The KKM Theorem in Modular Spaces and Applications to Minimax Inequalities

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Abstract

In this paper, we present a modular version of KKM and generalized KKM mappings and then we establish a characterization of generalized KKM mappings in modular spaces. Also we prove an analogue to KKM principle in modular spaces. Moreover, as an application, we give some sufficient conditions which guarantee existence of solutions of minimax problems in which we get Fan's minimax inequality in modular spaces.
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1 Introduction

In 1929, Knaster, Kuratowski, and Mazurkiewicz established the well known KKM theorem on the closed cover of a simplex [11] which has a fundamental importance in modern nonlinear analysis. Then it is generalized to a subset of any topological vector space in 1996 by Fan [3]. It is proved in H-spaces by Horvath in 1983 [4, 5]. Next the hyperconvex verion of this theorem is established in 1996 [7]. Recently, Khamsi *et al.* proved the KKM principle in modular function spaces from which he obtained an analogue to Ky Fan's fixed point theorem in theorem in modular function spaces [9]. Based on the idea of the work of Khamsi [9], we introduce KKM mappings and generalized KKM mappings and establish some minimax inequalities on modular spaces.

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The theory of modular spaces was initiated by Nakano [14] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [13] in 1959. Besides the idea of defining a norm and considering particular Banach spaces of functions, another direction is based on considering an abstractly given funtional defined on a linear space of functions which controls the growth of members of the space. Even though a metric is not defined, many problems in metric theory can be reformulated in modular spaces (see, for instance [6] and references therein).

In this work, we first define the generalized KKM mapping on a modular space, and then we apply the property of the modular space to get a characterization of the generalized KKM mapping and the KKM theorem. Next, by using our results, we get some minimax inequality theorems.

2 Preliminaries

Definition 2.1. A functional ρ on a real linear space X is said to be a modular on X if it satisfies the following conditions:

1. $\rho(x) = 0$ iff $x = 0$,
2. $\rho(x) = \rho(-x)$,
3. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, for all $x, y \in X$ and $\alpha, \beta \geq 0$, $\alpha + \beta = 1$.

If we replace (3) by

4. $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$,

for any $\alpha, \beta \in \mathbb{R}_0^+$, $\alpha + \beta = 1$, then ρ will be called a convex modular.

The modular ρ on X defines the corresponding modular space X_ρ , which is given by

$$X_\rho = \{x \in X : \rho(\alpha x) \to 0 \text{ as } \alpha \to 0\}.$$

Example 2.2. (a) Let $X = \mathbb{R}$. Then

$$\rho(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is a modular on X.

(b) It is clear that every norm is a modular. However, Example 3 in [15] shows the converse is not true.

(c) Let B be a set. Define

$$\ell^1(B) = \{f : B \to \mathbb{C} : \sum_{s \in B} |f(s)| < \infty\}.$$

It is not difficult to check $\rho : \ell^1(B) \to [0, \infty)$ defined by

$$\rho(f) = \begin{cases} \sum_{s\in B} |f(s)| + 1 & f \neq 0 \\ 0 & f = 0, \end{cases}$$

is a modular.

(d) As a classical example, we mention the Musielak-Orlicz space denoted by L^{ϕ}, for more details see [12].

Let X and Y be nonempty sets, $A \subseteq Y$, and $F : X \to 2^Y$ be a multivalued map with nonempty values where 2^Y denotes the set of all subsets of Y. Then we define

$$F^-(A) = \{x \in X : \ F(x) \cap A \neq \emptyset\},$$

and the convex hull of A is denoted by $\mathrm{co}(A)$.

Definition 2.3. Let X be a nonempty set, ρ be a modular on Y, and $C \subseteq Y_\rho$.

1. A multivalued mapping $G : C \to 2^{Y_\rho}$ is said KKM if

$$\mathrm{co}(\{x_1, \dots, x_n\}) \subseteq \bigcup_{i=1,\dots,n} G(x_i),$$

   for every finite subset $\{x_1, \dots, x_n\}$ of X.

2. A multivalued mapping $G : X \to 2^{Y_\rho}$ is said generalized KKM if for each nonempty finite subset $A = \{x_1, \dots, x_n\}$ of X there exists a nonempty subset $\{y_1, \dots, y_n\}$ (y_i's can be equal here) of Y_ρ such that for each subset $\{y_{i_1}, \dots, y_{i_j}\}$ of $\{y_1, \dots, y_n\}$ we have

$$\mathrm{co}(\{y_{i_1}, \dots, y_{i_j}\}) \subseteq \bigcup_{k=1}^{j} G(x_{i_k}).$$

The concept of generalized KKM maps is defined by Chang and Zhang in topological vector spaces [2] motivated by the works of Knaster, Kuratowski and Mazurkiewicz [11]. These notions also have been studied by Khanh *et al.* in GFC-spaces [10], and more recently by Khamsi *et al.* in metric type spaces [8], and Park in generalized convex spaces [16]. We considered generalized KKM mappings in modular spaces.

In this work, we will need the following definition.

Definition 2.4. Let ρ be a modular on X.

1. The sequence $\{x_n\} \subset X_\rho$ is said to be convergent to a point $x \in X_\rho$ and denoted by $x_n \to x$ whenever $\rho(x_n - x) \to 0$.

2. The sequence $\{x_n\} \subset X_\rho$ is called Cauchy if $\lim_{k,l\to\infty} \rho(x_k - x_l) = 0$.

3. X_ρ is said to be complete if each Cauchy sequence in X_ρ is convergent to a point of X_ρ.

4. The closure of a subset E of X_ρ is denoted by $\overline{E}$ and defined by the set of all $y \in X_\rho$ such that there is a sequence $\{y_k\}$ of E which is convergent to y. We say that E is closed if $E = \overline{E}$.

5. A subset B of X_ρ is said to be compact if every family $\{F_\alpha : \alpha \in G; F_\alpha \subseteq X_\rho\}$ of closed sets satisfying $B \subseteq \cup_{\alpha\in\Gamma} F_\alpha^c$ has a finite subfamily $F_{\alpha_1}, F_{\alpha_2}, \ldots, F_{\alpha_n}$ such that $B \subseteq \cup_{j=1}^{n} F_{\alpha_j}^c$.

3 The KKM Theory in Modular Spaces

Let us recall that for $n \geq 0$, $\triangle_n$ denotes the standard n-simplex of $\mathbb{R}^{n+1}$ with vertices $e_0, \ldots, e_n$, where e_i is the ith unit vector in $\mathbb{R}^{n+1}$, that is

$$\triangle_n = \Big\{(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} \alpha_i = 1, \forall i \ \alpha_i \geq 0\Big\}.$$

Let ρ be a modular on X. The mapping $f : \triangle_n \to X_\rho$ is said to be continuous if $a_m \to a$ in $\triangle_n$, then $\rho(f(a_m) - f(a)) \to 0$.

Lemma 3.1. *[1] Suppose $F_0, F_1, \ldots, F_n$ are closed subset of standard n-simplex $\triangle_n$ in $\mathbb{R}^{n+1}$. If for any nonempty subset I of $\{0, 1, \ldots, n\}$,*

$$\mathrm{co}(\{e_i : i \in I\}) \subseteq \bigcup_{i\in I} F_i,$$

then $\bigcap_{i=0}^{n} F_i \neq \emptyset$.

Lemma 3.2. *The mapping $f : \triangle_n \to X_\rho$ defined as*

$$f(t_0, t_1, \ldots, t_n) = \sum_{i=0}^{n} t_i x_i,$$

is continuous for each $x_0, \ldots, x_n \in X_\rho$ and $n \in \mathbb{N}$ where ρ is a modular on X.

Proof. Suppose $\{a_m\}$ is a sequence in $\triangle_n$ such that $a_m \to a$ as $m \to \infty$. We have

$$\begin{aligned}
\rho(f(a_0^m, \ldots, a_n^m) - f(a_0, \ldots, a_n)) &= \rho(\sum_{i=0}^{n} a_i^m x_i - \sum_{i=0}^{n} a_i x_i) \\
&= \rho(\sum_{i=0}^{n} (1/n)(a_i^m - a_i) n x_i) \\
&\leq \sum_{i=0}^{n} \rho\left(n(a_i^m - a_i) x_i\right) \to 0,
\end{aligned}$$

as $m \to \infty$ where $a_m = (a_0^m, \ldots, a_n^m)$ and $a = (a_0, \ldots, a_n)$. As a result,

$$f(a_0^m, \ldots, a_n^m) \to f(a_0, \ldots, a_n)$$

as $m \to \infty$ and so f is continuous. $\square$

A characterization for generalized KKM mappings in modular spaces is obtained in the following theorem.

Theorem 3.3. *Let X be a nonempty set, ρ be a modular on Y, and $F : X \to 2^{Y_\rho}$ be a multivalued mapping with closed values. Then the family*

$$\{F(x) : \ x \in X\},$$

has the finite intersection property if and only if the mapping F is a generalized KKM mapping.

Proof. Let F be a generalized KKM mapping. Take a finite subset $\{x_0, \ldots, x_n\}$ of X. It follows that there exist corresponding points $y_0, \ldots, y_n$ of Y_ρ such that for each subset $y_{i_0}, \ldots, y_{i_k}$, we have

$$\mathrm{co}(\{y_{i_0}, \ldots, y_{i_k}\}) \subset \bigcup_{j=0}^{k} F(x_{i_j}).$$

Let $C = \overline{\mathrm{co}}(\{y_0, y_1, \ldots, y_n\})$ and define $F_i = F(x_i) \cap C$ for every $i = 0, \ldots, n$. Define $\phi : \triangle_n \to C$ by $\phi(a) = \sum_{i=0}^{n} a_i y_i$ where $a = (a_0, a_1, \ldots, a_n)$. By Lemma 3.2, ϕ is continuous which follows $\phi^{-1}(F_i)$ is closed in $\triangle_n$ for each $i = 0, \ldots, n$.

On the other hand, we have

$$\phi(\mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}) \subseteq \overline{\mathrm{co}}\{y_{i_0}, y_{i_1}, \ldots, y_{i_k}\} \subseteq \bigcup_{j=0}^{k} F(x_{i_j}),$$

for each subset $\{e_{i_0}, \ldots, e_{i_k}\}$ of $\{e_0, \ldots, e_n\}$. It implies

$$\mathrm{co}(\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}) \subseteq \bigcup_{j=0}^{k} \phi^{-1}(F(x_{i_j})),$$

for each subset $\{e_{i_0}, \ldots, e_{i_k}\}$ of $\{e_0, \ldots, e_n\}$. Therefore, by Lemma 3.1

$$\bigcap_{i=0}^{n} \phi^{-1}(F_i) \neq \emptyset.$$

It implies there exists $a \in \triangle_n$ such that

$$a \in \bigcap_{i=0}^{n} \phi^{-1}(F(x_i) \cap C).$$

Then $\phi(a) \in \bigcap_{i=0}^n F(x_i) \cap C$. Finally, we have

$$\phi(a) \in \bigcap_{i=0}^{n} F(x_i).$$

We show that if the family $\{F(x) : x \in X\}$ has the finite intersection property, then F is a generalized KKM mapping. Suppose $\{x_0, \dots, x_n\}$ is a subset of X. Since $\cap_{i=0}^n F(x_i) \neq \emptyset$, choose $y^* \in \cap_{i=0}^n F(x_i)$. Set $y_i = y^*$ for $i = 0, \dots, n$. Then for any $0 \leq k \leq n$ and any subset $\{y_{i_0}, \dots, y_{i_k}\}$, it follows that

$$\mathrm{co}(\{y_{i_j} : j = 0, \dots, k\}) = \mathrm{co}(\{y^*\}) = \{y^*\} \subseteq \bigcup_{i=0}^{k} F(x_i),$$

which shows F is a generalized KKM mapping. □

Corollary 3.4. *Let ρ be a modular on Y, X be a nonempty set of Y_ρ, and $G : X \to 2^{Y_\rho}$ be a closed valued map. If G is KKM, then the family $\{G(x) : x \in X\}$ has the finite intersection property.*

Theorem 3.5. *Let ρ be a modular on Y, X be a nonempty set, and $G : X \to 2^{Y_\rho}$ be a map with closed values. Moreover, suppose there exists $x_0 \in X$ such that $G(x_0)$ is compact. Then $\bigcap_{x \in X} G(x) \neq \emptyset$ if and only if the mapping G is a generalized KKM mapping.*

Proof. Consider $\bigcap_{x \in X} G(x) \neq \emptyset$. So the family $\{G(x) : x \in X\}$ has the finite intersection property. By closedness of $G(x)$ for each $x \in X$ and Theorem 3.3, G is generalized KKM.

Now suppose G is a generalized KKM mapping and on the contrary $\bigcap_{x \in X} G(x) = \emptyset$. As a result, $\bigcup_{x \in X} G(x_0) \setminus G(x) = G(x_0)$. Compactness of $G(x_0)$ implies there exist $x_0, \dots, x_n$ such that

$$\bigcup_{i=0}^{n} G(x_0) \setminus G(x_i) = G(x_0).$$

It follows that $\bigcap_{i=0}^n G(x_i) = \emptyset$ which is a contradiction by Theorem 3.3. □

Let X be a nonempty set and ρ be a modular on Y. A mapping $G : X \to 2^{Y_\rho}$ is said to be generalized transfer closed-valued if $\bigcap_{x \in X} \overline{G(x)} \neq \emptyset$ implies $\bigcap_{x \in X} G(x) \neq \emptyset$.

Remark 3.6. *Let ρ be a modular on X and Y be a compact subset of X_ρ. By using Theorems 3.5 and 3.3, it is not difficult to check that for every family $\mathcal{C}$ of closed sets in Y having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ is nonempty.*

Theorem 3.7. *Let X be a nonempty set, ρ be a modular on Y, and $G : X \to 2^{Y_\rho}$ be generalized transfer closed valued. Moreover, suppose there exists a finite subset X_0 of X such that $\bigcap_{x\in X_0} \overline{G(x)}$ is nonempty and compact. Then $\bigcap_{x\in X} G(x)$ is nonempty if and only if the mapping $\overline{G}$ is a generalized KKM mapping.*

Proof. Using Theorem 3.3, we know $\overline{G}$ is a generalized KKM mapping if and only if

$$\bigcap_{x\in A} \overline{G(x)} \neq \emptyset, \tag{1}$$

for every finite subset A of X. On the other hand, we know

$$\begin{aligned}\bigcap_{x\in X} \overline{G(x)} &= \bigcap_{x\in X} \overline{G(x)} \cap \bigcap_{x\in X_0} \overline{G(x)} \\ &= \bigcap_{x\in X} \left[\overline{G(x)} \cap \bigcap_{x\in X_0} \overline{G(x)} \right]. \end{aligned} \tag{2}$$

Applying (2) and using Remark 3.6, we obtain (1) holds if and only if

$$\bigcap_{x\in X} \overline{G(x)} \neq \emptyset. \tag{3}$$

Since G is a generalized transfer closed valued multifunction, we deduce the inclusion (3) holds if and only if $\cap_{x\in X} G(x) \neq \emptyset$ and the theorem is proved. □

Corollary 3.8. *Let ρ be a modular on Y, X be a nonempty subset of Y_ρ, and the set-valued map $G : X \to 2^Y$ be generalized transfer closed valued and KKM. If there exists a nonempty finite subset X_0 of X such that $\bigcap_{x\in X_0} \overline{G(x)}$ is nonempty and compact, then*

$$\bigcap_{x\in X} \overline{G(x)} \neq \emptyset.$$

4 The Minimax Inequality

In this section as an application of Theorems 3.3, and 3.7, some minimax inequalities in modular spaces are proved. The main result of this section is the following.

Theorem 4.1. *Let ρ be a modular on X and Y be a nonempty subset of X_ρ. Suppose Z is a nonempty set and $\phi : Z \times Y \to \mathbb{R} \cup \{+\infty, -\infty\}$ is a mapping. Suppose that the following properties hold.*

1. *there exists a subset* Z_0 *of* Z *such that* $\bigcap_{z\in Z_0} \overline{\{y \in Y : \phi(z,y) \leq 0\}}$ *is nonempty and compact.*

2. *the mapping* $z \rightarrowtail \{y \in Y : \phi(z,y) \leq 0\}$ *is generalized transfer closed valued on* Z.

3. *the mapping* $z \rightarrowtail \overline{\{y \in Y : \phi(z,y) \leq 0\}}$ *is a generalized KKM map on* Z.

Then there exists $y^* \in Y$ *such that* $\phi(z, y^*) \leq 0$ *for each* $z \in Z$, $\min_{y\in Y}\sup_{z\in Z}\phi(z,y) \leq 0$.

Proof. Define $G : Z \to 2^Y$ by $G(z) = \{y \in Y : \phi(z,y) \leq 0\}$. It is not difficult to check G is a generalized transfer closed valued mapping and $\overline{G}$ is a generalized KKM mapping on Z. On the other hand, we know that

$$\bigcap_{z\in Z_0} \overline{G(z)},$$

is nonempty and compact for some subset Z_0 of Z. Therefore, the assumptions stated in Theorem 3.7 are satisfied, which in turn implies $\bigcap_{z\in Z} G(z) \neq \emptyset$. Now choose $y^* \in \bigcap_{z\in Z} G(z)$. By the definition of G we finally deduce $\phi(z, y^*) \leq 0$ for all $z \in Z$ as we wanted. □

Theorem 4.2. *Let* ρ *be a modular on* X *and* Y *be a nonempty subset of* X_ρ, *and* $Z \subseteq Y$. *Suppose* $\phi : Z \times Y \to \mathbb{R} \cup \{+\infty, -\infty\}$ *is a mapping and the following properties hold.*

1. *there exists a subset* Z_0 *of* Z *such that* $\bigcap_{z\in Z_0} \overline{\{y \in Y : \phi(z,y) \leq 0\}}$ *is nonempty and compact.*

2. *the mapping* $z \rightarrowtail \{y \in Y : \phi(z,y) \leq 0\}$ *is generalized transfer closed valued on* Z,

3. *for each nonempty finite subset* A *of* X *and each* $y \in \mathrm{co}(A)$, $\min_{x\in A}\phi(x,y) \leq 0$.

Then there exists $y^* \in Y$ *such that* $\phi(x, y^*) \leq 0$ *for each* $x \in X$; $\min_{y\in Y}\sup_{x\in X}\phi(x,y) \leq 0$.

Proof. Define the setvalued map G on Z at $z \in Z$ as $G(z) = \{y \in Y : \phi(z,y) \leq 0\}$. According to Theorem 4.1, it is enough to prove that $\overline{G}$ is generalized KKM. Suppose not, so there exists a nonempty finite subset A of X such that

$$\mathrm{co}(A) \nsubseteq \bigcup_{x\in A} \overline{G(x)},$$

which in turn implies there exists $y \in \mathrm{co}(A)$ such that

$$y \notin \bigcup_{x\in A} \overline{G(x)}.$$

It means that $\phi(x,y) > 0$ for each $x \in A$. As a result $\min_{x\in A}\phi(x,y) > 0$ which is a contradiction, and the thorem is proved. □

Theorem 4.3. *Let ρ be a modular on Z, Y be a nonempty subset of Z_ρ, and $X \subseteq Y$. Suppose $\phi : X \times Y \to \mathbb{R} \cup \{+\infty, -\infty\}$ is a mapping and the following properties hold.*

1. *for each $y \in Y$ and for each $\alpha \in \mathbb{R}$, the set $\{x \in X : \phi(x, y) \geq \alpha\}$ (resp., $\{x \in X : \phi(x, y) \geq \alpha\}$) is convex.*
2. *There exists $\gamma \in \mathbb{R}$ such that $\phi(x, x) \leq \gamma$ (resp., $\phi(x, x) \geq \gamma$) for each $x \in X$.*
3. *there exists a finite subset X_0 of X such that $\bigcap_{x \in X_0} \overline{\{y \in Y : \phi(x, y) \leq \gamma\}}$*

$$(resp. \bigcap_{x \in X_0} \overline{\{y \in Y : \phi(x, y) \geq \gamma\}}),$$

   is nonempty and compact.

4. *the map $x \to \{y \in Y : \phi(x, y) \leq \sup_{z \in X} \phi(z, z)\}$ (resp.$x \to \{y \in Y : \phi(x, y) \geq \sup_{z \in X} \phi(z, z)\}$), is generalized transfer closed valued.*

Then there exists $y^* \in Y$ such that $\phi(x, y^*) \leq \gamma$ (resp., $\phi(x, y^*) \geq \gamma$) for each $x \in X$ and hence $\sup_{x \in X} \phi(x, y^*) \leq \sup_{z \in X} \phi(x, x)$ (resp. $\inf_{x \in X} \phi(x, y^*) \geq \inf_{z \in X} \phi(x, x)$).

Proof. By hypothesis (2), $\lambda = \sup_{z \in X} \phi(x, x)$ exists. Define $G : X \to 2^Y$ as $G(x) = \{y \in Y : \phi(x, y) \leq \lambda\}$, we can find that the existence of $y^* \in Y$ such that $\sup_{z \in X} \phi(x, y^*) \leq \sup_{z \in X} \phi(x, x)$ requires that

$$y^* \in \bigcap_{x \in X} G(x).$$

To do this, it is enough to show that

$$\bigcap_{x \in X} G(x).$$

In order to show this we would like to apply Theorem 3.7. First we prove that $\overline{G}$ is a KKM map. Suppose not, as a result, there exist a subset $\{x_1, \ldots, x_n\}$ of X and $x^* \in \text{co}(\{x_1, \ldots, x_n\})$ such that

$$x^* \notin \bigcup_{i=1}^{n} G(x_i).$$

It means that $\phi(x_i, x^*) > \lambda$, for each $i = 1, \ldots, n$. Set $\alpha = \min_{i=1,\ldots,n} \phi(x_i, x^*)$ and $A = \{x \in X : \phi(x, x^*) \geq \alpha\}$. Since $x_i \in A$ for each $i = 1, \ldots, n$, by the hypothesis (1), $x^* \in A$. It follows that $\phi(x^*, x^*) \geq \alpha > \lambda$, which is a contrdiction. It implies that $\overline{G}$ is a KKM map.

Next we know that $\lambda \leq \gamma$ from which it follows that

$$\bigcap_{x\in X_0} \overline{G(x)} = \bigcap_{x\in X_0} \overline{\{y \in Y : \phi(x,y) \leq \lambda\}} \subseteq \bigcap_{x\in X_0} \overline{\{y \in Y : \phi(x,y) \leq \gamma\}},$$

for some finite subset X_0 of X. It implies that

$$\bigcap_{x\in X_0} \overline{G(x)}$$

is compact since $\bigcap_{x\in X_0} \overline{\{y \in Y : \phi(x,y) \leq \gamma\}}$ is nonempty and compact. On the other hand, by Theorem 3.3, the fact that $\bigcap_{x\in X_0} \overline{G(x)}$ is nonempty follows from $\overline{G}$ is a KKM map. □

The following is a modular version of the Ky Fan's minimax inequality.

Corollary 4.4. *Let ρ be a modular on X and Y be a nonempty subset of X_ρ. Suppose $\phi : Y \times Y \to \mathbb{R} \cup \{+\infty, -\infty\}$ is a mapping and the following properties hold.*

1. *for each $y \in Y$ and for each $\alpha \in \mathbb{R}$, the set $\{x \in Y : \phi(x,y) \geq \alpha\}$ (resp., $\{x \in Y : \phi(x,y) \geq \alpha\}$) is closed and convex.*
2. *There exists $\gamma \in \mathbb{R}$ such that $\phi(x,x) \leq \gamma$ (resp., $\phi(x,x) \geq \gamma$) for each $x \in Y$.*
3. *there exists $x_0 \in Y$ such that $\{y \in Y : \phi(x_0,y) \leq \gamma\}$ (resp. $\{y \in Y : \phi(x_0,y) \geq \gamma\}$) is compact.*

Then there exists $y^* \in Y$ such that $\phi(x,y^*) \leq \gamma$ (resp., $\phi(x,y^*) \geq \gamma$) for each $x \in Y$ and hence $\sup_{x\in Y} \phi(x,y^*) \leq \sup_{x\in Y} \phi(x,x)$ (resp. $\inf_{x\in Y} \phi(x,y^*) \geq \inf_{x\in Y} \phi(x,x)$).

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