CROSSED PRODUCTS OF PRO-C^*-ALGEBRAS AND HILBERT PRO-C^*-MODULES

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Abstract. In this paper, we prove a universal property for the crossed product of a pro-C^*-algebra by an inverse limit action of a locally compact group. Also, we prove a universal property of the crossed product of a Hilbert (pro-) C^*-module by an (inverse limit) action of a locally compact group.

1. Introduction

The crossed product of a C^*-algebra A by an action α of a locally compact group G is a new C^*-algebra, denoted by $G\times_\alpha A$, which contains, in a some suitable sense, A and G, and it is one of the most important construction in operator algebras. A topological dynamical system (G,θ,X), (that is, G is a locally compact group, X is a compact Hasdorff space and θ is a continuous action on X), induces an action α of G on the C^*-algebra $C(X)$ of all continuous complex-valued functions on X, and $G\times_\alpha C(X)$ encodes the topological dynamical system (G,θ,X). There is a vast literature on crossed products of C^*-algebras (see, e.g. [15]), but the corresponding theory in the context of non-normed topological algebras has still a long way to cover. Crossed products of pro-C^*-algebras by inverse limit actions of locally compact groups were considered first by Phillips [11] and secondly by Joiţa [3]. The crossed product of a pro-C^*-algebra A by an inverse limit action α is defined as the enveloping pro-C^*-algebra of the covariance algebra $L^1(G,\alpha,A)$. It is often easiest to exhibit such objects by verifying a particular representation has the required universal property, rather than working directly with definition. For example, Raeburn [14] proved Takai's theorem by exploiting the universal properties of crossed products, and Joiţa [6] used the universal property of crossed products of C^*-algebras to study of covariant completely positive maps on C^*-algebras. In Section 3, we prove a universal property for the crossed product of a pro-C^*-algebra by an inverse limit action of a locally compact group.

Hilbert C^*-modules appear as imprimitivity bimodules in the study of the Morita equivalence for C^*-algebras. Given two C^*-dynamical systems (G,α,A) and (G,β,B) such that A and B are strongly Morita equivalent and an (α,β)-compatible action η of G on the imprimitivity $A-B$ bimodule X, then the C^*-crossed products $G\times_\alpha A$ and $G\times_\beta B$ are strongly Morita equivalent. The imprimitivity $G\times_\alpha A-G\times_\beta B$ bimodule is called the crossed product of X by η, and it is denoted by $G\times_\eta X$. In [5], we show that for a dynamical system (G,η,X) on a full Hilbert C^*-module X, the Hilbert C^*-module X can be embedding in the multiplier module of $G\times_\eta X$, and G is isomorphic to a group of unitaries in the C^*-algebra of adjointable module morphisms on $G\times_\eta X$. In Section 4, we show that the crossed product of

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Hilbert C^*-modules, respectively Hilbert pro-C^*-modules by inverse limit actions, has the universal property with respect to the covariant representations of Hilbert C^*-modules, respectively Hilbert pro-C^*-modules.

2. Preliminaries

A pro-C^*-algebra is a Hausdorff complete complex topological $*$-algebra A whose topology is determined by its continuous C^*-seminorms in the sense that a net $\{a_i\}_{i\in I}$ converges to 0 in A if and only if the net $\{p(a_i)\}_{i\in I}$ converges to 0 for all continuous C^*-seminorms p on A. Other terms been used for a pro-C^*-algebra are: locally C^*-algebra (A. Inoue), b^*-algebra (C. Apostol) and LMC^*-algebra (G. Lassner, K. Schmüdgen). We refer the reader to [2] for further information about pro-C^*-algebras.

Let A be a pro-C^*-algebra. We denote by $S(A)$ the set of all continuous C^*-seminorms on A. For $p \in S(A)$, the quotient $*$-algebra $A_p = A/\ker p$, where $\ker p = \{a \in A; p(a) = 0\}$, is a C^*-algebra with respect to the C^*-norm $\|\cdot\|_p$ induced by p, and the canonical map from A to A_p is denoted by π_p^A. The set $S(A)$ is directed with the order $p \geq q$ if $p(a) \geq q(a)$ for all a in A. Then, for p and q in $S(A)$, with $p \geq q$, there is a canonical surjective C^*-morphism $\pi_{pq}^A : A_p \to A_q$ such that $\pi_{pq}^A(\pi_p^A(a)) = \pi_q^A(a)$ for all a in A and $\{A_p; \pi_{pq}^A\}_{p\geq q, p,q\in S(A)}$ is an inverse system of C^*-algebras. Moreover, the pro-C^*-algebras A and $\lim\limits_{\leftarrow p} A_p$ are isomorphic.

A multiplier on a pro-C^*-algebra A is a pair (l, r) of linear maps from A to A such that l and r are respectively left and right A-module morphisms, and $r(a)b = al(b)$ for all $a, b \in A$. The multiplier algebra $M(A)$ of A is a pro-C^*-algebra with respect to the topology determined by the family of C^*-seminorms $\{p_{M(A)}\}_{p\in S(A)}$ with $p_{M(A)}(l, r) = \sup\{p(l(a)); a \in A, p(a) \leq 1\}$, and it is isomorphic to $\lim\limits_{\leftarrow p} M(A_p)$. The strict topology on $M(A)$ is given by the family of seminorms $\{p_a\}_{(a,p)\in A\times S(A)}$, where $p_a(l, r) = p(l(a)) + p(r(a))$. If $\varphi : A \to B$ is a nondegenerate morphism of pro-C^*-algebras (that is, φ is a continuous $*$-morphism and $[\varphi(A) B] = B$, where $[\varphi(A) B]$ denotes the closed subspace of B generated by $\{\varphi(a) b; a \in A, b \in B\}$), then it extends to a unital morphism of pro-C^*-algebras $\overline{\varphi} : M(A) \to M(B)$.

A representation of A on a Hilbert space $\mathcal{H}$ is a pair $(\varphi, \mathcal{H})$, where φ is a continuous $*$-morphism from A to $L(\mathcal{H})$. We say that $(\varphi, \mathcal{H})$ is nondegenerate if $[\varphi(A) \mathcal{H}] = \mathcal{H}$.

Hilbert modules are generalizations of Hilbert spaces by allowing the inner product to take values in a (pro-) C^*-algebra rather than the field of complex numbers. They are useful tools in theory of operator algebras, operator K-theory, KK-theory of C^*-algebras, group representation theory, C^*-algebraic theory of quantum groups and theory of operator spaces.

Here we recall some definitions and simple facts about Hilbert pro-C^*-modules and the module maps between them (see [7, 10, 12]).

A Hilbert pro-C^*-module X over a pro-C^*-algebra A (or a Hilbert A-module) is a linear space that is also a right A-module, equipped with an A-valued inner product $\langle\cdot,\cdot\rangle$ that is $\mathbb{C}$- and A-linear in the second variable and conjugate linear in the first variable such that X is complete with the family of seminorms $\{p_X\}_{p\in S(A)}$, where $p_X(x) = p(\langle x, x\rangle)^{\frac{1}{2}}$. A Hilbert A-module X is full if the pro-C^*-subalgebra of A generated by $\{\langle x, y\rangle; x, y \in X\}$ coincides with A.

Given a Hilbert pro-C^*module X over A, for $p \in S(A)$, $X_p = X/\ker p_X$ has a canonical structure of Hilbert C^*-module over A_p and the canonical map from X onto X_p is denoted by σ_p^X. For $p, q \in S(A)$, with $p \geq q$, there is a canonical surjective linear map $\sigma_{pq}^X : X_p \to X_q$ such that $\sigma_{pq}^X \circ \sigma_p^X = \sigma_q^X$. Then $\{X_p; A_p; \sigma_{pq}^X; \pi_{pq}^A\}_{p\geq q,\ p,q\in S(A)}$ is an inverse system of Hilbert C^*-modules, and X can be identified to $\lim\limits_{\leftarrow p} X_p$.

Let X and Y be Hilbert A-modules. A map $T : X \to Y$ is adjointable if there is a map $T^* : Y \to X$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in X$ and for all $y \in Y$. The set $L(X)$ of all adjointable A-module morphisms from X to X is a pro-C^*-algebra with respect to the topology determined by the family of C^*-seminorms $\{p_{L(X)}\}_{p\in S(A)}$, where $p_{L(X)}(T) = \sup\{p_X(T(x)); p_X(x) \leq 1\}$. The closed linear subspace $K(X)$ of $L(X)$ spanned by $\{\theta_{y,x} : X \to X; x, y \in X\}$, where $\theta_{y,x}(z) = y\langle x, z\rangle$ is a closed two-sided ideal of $L(X)$, and $M(K(X)) = L(X)$.

A morphism of Hilbert pro-C^*-modules is a map $\Phi : X \to Y$ from a Hilbert A-module X to a Hilbert B-module Y with the property that there is a pro-C^*-morphism $\varphi : A \to B$ such that

$$\langle \Phi(x), \Phi(y)\rangle = \varphi(\langle x, y\rangle)$$

for all x and y in X. A map $\Phi : X \to Y$ is an isomorphism of Hilbert pro-C^*-modules if it is invertible, and if Φ and Φ^{-1} are morphisms of Hilbert pro-C^*-modules.

A representation of X on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is a morphism of Hilbert C^*-modules Φ_X from X to the Hilbert $L(\mathcal{H})$-module $L(\mathcal{H}, \mathcal{K})$. If X is full, then the representation φ_A associated to Φ_X is unique. A representation $\Phi_X : X \to L(\mathcal{H}, \mathcal{K})$ of X is nondegenerate if $[\Phi_X(X)\mathcal{H}] = \mathcal{K}$ and $[\Phi_X(X)^*\mathcal{K}] = \mathcal{H}$.

The pro-C^*-algebra $L(A)$ of all adjointable module morphisms on A can be identified to the multiplier algebra $M(A)$ of A (see, for example, [10, Theorem 4.2(6)]). Then the vector space of all adjointable module morphisms from A to X, denoted by $M(X)$, has a natural structure of Hilbert $M(A)$-module (see, for example, [13]), and it is called the multiplier module of X. The strict topology on $M(X)$ is given by the family of seminorms $\{\|\cdot\|_{p,a,x}\}_{(p,a,x)\in S(A)\times A\times X}$, where $\|h\|_{p,a,x} = p_X(h(a)) + p(h^*(x))$. A nondegenerate morphism of Hilbert pro-C^*-modules Φ_X from a full Hilbert pro-C^*-module X to $M(Y)$ (that is, $[\Phi_X(X)B] = Y$ and $\left[\Phi_X(X)^* Y\right] = B$) extends to a unique morphism $\overline{\Phi_X}$ from $M(X)$ to $M(Y)$, and its underlying pro-C^*-morphism $\overline{\varphi_A}$ is the extension of the underlying pro-C^*-morphism φ_A of Φ_X to $M(A)$.

3. The universal property of the crossed products of pro-C^*-algebras

Let A be a pro-C^*-algebra, G a locally compact group and Δ the modular function of G with respect to the left invariant Haar measure ds.

An action α of G on A is an inverse limit action if there is a cofinal subset of G-invariant continuous C^*-seminorms on A (this is, $p(\alpha_t(a)) = p(a)$ for all a in A and for all t in G). Therefore, if α is an inverse limit action of G on A, we can suppose that $\alpha_t = \lim\limits_{\leftarrow p} \alpha_t^p$, where $\alpha^p, p \in S(A)$ are actions of G on A_p, $p \in S(A)$ (see [3]).

A triple (G, α, A), consisting of a locally compact group G, a pro-C^*-algebra A and an inverse limit action α of G on A is called a pro-C^*-dynamical system.

The vector space $C_c(G, A)$ of all continuous functions from G to A with compact support is a $*$-algebra with the convolution as product, given by

$$(f \times h)(s) = \int\limits_G f(t)\alpha_t\left(h(t^{-1}s)\right)dt$$

and involution defined by

$$f^\sharp(t) = \Delta(t)^{-1}\alpha_t\left(f(t^{-1})^*\right).$$

For any $p \in S(A)$, the map $N_p : C_c(G, A) \to [0, \infty)$ given by

$$N_p(f) = \int\limits_G p(f(t))dt$$

is a submultiplicative $*$-seminorm on $C_c(G, A)$.

Let $L^1(G, \alpha, A)$ be the Hausdorff completion of $C_c(G, A)$ with respect to the topology defined by the family of submultiplicative $*$-seminorms $\{N_p\}_{p\in S(A)}$. Then, $L^1(G, \alpha, A)$, called the covariant algebra associated with the pro-C^*-dynamical system (G, A, α), is a locally m-convex $*$-algebra with bounded approximate unit. Its enveloping pro-C^*-algebra is called the crossed product of A by α, and it is denoted by $G \times_\alpha A$. Moreover, $G \times_\alpha A = \lim\limits_{\leftarrow p} G \times_{\alpha^p} A_p$ (see [3]).

Let (G, α, A) be a pro-C^*-dynamical system. A covariant morphism from A to a pro-C^*-algebra B is a pair (φ, u) consisting of a morphism of pro-C^*-algebras φ from A to $M(B)$ and a strictly continuous group morphism u from G to $\mathcal{U}(M(B))$, the group of unitaries in $M(B)$, such that

$$\varphi(\alpha_t(a)) = u_t\varphi(a)u_{t^{-1}}$$

for all $t \in G$ and for all $a \in A$. The covariant morphism (φ, u) is nondegenerate if $[\varphi(A)B] = B$. A (nondegenerate) covariant representation of (G, α, A) on a Hilbert space $\mathcal{H}$ is a triple $(\varphi, u, \mathcal{H})$ consisting of a (nondegenerate) representation $(\varphi, \mathcal{H})$ of A and a unitary representation u of G on $\mathcal{H}$ such that $\varphi(\alpha_t(a)) = u_t\varphi(a)u_{t^{-1}}$ for all $t \in G$ and for all $a \in A$.

Given a C^*-dynamical system (G, α, A), it is well known that the C^*-algebra A can be identified with a C^*-subalgebra in the multiplier algebra $M(G \times_\alpha A)$ of the crossed product of A by α, and G is isomorphic to a group of unitaries in $M(G \times_\alpha A)$.

In the following proposition we show that this result is also true for crossed products of pro-C^*-algebras.

Proposition 3.1. *Let (G, α, A) be a pro-C^*-dynamical system. Then there is a nondegenerate covariant morphism (i_A, i_G) from A to $G \times_\alpha A$ such that:*

(1) *for any nondegenerate covariant representation $(\varphi, u, \mathcal{H})$ of (G, α, A), there is a nondegenerate representation $(\Phi, \mathcal{H})$ of $G \times_\alpha A$ such that $\overline{\Phi} \circ i_A = \varphi$ and $\overline{\Phi} \circ i_G = u$;*
(2) *$G \times_\alpha A = \overline{\mathrm{span}\{i_A(a)i_G(f); a \in A, f \in C_c(G)\}}$.*

Moreover, i_A and i_G are injective,

$$(i_A(a)(f))(s) = af(s)$$

and

$$\left(i_G\left(t\right)\left(f\right)\right)\left(s\right)=\alpha_t\left(f\left(t^{-1}s\right)\right)$$

for all $a\in A$*,* $t\in G$ *and* $f\in C_c\left(G,A\right)$*.*

Proof. Let $p\in S(A)$. By [15, Proposition 2.34], there is a nondegenerate covariant morphism $\left(i_{A_p},i_G^p\right)$ from A_p to $G\times_{\alpha^p}A_p$ which verifies (1) and (2). Moreover, i_{A_p} and i_G^p are injective, $\left(i_{A_p}\left(\pi_p^A\left(a\right)\right)\left(f\right)\right)\left(s\right)=\pi_p^A\left(a\right)f\left(s\right)$ and $\left(i_G^p\left(t\right)\left(f\right)\right)\left(s\right)=\alpha_t^p\left(f\left(t^{-1}s\right)\right)$ for all $a\in A$, $t\in G$ and $f\in C_c\left(G,A_p\right)$. Since we have

$$\begin{aligned}
&\left(\left(\pi_{pq}^{M(G\times_\alpha A)}\circ i_{A_p}\right)\left(\pi_p^A\left(a\right)\right)\right)\left(\pi_q^A\left(b\right)\otimes f\right)\left(s\right)\\
=\;&\left(\pi_{pq}^{M(G\times_\alpha A)}\left(i_{A_p}\left(\pi_p^A\left(a\right)\right)\right)\right)\left(\pi_q^A\left(b\right)\otimes f\right)\left(s\right)\\
=\;&\pi_{pq}^{G\times_\alpha A}\left(\left(i_{A_p}\left(\pi_p^A\left(a\right)\right)\right)\left(\pi_p^A\left(b\right)\otimes f\right)\right)\left(s\right)\\
=\;&\pi_{pq}^A\left(\pi_p^A\left(a\right)\pi_p^A\left(b\right)f\left(s\right)\right)=\pi_q^A\left(a\right)\pi_q^A\left(b\right)f\left(s\right)\\
=\;&i_{A_q}\left(\pi_q^A\left(a\right)\right)\left(\pi_q^A\left(b\right)\otimes f\right)\left(s\right)
\end{aligned}$$

for all $a,b\in A$, $f\in C_c(G),s\in G$ and for all $p,q\in S(A)$ with $p\geq q$, taking into account that $A_q\otimes_{\mathrm{alg}}C_c(G)$ is dense in $G\times_{\alpha^q}A_q$, we deduce that $\left(i_{A_p}\right)_{p\in S(A)}$ is an inverse system of injective C^*-morphisms. Then $i_A=\lim\limits_{\leftarrow p}i_{A_p}$ is an injective pro-C^*-morphism.

Let $t\in G$. Since we have

$$\begin{aligned}
&\left(\pi_{pq}^{M(G\times_\alpha A)}\left(i_G^p\left(t\right)\right)\right)\left(\pi_q^A\left(b\right)\otimes f\right)\left(s\right)\\
=\;&\left(\pi_{pq}^{G\times_\alpha A}\left(i_G^p\left(t\right)\left(\pi_q^A\left(b\right)\otimes f\right)\right)\right)\left(s\right)\\
=\;&\pi_{pq}^A\left(i_G^p\left(t\right)\left(\pi_p^A\left(b\right)\otimes f\right)\left(s\right)\right)\\
=\;&\pi_{pq}^A\left(\alpha_t^p\left(\pi_p^A\left(b\right)\right)f\left(t^{-1}s\right)\right)\\
=\;&\alpha_t^q\left(\pi_{pq}^A\left(\pi_p^A\left(b\right)\right)\right)f\left(t^{-1}s\right)=\alpha_t^q\left(\pi_q^A\left(b\right)\right)f\left(t^{-1}s\right)\\
=\;&i_G^q\left(t\right)\left(\pi_q^A\left(b\right)\otimes f\right)\left(s\right)
\end{aligned}$$

for all $b\in A$, $f\in C_c(G),s\in G$ and for all $p,q\in S(A)$ with $p\geq q$, taking into account that $A_q\otimes_{\mathrm{alg}}C_c(G)$ is dense in $G\times_{\alpha^q}A_q$, we deduce that $\left(i_G^p\left(t\right)\right)_{p\in S(A)}$ is an inverse system of unitaries in $M(G\times_\alpha A)$. Let $i_G(t)=\lim\limits_{\leftarrow p}i_G^p(t)$. Clearly, the map $t\rightarrow i_G(t)$ is an injective strictly continuous morphism of groups from G to $\mathcal{U}\left(M(G\times_\alpha A)\right)$.

It is easy to verify that (i_A,i_G) is a nondegenerate covariant morphism from A to $G\times_\alpha A$.

Let $(\varphi,u,\mathcal{H})$ be a nondegenerate covariant representation of (G,α,A). Then there is a nondegenerate covariant representation $\left(\varphi_p,u,\mathcal{H}\right)$ of (G,α^p,A_p) such that $\varphi=\varphi_p\circ\pi_p^A$, and by [15, Proposition 2.34 (1)], there is a nondegenerate representation $(\Phi_p,\mathcal{H})$ of $G\times_{\alpha^p}A_p$ such that $\overline{\Phi_p}\circ i_{A_p}=\varphi_p$ and $\overline{\Phi_p}\circ i_G^p=u$. Let $\Phi=\Phi_p\circ\pi_p^{G\times_\alpha A}$. Clearly, $(\Phi,\mathcal{H})$ is a nondegenerate representation of $G\times_\alpha A$. Moreover,

$$\overline{\Phi}\circ i_A=\overline{\Phi_p\circ\pi_p^{G\times_\alpha A}}\circ i_A=\overline{\Phi_p}\;\circ\overline{\pi_p^{G\times_\alpha A}}\circ i_A=\overline{\Phi_p}\circ i_{A_p}\circ\pi_p^A=\varphi_p\circ\pi_p^A=\varphi$$

and

$$\overline{\Phi}\circ i_G=\overline{\Phi_p}\;\circ\overline{\pi_p^{G\times_\alpha A}}\circ i_G=\overline{\Phi_p}\;\circ i_G^p=u.$$

Therefore, the statement (1) is verified.

By [9, Lemma III 3.2], we have

$$\begin{aligned}
& \overline{\operatorname{span}\{i_A(a)i_G(f); a\in A, f\in C_c(G)\}} \\
= \; & \lim_{\leftarrow p} \overline{\pi_p^{G\times_\alpha A}\left(\overline{\operatorname{span}\{i_A(a)i_G(f); a\in A, f\in C_c(G)\}}\right)} \\
= \; & \lim_{\leftarrow p} \overline{\operatorname{span}\{i_{A_p}(\pi_p^A(a))i_G^p(f); a\in A, f\in C_c(G)\}} \\
& [15, \text{Proposition } 2.34\ (1)] \\
= \; & \lim_{\leftarrow p} G\times_{\alpha^p} A_p = G\times_\alpha A
\end{aligned}$$

and so, the statement (2) is verified too. □

Corollary 3.2. *Let (G,α,A) be a pro-C^*-dynamical system. If G is discrete, then A can be identified with a pro-C^*-subalgebra of $G\times_\alpha A$, and if, moreover, A is unital, then G can be identified with a group of unitaries in $G\times_\alpha A$.*

Proof. If G is discrete, then A_p can be identified with a C^*-subalgebra of $G\times_{\alpha^p} A_p$ for each $p\in S(A)$. Thus, we have

$$\begin{aligned}
i_A(A) &= \lim_{\leftarrow p} \overline{\pi_p^{M(G\times_\alpha A)}(i_A(A))} \\
&\quad [9, \text{Lemma III } 3.2] \\
&= \lim_{\leftarrow p} \overline{i_{A_p}\left(\pi_p^A(A)\right)} = \lim_{\leftarrow p} \overline{i_{A_p}(A_p)} \\
&\subseteq \lim_{\leftarrow p} G\times_{\alpha_p} A_p = G\times_\alpha A.
\end{aligned}$$

If A is unital, then the C^*-algebras $A_p, p\in S(A)$ are unital, and since G is discrete, the C^*-algebras $G\times_{\alpha^p} A_p, p\in S(A)$ are unital. Therefore, $M(G\times_\alpha A) = G\times_\alpha A$. □

In the following theorem we show that the crossed product of a pro-C^*-algebra by an inverse limit action of a locally compact group is a universal object for covariant representations of pro-C^*-dynamical systems.

Theorem 3.3. *Let (G,α,A) be a pro-C^*-dynamical system and B a pro-C^*-algebra with the property that there is a nondegenerate covariant morphism (j_A, j_G) from A to B which satisfies the following:*

1. *for any nondegenerate covariant representation $(\varphi,u,\mathcal{H})$ of (G,α,A), there is a nondegenerate representation $(\Phi,\mathcal{H})$ of B such that $\overline{\Phi}\circ j_A=\varphi$ and $\overline{\Phi}\circ j_G = u$,*
2. *$B=\overline{\operatorname{span}\{j_A(a)\, j_G(f); a\in A, f\in C_c(G)\}}$, where $j_G(f)=\int_G f(t)\, j_G(t)dt$.*

Then there is a pro-C^*-isomorphism $j: B\to G\times_\alpha A$ such that

$$\overline{j}\circ j_A = i_A \text{ and } \overline{j}\circ j_G = i_G.$$

Proof. Let $q\in S(B)$. Then there is $p_q\in S(A)$ such that $q_{M(B)}(j_A(a))\le p_q(a)$ for all $a\in A$, and so there is a C^*-morphism $j_{A_{p_q}}: A_{p_q}\to M(B_q)$ such that $j_{A_{p_q}}\circ\pi_{p_q}^A = \overline{\pi_q^B}\circ j_A$.

Let $(\theta_q,\mathcal{H}_\theta)$ be a faithful nondegenerate representation of B_q. Then $\left(\overline{\theta_q}\circ j_{A_{p_q}},\mathcal{H}_\theta\right)$ is a nondegenerate representation of A_{p_q}. For $t\in G$, $u_t^q=\overline{\theta_q}\left(\overline{\pi_q^B}(j_G(t))\right)$ is a unitary operator in $L(\mathcal{H}_\theta)$ and $t \to u_t^q$ is a unitary representation of G on $\mathcal{H}_\theta$. Moreover, $\left(\overline{\theta_q} \circ j_{A_{p_q}}, u^q, \mathcal{H}_\theta\right)$ is a nondegenerate covariant representation of $\left(G, \alpha^{p_q}, A_{p_q}\right)$. Then, there is a nondegenerate representation $(\Theta_q, \mathcal{H}_\theta)$ of $G \times_{\alpha^{p_q}} A_{p_q}$ such that

$$\overline{\Theta_q} \circ i_{A_{p_q}} = \overline{\theta_q} \circ j_{A_{p_q}} \text{ and } \overline{\Theta_q} \circ i_G^q = u^q.$$

We have

$$\begin{aligned}
\Theta_q\left(i_{A_{p_q}}\left(\pi_{p_q}^A(a)\right) i_G^q(f)\right) &= \overline{\Theta_q}\left(i_{A_{p_q}}\left(\pi_{p_q}^A(a)\right)\right) \overline{\Theta_q}\left(i_G^q(f)\right) \\
&= \left(\overline{\theta_q} \circ j_{A_{p_q}}\right)\left(\pi_{p_q}^A(a)\right) \overline{\theta_q}\left(\overline{\pi_q^B}\left(j_G(f)\right)\right) \\
&= \theta_q\left(j_{A_{p_q}}\left(\pi_{p_q}^A(a)\right) \overline{\pi_q^B}\left(j_G(f)\right)\right)
\end{aligned}$$

for all $a \in A$ and $f \in C_c(G)$, and so Θ_q and θ_q have the same range. Let $\Phi_q = \theta_q^{-1} \circ \Theta_q \circ \pi_{p_q}^{G \times_\alpha A}$. Clearly, Φ_q is a continuous $*$-morphism from $G \times_\alpha A$ to B_q, $\overline{\Phi_q} \circ i_A = \overline{\pi_q^B} \circ j_A$ and $\overline{\Phi_q} \circ i_G = \overline{\pi_q^B} \circ j_G$. Since we have

$$\begin{aligned}
\left(\pi_{q_1 q_2}^B \circ \Phi_{q_1}\right)\left(i_A(a) i_G(f)\right) &= \left(\pi_{q_1 q_2}^B \circ \theta_{q_1}^{-1} \circ \Theta_{q_1} \circ \pi_{p_{q_1}}^{G \times_\alpha A}\right)\left(i_A(a) i_G(f)\right) \\
&= \left(\pi_{q_1 q_2}^B \circ \theta_{q_1}^{-1} \circ \Theta_{q_1}\right)\left(i_{A_{p_1}}\left(\pi_{p_1}^A(a)\right) i_G^{p_1}(f)\right) \\
&= \left(\pi_{q_1 q_2}^B \circ \theta_{q_1}^{-1}\right)\left(\theta_{q_1}\left(j_{A_{p_{q_1}}}\left(\pi_{p_{q_1}}^A(a)\right) \overline{\pi_{q_1}^B}\left(j_G(f)\right)\right)\right) \\
&= \pi_{q_1 q_2}^B\left(j_{A_{p_{q_1}}}\left(\pi_{p_{q_1}}^A(a)\right) \overline{\pi_{q_1}^B}\left(j_G(f)\right)\right) \\
&= \pi_{q_1 q_2}^B\left(\pi_{q_1}^B\left(j_A(a) j_G(f)\right)\right) = \pi_{q_2}^B\left(j_A(a) j_G(f)\right) \\
&= \Phi_{q_2}\left(i_A(a) i_G(f)\right)
\end{aligned}$$

for all $a \in A$ and $f \in C_c(G)$ and for all $q_1, q_2 \in S(B)$ with $q_1 \geq q_2$, we deduce that there is a pro-C^*-morphism $\Phi : G \times_\alpha A \to B$ such that $\pi_q^B \circ \Phi = \Phi_q$. Moreover, since

$$\overline{\pi_q^B}\left(\overline{\Phi}\left(i_A(a)\right)\right) = \overline{\Phi_q}\left(i_A(a)\right) = \overline{\pi_q^B}\left(j_A(a)\right)$$

for all $a \in A$ and for all $q \in S(B), \overline{\Phi} \circ i_A = j_A$, and since

$$\overline{\pi_q^B}\left(\overline{\Phi}\left(i_G(f)\right)\right) = \overline{\Phi_q}\left(i_G(f)\right) = \overline{\pi_q^B}\left(j_G(f)\right)$$

for all $f \in C_c(G)$ and for all $q \in S(B), \overline{\Phi} \circ i_G = j_G$.

In the same manner, we obtain a pro-C^*-morphism $\Psi : B \to G \times_\alpha A$ such that $\overline{\Psi} \circ j_A = i_A$ and $\overline{\Psi} \circ j_G = i_G$. Clearly,

$$(\Psi \circ \Phi)\left(i_A(a) i_G(f)\right) = i_A(a) i_G(f) \text{ and } (\Phi \circ \Psi)\left(j_A(a) j_G(f)\right) = j_A(a) j_G(f)$$

for all $a \in A$ and $f \in C_c(G)$, and so $\Psi : B \to G \times_\alpha A$ is a pro-C^*-isomorphism. We put $j = \Psi$ and the theorem is proved. □

4. The universal property of the crossed products of Hilbert pro-C^*-modules

An action of a locally compact group G on a full Hilbert pro-C^*-module X over a pro-C^*-algebra A is a group morphism $t \mapsto \eta_t$ from G to $\mathrm{Aut}(X)$, the group of all isomorphisms of Hilbert pro-C^*-modules from X to X, such that the map $t \mapsto \eta_t(x)$ from G to X is continuous for each $x \in X$. An action η of G on X is an inverse limit action if we can write X as an inverse limit $\lim\limits_{\leftarrow \lambda} X_\lambda$ of Hilbert

C^*-modules in such a way that there are actions η^λ of G on X_λ, $\lambda \in \Lambda$ such that $\eta_t = \lim\limits_{\leftarrow \lambda} \eta_t^\lambda$ for all t in G. The triple (G, η, X) consisting of a locally compact group G, a full Hilbert pro-C^*-module X and an inverse limit action η of G on X is called a dynamical system on a Hilbert pro-C^*-module X. Clearly, any pro-C^*-dynamical system (G, α, A) can be regarded as a dynamical system on Hilbert pro-C^*-module in the sense of the above definition.

An inverse limit action $t \mapsto \eta_t$ of G on X induces a unique inverse limit action $t \mapsto \alpha_t^\eta$ of G on A such that

$$\alpha_t^\eta\left(\langle x, y\rangle\right) = \langle \eta_t\left(x\right), \eta_t\left(x\right)\rangle$$

for all $x, y \in X$ and for all $t \in G$ (see [3, 4, 8]).

Let (G, η, X) be a dynamical system on a Hilbert pro-C^*-module X. The linear space $C_c(G, X)$ of all continuous functions from G to X with compact support has the pre-Hilbert $G \times_{\alpha^\eta} A$-module structure with the action of $G \times_{\alpha^\eta} A$ on $C_c(G, X)$ given by

$$\left(\widehat{x} f\right)(s) = \int\limits_G \widehat{x}\left(t\right) \alpha_t^\eta\left(f\left(t^{-1}s\right)\right) dt$$

for all $\widehat{x} \in C_c(G, X)$ and $f \in C_c(G, A)$, and the inner product given by

$$\langle \widehat{x}, \widehat{y}\rangle(s) = \int\limits_G \langle \eta_{t^{-1}}\left(\widehat{x}(t)\right), \eta_{t^{-1}}\left(\widehat{y}\left(ts\right)\right)\rangle dt.$$

The crossed product of X by η, denoted by $G \times_\eta X$, is the Hilbert $G \times_{\alpha^\eta} A$-module obtained by the completion of the pre-Hilbert $G \times_{\alpha^\eta} A$-module $C_c(G, X)$ (see [3, 4, 8]).

Let (G, η, X) be a dynamical system on a Hilbert pro-C^*-module X. A covariant morphism from X to a Hilbert pro-C^*-module Y over B is a triple (v, Φ_X, u) consisting of a morphism Φ_X from X to $M(Y)$, a strictly continuous group morphism u from G to $\mathcal{U}\left(M(B)\right)$ and a strictly continuous group morphism v from G to $\mathcal{U}\left(M(K(Y))\right)$ such that

$$v_t \Phi_X\left(x\right) u_{t^{-1}} = \Phi_X\left(\eta_t\left(x\right)\right)$$

for all $x \in X$ and for all $t \in G$. The covariant morphism (v, Φ_X, u) is nondegenerate if Φ_X is nondegenerate. If (v, Φ_X, u) is a (nondegenerate) covariant morphism from X to Y, then $\left(\varphi_{A,} u\right)$, where φ_A is the underlying pro-C^*-morphism of Φ_X, is a (nondegenerate) covariant morphism from A to B.

A (nondegenerate) covariant representation of (G, η, X) on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is a quintuple $(v, \Phi_X, u, \mathcal{H}, \mathcal{K})$ consisting of a (nondegenerate) representation $(\Phi_X, \mathcal{H}, \mathcal{K})$ of X and two unitary representations v and u of G on the Hilbert spaces $\mathcal{K}$ respectively $\mathcal{H}$ such that $v_t \Phi_X\left(x\right) u_{t^{-1}} = \Phi_X\left(\eta_t\left(x\right)\right)$ for all $x \in X$ and for all $t \in G$.

As in the case of crossed products of pro-C^*-algebras, we show that the crossed product of a Hilbert pro-C^*-module is a universal object for covariant representations of dynamical systems on a Hilbert pro-C^*-module.

Proposition 4.1. *Let (G, η, X) be a dynamical system on a Hilbert C^*-module X. Then there is a nondegenerate covariant morphism $\left(i_G^X, i_X, i_G\right)$ from X to $G \times_\eta X$ such that*

1. *for any nondegenerate covariant representation $(v, \Phi_X, u, \mathcal{H}, \mathcal{K})$ of (G, η, X), there is a nondegenerate representation $\left(\Phi_{G \times_\eta X}, \mathcal{H}, \mathcal{K}\right)$ of $G \times_\eta X$ such that $\overline{\Phi_{G \times_\eta X}} \circ i_X = \Phi_X$ and $\overline{\varphi_{G \times_{\alpha^\eta} A}} \circ i_G = u$;*

(2) $G \times_\eta X = \overline{span\{i_X(x)\, i_G(f)\, ; x \in X, f \in C_c(G)\}}$.

Moreover, the maps i_G^X, i_X *and* i_G *are injective,*

$$i_X(x)(f)(s) = xf(s) \text{ and } i_G(t)(f) = \alpha_t^\eta\left(f\left(t^{-1}s\right)\right)$$

for all $f \in C_c(G, A)$ *and* $s, t \in G$ *and*

$$i_G^X(t)(x \otimes f)(s) = \eta_t(x)\, f\left(t^{-1}s\right)$$

for all $x \in X$*, for all* $f \in C_c(G)$ *and for all* $s, t \in G$.

Proof. By [5, Theorem 3.5], there is $\left(i_G^X, i_X, i_G\right)$ a covariant morphism from X to $G \times_\eta X$ such that i_G^X, i_X and i_G are injective. The assertion (1) follows from the proof of Proposition 3.8 [5].

(2) Let $\left(\Phi_{G\times_\eta X}, \mathcal{H}, \mathcal{K}\right)$ be a faithful nondegenerate representation of $G \times_\eta X$ (see [1, Theorem 3.11]). Then there is a nondegenerate covariant representation $(v, \Phi_X, u, \mathcal{H}, \mathcal{K})$ of (G, η, X) such that $\Phi_{G\times_\eta X} = \Phi_X \times u$, where

$$(\Phi_X \times u)(f) = \int_G \Phi_X(f(t))\, u_t dt$$

for all $f \in C_c(G, X)$, and $\varphi_{G\times_{\alpha^\eta} A} = \varphi_A \times u$. Moreover,

$$\overline{\Phi_X \times u} \circ i_X = \Phi_X \text{ and } \overline{\varphi_A \times u} \circ i_G = u,$$

(see the proof of Proposition 3.8 [5]). Thus, we have

$$\begin{aligned}(\Phi_X \times u)(i_X(x)\, i_G(f)) &= \overline{\Phi_X \times u}(i_X(x))\, \overline{\varphi_A \times u}(i_G(f)) \\ &= \Phi_X(x)\, u(f) = (\Phi_X \times u)(x \otimes f),\end{aligned}$$

whence, it follows that $i_X(x)\, i_G(f) = x \otimes f$ for all $f \in C_c(G)$ and for all $x \in X$. Since $X \otimes_{\text{alg}} C_c(G)$ is dense in $G \times_\eta X$,

$$\overline{\text{span}\{i_X(x)\, i_G(f)\, ; f \in C_c(G), x \in X\}} = G \times_\eta X.$$

□

Theorem 4.2. *Let* (G, η, X) *be a dynamical system on a Hilbert* C^**-module* X *and* Y *a full Hilbert* C^**-module over* B *with the property that there is a nondegenerate covariant morphism* $\left(j_G^X, j_X, j_G\right)$ *from* X *to* Y *which satisfies the following:*

(1) *for any nondegenerate covariant representation* $(v, \Phi_X, u, \mathcal{H}, \mathcal{K})$ *of* (G, η, X)*, there is a nondegenerate representation* $(\Phi_Y, \mathcal{H}, \mathcal{K})$ *of* Y *such that* $\overline{\Phi_Y} \circ j_X = \Phi_X$ *and* $\overline{\varphi_B} \circ j_G = u$*;*

(2) $Y = \overline{span\{j_X(x)\, j_G(f)\, ; x \in X, f \in C_c(G)\}}$.

Then there is an isomorphism of Hilbert C^**-modules* $J : Y \to G \times_\eta X$ *such that*

$$\overline{J} \circ j_X = i_X \text{ and } \overline{j} \circ j_G = i_G,$$

where j *is the underlying* C^**-morphism of* J.

Proof. Let $\left(\Phi_{G\times_\eta X}, \mathcal{H}, \mathcal{K}\right)$ be a faithful nondegenerate representation of $G \times_\eta X$ (see [1, Theorem 3.11]). Then $\overline{\Phi_{G\times_\eta X}} \circ i_X$ is a morphism of Hilbert C^*-modules and its underlying C^*-morphism is $\overline{\varphi_{G\times_{\alpha^\eta} A}} \circ i_A$. Since

$$\begin{aligned}&\left(\overline{\varphi_{G\times_{\alpha^\eta} A}} \circ i_G\right)(t)\left(\overline{\varphi_{G\times_{\alpha^\eta} A}} \circ i_A\right)(a)\left(\overline{\varphi_{G\times_{\alpha^\eta} A}} \circ i_G\right)\left(t^{-1}\right) \\ =\;& \overline{\varphi_{G\times_{\alpha^\eta} A}}\left(i_G(t)\, i_A(a)\, i_G\left(t^{-1}\right)\right) \\ =\;& \left(\overline{\varphi_{G\times_{\alpha^\eta} A}} \circ i_A\right)\left(\alpha_t^\eta(a)\right)\end{aligned}$$

for all $a \in A$ and for all $t \in G$, there is a strictly continuous morphism v from G to $\mathcal{U}(L(\mathcal{K}))$ such that $\left(v, \overline{\varphi_{G\times_\eta X}} \circ i_X, \overline{\varphi_{G\times_{\alpha^\eta} A}} \circ i_G, \mathcal{H}, \mathcal{K}\right)$ is a nondegenerate covariant representation of (G, η, X) (see [5, Lemma 3.4]). Then there is a nondegenerate representation $(\Phi_Y, \mathcal{H}, \mathcal{K})$ of Y such that $\overline{\Phi_Y} \circ j_X = \overline{\Phi_{G\times_\eta X}} \circ i_X$ and $\overline{\varphi_B} \circ j_G = \overline{\varphi_{G\times_{\alpha^\eta} A}} \circ i_G$. Since we have

$$\Phi_Y\left(j_X(x)\, j_G(f)\right) = \Phi_{G\times_\eta X}\left(i_X(x)\, i_G(f)\right)$$

for all $x \in X$ and for all $f \in C_c(G)$, it follows from the assertion (2) and Proposition 4.1 (2) that $[\Phi_Y(Y)] = \left[\Phi_{G\times_\eta X}(G \times_\eta X)\right]$. Let $J = \left(\Phi_{G\times_\eta X}\right)^{-1} \circ \Phi_Y$. Then J is a morphism of Hilbert C^*-modules from Y to $G \times_\eta X$ and its underlying C^*-morphism is $j = \left(\varphi_{G\times_{\alpha^\eta} A}\right)^{-1} \circ \varphi_B$. Moreover, $\overline{J} \circ j_X = i_X$ and $\overline{j} \circ j_G = i_G$.

In the same manner, we obtain a morphism of Hilbert C^*-modules $\Psi : G \times_\eta X \to Y$ such that $\overline{\Psi} \circ i_X = j_X$, $\overline{\psi} \circ i_G = j_G$. Then,

$$(J \circ \Psi)\left(i_X(x)\, i_G(f)\right) = i_X(x)\, i_G(f) \text{ and } (\Psi \circ J)\left(j_X(x)\, j_G(f)\right) = j_X(x)\, j_G(f)$$

for all $x \in X$ and for all $f \in C_c(G)$, and so J is an isomorphism of Hilbert C^*-modules. □

Suppose that (G, η, X) is a dynamical system on a Hilbert pro-C^*-module X. Then $G \times_\eta X = \lim\limits_{\leftarrow p} G \times_{\eta^p} X_p$ with $G \times_{\eta^p} X_p = (G \times_\eta X)_p$ for all $p \in S(A)$ (see, for example, [4, Lemma 5.3]). It is easy to check that $\left(i_{X_p}\right)_{p\in S(A)}$ is an inverse limit of injective linear maps. Let $i_X = \lim\limits_{\leftarrow p} i_{X_p}$. Then i_X is an injective morphism of Hilbert pro-C^*-modules and its underlying pro-C^*-morphism is $i_A = \lim\limits_{\leftarrow p} i_{A_p}$. For $t \in G$, $\left(i_{X_p}^G(t)\right)_{p\in S(A)}$ and $\left(i_G^p(t)\right)_{p\in S(A)}$ are inverse systems of unitaries in $L\left(G \times_{\eta^p} X_p\right)$, $p \in S(A)$, respectively $M(G \times_{\alpha^{\eta^p}} A_p), p \in S(A)$, and then $i_X^G(t) = \lim\limits_{\leftarrow p} i_{X_p}^G(t)$ and $i_G(t) = \lim\limits_{\leftarrow p} i_G^p(t)$ are unitaries in $L(G \times_\eta X)$, respectively $M(G \times_{\alpha^\eta} A)$, and the maps $t \to i_X^G(t)$ and $t \to i_G(t)$ are strictly continuous group morphisms from G to $\mathcal{U}(K(G \times_\eta X))$ respectively $\mathcal{U}(M(G \times_{\alpha^\eta} A))$. Since we have

$$\begin{aligned} i_X^G(t)\, i_X(x)\, i_G\left(t^{-1}\right) &= \lim_{\leftarrow p} i_{X_p}^G(t)\, i_{X_p}\left(\sigma_p^X(x)\right) i_G^p\left(t^{-1}\right) = \lim_{\leftarrow p} i_{X_p}\left(\eta_t^p\left(\sigma_p^X(x)\right)\right) \\ &= \lim_{\leftarrow p} i_{X_p}\left(\sigma_p^X\left(\eta_t(x)\right)\right) = i_X\left(\eta_t(x)\right) \end{aligned}$$

for all $t \in G$ and for all $x \in X$, we deduce that $\left(i_X^G, i_X, i_G\right)$ is an injective covariant morphism from X to $G \times_\eta X$, and since i_{X_p} is nondegenerate for all $p \in S(A)$, it is nondegenerate. Moreover,

$$\begin{aligned} & \overline{\operatorname{span}\{i_X(x)\, i_G(f), x \in X, f \in C_c(G)\}} \\ = & \lim_{\leftarrow p} \overline{\sigma_p^{G\times_\eta X}\left(\overline{\operatorname{span}\{i_X(x)\, i_G(f), x \in X, f \in C_c(G)\}}\right)} \\ = & \lim_{\leftarrow p} \overline{\operatorname{span}\{i_{Xp}\left(\sigma_p^X(x)\right) i_G^p(f), x \in X, f \in C_c(G)\}} \\ = & \lim_{\leftarrow p} G \times_{\eta^p} X_p = G \times_\eta X. \end{aligned}$$

Theorem 4.3. *Let (G, η, X) be a dynamical system on a Hilbert pro-C^*-module X and Y a full Hilbert pro-C^*-module over B with the property that there is a* *nondegenerate covariant morphism* $\left(j_{X,}^{G} j_X, j_G\right)$ *from* X *to* Y *which satisfies the following:*

(1) *for any nondegenerate covariant representation* $(v, \Phi_X, u, \mathcal{H}, \mathcal{K})$ *of* (G, η, X), *there is a nondegenerate representation* $(\Phi_Y, \mathcal{H}, \mathcal{K})$ *of* Y *such that* $\overline{\Phi_Y} \circ j_X = \Phi_X$ *and* $\overline{\varphi_B} \circ j_G = u$;

(2) $Y = \overline{span\{j_X(x)\, j_G(f)\, ; x \in X, f \in C_c(G)\}}$.

Then there is an isomorphism of pro-C^**-modules* $J : Y \to G \times_\eta X$ *such that*

$$\overline{J} \circ j_X = i_X \text{ and } \overline{j} \circ j_G = i_G,$$

where j *is the underlying pro-*C^**-morphism of* J.

Proof. The proof it is similar to the proof of Theorem 3.3. □

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