+1}^{(i)},w_k^{(i)}\right)=F\left(k,z^{(i)},z^{(i)}\right)$, then $F\left(k,w_{k+1}^{(i)},w_k^{(i)}\right)\to+\infty$ as $i\to\infty$. This contradicts the fact that $\left\{\sum\limits_{k=1}^{mT}F\left(k,w_{k+1}^{(i)},w_k^{(i)}\right)\right\}$ is bounded. Thus the P.S. condition is verified. $\square$

Lemma 2.3. *Assume that* (F_1), (F_4) *and* (F_5) *are satisfied. Then* J *satisfies the P.S. condition.*

Proof. Let $\left\{u^{(i)}\right\}\subset E_{mT}$ be such that $\left\{J\left(u^{(i)}\right)\right\}$ is bounded and $J'\left(u^{(i)}\right)\to 0$ as $i\to\infty$. Then there exists a positive constant M_3 such that $\left|J\left(u^{(i)}\right)\right|\leq M_3$.

For i large enough, we have

$$\left|\left\langle J'\left(u^{(i)}\right),u^{(i)}\right\rangle\right|\leq\left\|u^{(i)}\right\|_2.$$

So

$$M_3+1/p\left\|u^{(i)}\right\|_2$$

$$
\begin{aligned}
&\geq J\left(u^{(i)}\right) - 1/p\left\langle J'\left(u^{(i)}\right), u^{(i)}\right\rangle \\
&= \sum_{k=1}^{mT}\left[F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right) - 1/p\left(\partial F\left(k-1, u_k^{(i)}, u_{k-1}^{(i)}\right)/\partial v_2 \cdot u_k^{(i)} + \partial F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right)/\partial v_2 \cdot u_k^{(i)}\right)\right] \\
&= \sum_{k=1}^{mT}\left[F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right) - 1/p\left(\partial F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right)/\partial v_1 \cdot u_{k+1}^{(i)} + \partial F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right)/\partial v_2 \cdot u_k^{(i)}\right)\right].
\end{aligned}
$$

Take

$$
I_1 = \left\{k \in \mathbf{Z}(1, mT) \middle| \sqrt{\left(u_{k+1}^{(i)}\right)^2 + \left(u_k^{(i)}\right)^2} \geq R_1\right\},\ I_2 = \left\{k \in \mathbf{Z}(1, mT) \middle| \sqrt{\left(u_{k+1}^{(i)}\right)^2 + \left(u_k^{(i)}\right)^2} < R_1\right\}.
$$

By (F_4), we have

$$
\begin{aligned}
& M_3 + 1/p\left\|u^{(i)}\right\|_2 \\
\geq & \sum_{k=1}^{mT} F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right) - 1/p\sum_{k\in I_1}\left[\partial F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right)/\partial v_1 \cdot u_{k+1}^{(i)} + \partial F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right)/\partial v_2 \cdot u_k^{(i)}\right] \\
& -1/p\sum_{k\in I_2}\left[\partial F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right)/\partial v_1 \cdot u_{k+1}^{(i)} + \partial F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right)/\partial v_2 \cdot u_k^{(i)}\right] \\
\geq & \sum_{k=1}^{mT} F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right) - \alpha/2\sum_{k\in I_1} F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right) \\
& -1/p\sum_{k\in I_2}\left[\partial F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right)/\partial v_1 \cdot u_{k+1}^{(i)} + \partial F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right)/\partial v_2 \cdot u_k^{(i)}\right] \\
= & (1-\alpha/2)\sum_{k=1}^{mT} F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right) \\
& +1/p\sum_{k\in I_2}\left[\alpha/2 \cdot pF\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right) - \partial F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right)/\partial v_1 \cdot u_{k+1}^{(i)} - \partial F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right)/\partial v_2 \cdot u_k^{(i)}\right].
\end{aligned}
$$

The continuity of $\alpha/2 \cdot pF(k, v_1, v_2) - \partial F(k, v_1, v_2)/\partial v_1 \cdot v_1 - \partial F(k, v_1, v_2)/\partial v_2 \cdot v_2$ with respect to the second and third variables implies that there exists a constant $M_4 > 0$ such that

$$
\alpha/2 \cdot pF(k, v_1, v_2) - \partial F(k, v_1, v_2)/\partial v_1 \cdot v_1 - \partial F(k, v_1, v_2)/\partial v_2 \cdot v_2 \geq -M_6,
$$

for $k \in \mathbf{Z}(1, mT)$ and $\sqrt{v_1^2 + v_2^2} \leq R_1$. Therefore,

$$
M_3 + 1/p\left\|u^{(i)}\right\|_2 \geq (1-\alpha/2)\sum_{k=1}^{mT} F\left(k, u_{k+1}^{(i)}, u_k^{(i)}\right) - 1/p \cdot mTM_4.
$$

By (F_5), we get

$$
\begin{aligned}
M_3 + 1/p\left\|u^{(i)}\right\|_2 &\geq (1-\alpha/2)\, a_1\sum_{k=1}^{mT}\left[\sqrt{\left(u_{k+1}^{(i)}\right)^2 + \left(u_k^{(i)}\right)^2}\right]^{\gamma/2\cdot p} - (1-\alpha/2)\, a_2 mT - 1/p \cdot mTM_4 \\
&\geq (1-\alpha/2)\, a_1\sum_{k=1}^{mT}\left|u_k^{(i)}\right|^{\gamma/2\cdot p} - M_5,
\end{aligned}
$$

where $M_5 = (1-\alpha/2)\, a_2 mT + 1/p \cdot mTM_4$.

Combining with (2.4), we have

$$M_3 + 1/p \left\| u^{(i)} \right\|_2 \geq (1-\alpha/2)\, a_1 c_1^{\gamma/2\cdot p} \left\| u^{(i)} \right\|_2^{\gamma/2\cdot p} - M_5.$$

Thus,

$$(1-\alpha/2)\, a_1 c_1^{\gamma/2\cdot p} \left\| u^{(i)} \right\|_2^{\gamma/2\cdot p} - 1/p \left\| u^{(i)} \right\|_2 \leq M_3 + M_5.$$

This implies that $\left\{ \left\| u^{(i)} \right\|_2 \right\}$ is bounded on the finite dimensional space E_{mT}. As a consequence, it has a convergent subsequence. □

3. Proof of the main results

In this Section, we shall prove our main results by using the critical point method.

Proof of Theorem 1.1. By Lemma 2.2, we know that J satisfies the P.S. condition. In order to prove Theorem 1.1 by using the Saddle Theorem, we shall prove the conditions (J_1) and (J_2).

From (2.8) and (F_2'), for any $v \in V$,

$$\begin{aligned} J(v) &= -H(v) + \sum_{k=1}^{mT} F(k, v_{k+1}, v_k) \\ &\leq -\underline{r}/p \cdot c_1^p \lambda_{\min}^{np/2} \|v\|_2^p + mTM_1 + M_0 \sum_{k=1}^{mT} \left(|v_{k+1}| + |v_k|\right) \\ &\leq -\underline{r}/p \cdot c_1^p \lambda_{\min}^{np/2} \|v\|_2^p + mTM_1 + 2M_0\sqrt{mT}\|v\|_2 \to -\infty \text{ as } \|v\|_2 \to +\infty. \end{aligned}$$

Therefore, it is easy to see that the condition (J_1) is satisfied.

In the following, we shall verify the condition (J_2). For any $w \in W$, $w = (w_1, w_2, \cdots, w_{mT})^*$, there exists $z \in \mathbf{R}$ such that $w_k = z$, for all $k \in \mathbf{Z}(1, mT)$. By (F_3), we know that there exists a constant $R_0 > 0$ such that $F(k, z, z) > 0$ for $k \in \mathbf{Z}$ and $|z| > R_0/\sqrt{2}$. Let $M_6 = \min\limits_{k\in\mathbf{Z}, |z| \leq R_0/\sqrt{2}} F(k, z, z)$, $M_7 = \min\{0, M_6\}$. Then

$$F(k, z, z) \geq M_7, \ \forall (k, z, z) \in \mathbf{Z} \times \mathbf{R}^2.$$

So we have

$$J(w) = \sum_{k=1}^{mT} F(k, w_{k+1}, w_k) = \sum_{k=1}^{mT} F(k, z, z) \geq mTM_7, \ \forall w \in W.$$

The conditions of (J_1) and (J_2) are satisfied. □

Proof of Theorem 1.2. By Lemma 2.3, J satisfies the P.S. condition. To apply the Saddle Point Theorem, it suffices to prove that J satisfies the conditions (J_1) and (J_2).

For any $w \in W$, since $H(w) = 0$, we have

$$J(w) = \sum_{k=1}^{mT} F(k, w_{k+1}, w_k).$$

By (F_5),

$$J(w) \geq a_1 \sum_{k=1}^{mT} \left(\sqrt{w_{k+1}^2 + w_k^2} \right)^{\gamma/2\cdot p} - a_2 mT \geq -a_2 mT.$$

Combining with (F_4'), (2,4) and (2.8), for any $v \in V$, we get, like before,

$$\begin{aligned} J(v) &\leq -\underline{r}/p \cdot c_1^p \lambda_{\min}^{np/2} \|v\|_2^p + a_3 \sum_{k=1}^{mT} \left(\sqrt{v_{k+1}^2 + v_k^2} \right)^{\alpha/2 \cdot p} + a_4 mT \\ &\leq -\underline{r}/p \cdot c_1^p \lambda_{\min}^{np/2} \|v\|_2^p + a_3 c_2^{\alpha/2 \cdot p} \left[\sum_{k=1}^{mT} \left(v_{k+1}^2 + v_k^2 \right) \right]^{\alpha/4 \cdot p} + a_4 mT \\ &\leq -\underline{r}/p \cdot c_1^p \lambda_{\min}^{np/2} \|v\|_2^p + 2^{\alpha/4 \cdot p} a_3 c_2^{\alpha/2 \cdot p} \|v\|_2^{\alpha/2 \cdot p} + a_4 mT. \end{aligned}$$

Let $\mu = -a_2 mT$, since $1 < \alpha < 2$, there exists a constant $\rho > 0$ large enough such that

$$J(v) \leq \mu - 1 < \mu, \ \forall v \in V, \ \|v\|_2 = \rho.$$

Thus, by Lemma 2.1, (1.1) has at least one mT-periodic solution. □

Proof of Theorem 1.3. Similarly to the proof of Lemma 2.3, we can prove that J satisfies the P.S. condition. We shall prove this theorem by the Saddle Point Theorem. Firstly, we verify the condition (J_1).

In fact, (F_4) clearly implies (F_4'). For any $v \in V$, by (F_4') and (2.4), we have again $J(v) \to -\infty$ as $\|v\|_2 \to +\infty$.

Next, we show that J satisfies the condition (J_2). For any given $v_0 \in V$ and $w \in W$. Let $u = v_0 + w$. So

$$\begin{aligned} J(u) &= -H(u) + \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\ &= -H(v_0) + \sum_{k=1}^{mT} F(k, (v_0)_{k+1} + w_{k+1}, (v_0)_k + w_k) \\ &\geq -\bar{r}/p \cdot c_2^p \lambda_{\max}^{np/2} \|v_0\|_2^p + a_5 \sum_{k=1}^{mT} \left[\sqrt{((v_0)_{k+1} + w_{k+1})^2 + ((v_0)_k + w_k)^2} \right]^{\gamma/2 \cdot p} \\ &\geq -\bar{r}/p \cdot c_2^p \lambda_{\max}^{np/2} \|v_0\|_2^p + a_5 \sum_{k=1}^{mT} |(v_0)_k + w_k|^{\gamma/2 \cdot p} \\ &\geq -\bar{r}/p \cdot c_2^p \lambda_{\max}^{np/2} \|v_0\|_2^p + a_5 c_1^{\gamma/2 \cdot p} \left[\sum_{k=1}^{mT} |(v_0)_k + w_k|^2 \right]^{\gamma/4 \cdot p} \\ &= -\bar{r}/p \cdot c_2^p \lambda_{\max}^{np/2} \|v_0\|_2^p + a_5 c_1^{\gamma/2 \cdot p} \left[\|v_0\|_2^2 + \|w\|_2^2 \right]^{\gamma/4 \cdot p} \\ &\geq -\bar{r}/p \cdot c_2^p \lambda_{\max}^{np/2} \|v_0\|_2^p + a_5 c_1^{\gamma/2 \cdot p} \|v_0\|_2^{\gamma/2 \cdot p} + a_5 c_1^{\gamma/2 \cdot p} \|w\|_2^{\gamma/2 \cdot p}. \end{aligned}$$

Since $1 < \gamma < 2$, there exists a constant $\delta > 0$ small enough such that

$$J(v_0 + w) \geq \delta^{\gamma/2 \cdot p} \left(a_5 c_1^{\gamma/2 \cdot p} - \bar{r}/p \cdot c_2^p \lambda_{\max}^{np/2} \delta^{p - \gamma/2 \cdot p} \right) > 0,$$

for $v_0 \in V$, $\|v_0\|_2 = \delta$ and for any $w \in W$.

Take $\nu = \delta^{\gamma/2 \cdot p} \left(a_5 c_1^{\gamma/2 \cdot p} - \bar{r}/p \cdot c_2^p \lambda_{\max}^{np/2} \delta^{p - \gamma/2 \cdot p} \right)$. Then for $v_0 \in V$ and for any $w \in W$, we get $\|v_0\|_2 = \delta$ and $J(v_0 + w) \geq \nu > 0$.

By the Saddle Point Theorem, there exists a critical point $\bar{u} \in E_{mT}$, which corresponds to a mT-periodic solution of (1.1).

In the following, we shall prove that $\bar{u}$ is nontrivial, i.e., $\bar{u} \notin W$. Otherwise, $\bar{u} \in W$. Since $J'(\bar{u}) = 0$, then

$$\Delta^n \left(r_{k-n}\varphi_p\left(\Delta^n \bar{u}_{k-1}\right)\right) = (-1)^n f(k, \bar{u}_{k+1}, \bar{u}_k, \bar{u}_{k-1}).$$

On the other hand, $\bar{u} \in W$ implies that there is a point $z \in \mathbf{R}$ such that $\bar{u}_k = z$, for all $k \in \mathbf{Z}(1, mT)$. That is, $\bar{u}_1 = \bar{u}_2 = \cdots = \bar{u}_k = \cdots = z$. Thus, $f(k, \bar{u}_{k+1}, \bar{u}_k, \bar{u}_{k-1}) = f(k, z, z, z) = 0$, for all $k \in \mathbf{Z}(1, mT)$. From (F_6), we know that $z = 0$. Therefore, by (F_6), we have

$$J(\bar{u}) = \sum_{k=1}^{mT} F(k, \bar{u}_{k+1}, \bar{u}_k) = \sum_{k=1}^{mT} F(k, 0) = 0.$$

This contradicts $J(\bar{u}) \geq \nu > 0$. The proof of Theorem 1.3 is finished. □

Remark 3.1. The techniques of the proof of the Theorem 1.4 are just the same as those carried out in the proof of Theorem 1.3. We do not repeat them here.

Remark 3.2. Due to Theorems 1.3 and 1.4, the conclusion of Theorems 1.5 and 1.6 is obviously true.

4. Examples

As an application of the main theorems, finally, we give two examples to illustrate our results.

Example 4.1. For all $n \in \mathbf{Z}(1)$, $k \in \mathbf{Z}$, assume that

$$\Delta^n \left(r_{k-n}\varphi_p\left(\Delta^n u_{k-1}\right)\right) = (-1)^n \alpha p u_k \left[\psi(k)\left(u_{k+1}^2 + u_k^2\right)^{\alpha/2\cdot p-1} + \psi(k-1)\left(u_k^2 + u_{k-1}^2\right)^{\alpha/2\cdot p-1}\right], \tag{4.2}$$

where r_k is real valued for each $k \in \mathbf{Z}$, ψ is continuously differentiable and $\psi(k) > 0$, T is a given positive integer, $r_{k+T} = r_k > 0$, $\psi(k+T) = \psi(k)$, $1 < p < \infty$, $1 < \alpha < 2$. We have

$$f(k, v_1, v_2, v_3) = 2\alpha v_2 \left[\psi(k)\left(v_1^2 + v_2^2\right)^{\alpha/2\cdot p-1} + \psi(k-1)\left(v_2^2 + v_3^2\right)^{\alpha/2\cdot p-1}\right]$$

and

$$F(k, v_1, v_2) = \psi(k)\left(v_1^2 + v_2^2\right)^{\alpha/2\cdot p}.$$

Then

$$\partial F(k-1, v_2, v_3)/\partial v_2 + \partial F(k, v_1, v_2)/\partial v_2 = \alpha p v_2 \left[\psi(k)\left(v_1^2 + v_2^2\right)^{\alpha/2\cdot p-1} + \psi(k-1)\left(v_2^2 + v_3^2\right)^{\alpha/2\cdot p-1}\right].$$

It is easy to verify all the assumptions of Theorem 1.3 are satisfied. Consequently, for any given positive integer $m > 0$, (4.1) has at least one nontrivial mT-periodic solution.

Example 4.2. For all $n \in \mathbf{Z}(1)$, $k \in \mathbf{Z}$, assume that

$$\begin{aligned} &\Delta^n \left(r_{k-n}\varphi_p\left(\Delta^n u_{k-1}\right)\right) \\ &= (-1)^n \theta p u_k \left[\left(6 + \sin^2(k\pi/T)\right)\left(u_{k+1}^2 + u_k^2\right)^{\theta/2\cdot p-1} + \left(6 + \sin^2((k-1)\pi/T)\right)\left(u_k^2 + u_{k-1}^2\right)^{\theta/2\cdot p-1}\right], \end{aligned} \tag{4.3}$$

where r_k is real valued for each $k \in \mathbf{Z}$, T is a given positive integer, $r_{k+T} = r_k > 0$, $1 < p < \infty$, $0 < \theta < 2$. We have

$$f(k, v_1, v_2, v_3) = \theta p v_2 \left[\left(6 + \sin^2(k\pi/T)\right)\left(v_1^2 + v_2^2\right)^{\theta/2\cdot p-1} + \left(6 + \sin^2((k-1)\pi/T)\right)\left(v_2^2 + v_3^2\right)^{\theta/2\cdot p-1}\right]$$

and

$$F(k, v_1, v_2) = \left(6 + \sin^2(k\pi/T)\right)\left(v_1^2 + v_2^2\right)^{\theta/2\cdot p}.$$

Then

$$\begin{aligned}&\partial F(k-1, v_2, v_3)/\partial v_2 + \partial F(k, v_1, v_2)/\partial v_2\\ &= \theta p v_2\left[\left(6 + \sin^2(k\pi/T)\right)\left(v_1^2 + v_2^2\right)^{\theta/2\cdot p-1} + \left(6 + \sin^2((k-1)\pi/T)\right)\left(v_2^2 + v_3^2\right)^{\theta/2\cdot p-1}\right].\end{aligned}$$

It is easy to verify all the assumptions of Theorem 1.4 are satisfied. Consequently, for any given positive integer $m > 0$, (4.2) has at least one nontrivial mT-periodic solution.

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