MUTUALLY ESSENTIALLY PSEUDO-INJECTIVE MODULES

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Abstract. Let M and N be two modules. M is called essentially pseudo N-injective if for any essential submodule A of N, any monomorphism $f : A \to M$ can be extended to some $g \in Hom(N, M)$. M is called essentially pseudo injective if M is essentially pseudo M-injective. Basic properties of mutually essentially pseudo injective modules and essentially pseudo injective modules are proved and their connections with pseudo-injective modules are addressed.

1. Introduction

Let M and N be two right R-modules over a ring R. M is called p*seudo-N-injective* if, for any submodule A of N, every monomorphism in $Hom_R(A, M)$ can be extended to an element of $Hom_R(N, M)$. M is called *pseudo-injective* if it is pseudo-M-injective [7]. For pseudo-injective modules and generalizations of pseudo-injective modules, we direct the reader to papers [1], [3], [7] and [10] for nice introduction to these topics in the literature. Following, [1], a module M is called *essentially pseudo N-injective* if for any essential submodule A of N, any monomorphism $f : A \to M$ can be extended to some $g \in Hom(N, M)$. M is called *essentially pseudo-injective* if M is essentially pseudo M-injective. They provided some properties of essentially pseudo-injective modules and it's applications in quasi-Frobenius ring theory. In this paper, we show some other characterizations of mutually essentially pseudo-injective modules and essentially pseudo-injective modules. For quasi Frobenius rings and V-rings via essentially pseudo-injective modules are presented.

In [1], they proved that a module M is essentially pseudo-injective if and only if it is invariant under monomorphismsin $End(E(M))$. In Theorem 2.3, we show that a module M is essentially pseudo N-injective if and only if $\alpha(N) \leq M$ for every monomorphism $\alpha : E(N) \to E(M)$.

In this paper, R will present an associative ring with identity and all modules over R are unitary right modules. We also write M_R to indicate that M is a right R-module. For a submodule N of M, we use $N \leq M$ $(N < M)$ and $N \leq^{\oplus} M$ to mean that N is a submodule of M (respectively, proper submodule), N is a direct summand of M, and we write $N \leq^e M$ to indicate that N is an essential submodule

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of M. Throughout this paper, homomorphisms of modules are written on the left of their arguments.

2. Mutually essentially pseudo-injective and essentially pseudo N-injective modules

Let M and N be two modules. M is called *essentially pseudo N-injective* if for any essential submodule A of N, any monomorphism $f : A \to M$ can be extended to some $g \in Hom(N, M)$. M is called *essentially pseudo-injective* if M is essentially pseudo M-injective.

It is easy to see that if M is pseudo N-injective then M is essentially pseudo N-injective. But the converse is not true in general.

Example 2.1. *Let p be a prime. Then $\mathbb{Z}$-module $\mathbb{Z}_{p^2}$ is essentially pseudo $\mathbb{Z} \oplus \mathbb{Z}_{p^3}-$injective and not pseudo $\mathbb{Z} \oplus \mathbb{Z}_{p^3}$ -injective.*

We first characterize essentially pseudo N-injective modules.

Proposition 2.2. *The following are equivalent for modules M and N:*
1. *M is essentially pseudo N-injective.*
2. *For any right R-module A, any essential monomorphism $g : A \to N$ and any monomorphism $f : A \to M$, there exists a homomorphism $h : N \to M$ such that $f = gh$.*

Proof. (1) $\Rightarrow$ (2). Let A be right R-module, $g : A \to N$ be an essential monomorphism and $f : A \to M$ be a monomorphism. Since $g : A \to N$ be an essential monomorphism, we have $g(A) \leq^e N$. We choose a homomorphisms $f' : g(A) \to M$ such that $f'(g(a)) = f(a)$ for all $a \in A$. It is clear that f' is a monomorphism. Since M is essentially pseudo N-injective, there exists a homomorphism $h : N \to M$ such that $h_{|_{g(A)}} = f'$. Hence, we have $(hg)(a) = h(g(a)) = f'(g(a)) = f(a)$. Thus $f = gh$.

(2) $\Rightarrow$ (1) is obvious. □

Theorem 2.3. *The following are equivalent for modules M and N:*
1. *M is essentially pseudo N-injective.*
2. *$\alpha(N) \leq M$ for every monomorphism $\alpha : E(N) \to E(M)$.*

Proof. (1) $\Rightarrow$ (2). Let $\alpha : E(N) \to E(M)$ be a monomorphism. Let $A = N \cap \alpha^{-1}(M)$. Note that $A \leq^e N$ and $\alpha(A) \leq M$. Then there exists some $g : N \to M$ such that $g(a) = \alpha(a)$ for all $a \in A$. Now we show that $g(n) = \alpha(n)$ for all $n \in N$. Assume that $g(n_0) \neq \alpha(n_0)$ for some $n_0 \in N$. Let $x = g(n_0) - \alpha(n_0) \in E(M)$. Since $M \leq^e E(M)$, there exists $r \in R$ such that $0 \neq xr = g(n_0r) - \alpha(n_0r) \in M$. It follows that $\alpha(n_0r) \in M$. That means $n_0r \in A$. Therefore $\alpha(n_0r) = g(n_0r)$ and hence $xr = 0$, a contradiction.

(2) $\Rightarrow$ (1) Let $f : A \to M$ be a monomorphism with $A \leq^e N$. It is clear that $E(A) = E(N)$. Since $A \leq^e N$, there exists some monomorphism $g : E(N) \to E(M)$ such that $g_{|_A} = f$. Therefore $g(N) \leq M$ and g is the desired extension of f, i.e., M is essentially pseudo N-injective. □

Corollary 2.4 ([1, Corollary 2.12])**.** *The following conditions are equivalent:*

(1) M is essentially pseudo-injective.
(2) $\alpha(M) \leq M$ for every monomorphism α of $E(M)$.

A submodule N of M is said to be a *fully invariant* if $f(N)$ is contained in N for every $f \in End(M_R)$. Clearly, 0 and M are fully invariant submodules of M.

Theorem 2.5. *The followings are equivalent for module M:*

(1) Every submodule of M is essentially pseudo-injective.
(2) M is essentially pseudo-injective and every essential submodule of M is fully invariant under monomorphism of M.
(3) Every essential submodule of M is essentially pseudo-injective.

Proof. (1) $\Rightarrow$ (2). Let f be a monomorphism of M. There exists a monomorphism g of $E(M)$ such that g extends of f. Then for every essential submodule H of M, $g(H) \leq H$ or $f(H) \leq H$ (since $E(H) = E(M)$).

(2) $\Rightarrow$ (3). Let H be an essential submodule of M. Let $f : A \to H$ be a monomorphism with $A \leq^e H$. There exists a monomorphism g of $E(M)$ that is extension of f. It follows that $g(H) \leq H$ and so $g|_H$ extends of f.

(3) $\Rightarrow$ (1). Assume that H be a submodule of M. There exists a module K of M such that $H \oplus K \leq^e M$. By (3), $H \oplus K$ is essentially pseudo-injective. Thus H is too. □

It is well known in the literature that many of the basic properties of pseudo-injective modules. We first list here in Proposition 2.6 several such properties, and the proof for the sake of completeness.

Two modules M, N are called *mutually essentially pseudo-injective* if M is essentially pseudo N-injective and N is essentially pseudo M-injective.

Proposition 2.6. *Let M and N be modules.*

(1) M is essentially pseudo N-injective if and only if M is essentially pseudo K-injective for all essential submodules K of N.
(2) If M is essentially pseudo N-injective and $K \simeq N$, then M is essentially pseudo K-injective.
(3) Assume that M and N are essentially pseudo-injective modules. If there exists isomorphism between submodules A and B such that $A \leq^e N$ and $B \leq^e M$, then $M \simeq N$.
(4) Assume that A and B be are mutually essentially pseudo-injective modules. If $E(A) \simeq E(B)$, then every isomorphism $E(A) \to E(B)$ reduces an isomorphism $A \to B$, in particular $A \simeq B$. Consequently, A and B are essentially pseudo-injective.
(5) If M is essential pseudo N-injective then essential monomorphism $f : M \to N$ splits.

Proof. (1) Let $L \leq^e K \leq^e N$ and $f : L \to M$ be a monomorphism. First we note that $E(L) = E(K) = E(N)$. Then there exists some monomorphism $g : E(A) \to E(B)$ such that $g|_L = f$. Since M is essentially pseudo N-injective, we can obtain that $g(N) \leq M$ by Theorem 2.3. Therefore $g(K) \leq M$.

(2) Let $L \leq^e K$ and $g : K \to N$ be an isomorphism. Clearly, $g(L) \leq^e N$. Let $f : L \to M$ be a monomorphism. It is clear that there exists a monomorphism $fg' : g(L) \to M$, where $g' : g(L) \to L$ is a monomorphism. Since M is essentially pseudo N-injective, the composition map fg' can be extended to $h : N \to M$. Hence $hg : K \to M$ is desired homomorphism.

(3) Let $f : A \to B$ be an isomorphism. Since M is essentially pseudo N-injective, there is a homomorphism $g : E(N) \to E(M)$ such that $g|_A = f$. Since $A \leq^e N$ and $B \leq^e M$, we can obtain that g is an isomorphism. Hence $g(N) \leq M$ and $g^{-1}(M) \leq N$ by Theorem 2.3. Thus $g|_N : N \to M$ is an isomorphism.

(4) Let $g : E(A) \to E(B)$ be an isomorphism. Since B is essentially pseudo A-injective, $g(A) \leq B$ by Theorem 2.3. Similarly $g^{-1}(B) \leq A$. Hence $B = (gg^{-1})(B) = g((g^{-1})(B)) \leq g(A) \leq B$. Consequently $g(A) = B$, and hence $g|_A : A \to B$ is an isomorphism. Since A is essentially pseudo B-injective and $B \simeq A$, we can obtain that A is essentially pseudo A-injective, i.e., A is essentially pseudo-injective.

(5) Let $f : M \to N$ be an essential monomorphism, and $f^{-1} : f(M) \to M$ be the inverse of f. Since M is essential pseudo N-injective and $f(M) \leq^e N$, there is a homomorphism $g : N \to M$ that extends f^{-1}. Let $h = gf$. Then h is clearly an identity map of M. Thus $f(M)$ splits in N, as desired. □

Our aim in the following proposition is to investigate sufficient and necessary conditions which ensure that a module is essentially pseudo N-injective.

Theorem 2.7. *Let M and N be modules. Then*

1. *N is a semisimple module if and only if M is essentially pseudo N-injective for all module M.*
2. *Assume that $N = A \oplus B$ and $M = C \oplus D$ such that B is embedded in D. If M is essentially pseudo N-injective, then C is essentially pseudo A-injective.*

Proof. (1) Let $A \leq N$ and $C \leq N$ such that $A \oplus C \leq^e N$. Assume that $\iota : A \oplus C \to N$ is the inclusion homomorphism. Since $A \oplus C$ is essentially pseudo N-injective, there exists $f : N \to A \oplus C$ such that $f\iota = 1_{A \oplus C}$. It follows that $N = A \oplus C$.

The converse is clear.

(2) Assume that $\alpha : B \to D$ is a monomorphism. Let $f : H \to C$ be a monomorphism with $H \leq^e A$. Then $f \oplus \alpha : H \oplus B \to M$ is a monomorphism. Since M is essentially pseudo N-injective, there exists a homomorphism $g : N \to M$ that such g is extension of $f \oplus \alpha$. Let $\bar{f} = \pi g \iota : A \to C$ with $\pi : M \to C$ be the projection and $\iota : A \to C$ be the inclusion. Then $\bar{f}|_H = f$. Thus C is essentially pseudo A-injective.

□

Corollary 2.8. *Every direct summand of an essentially pseudo-injective is essentially pseudo-injective.*

Proof. Clear from Theorem 2.7. □

Let's continue to obtain other properties of essentially pseudo injective modules.

Theorem 2.9. *Let $M = M_1 \oplus M_2$ and $E(M_1), E(M_2)$ be invariant submodules under any monomorphism of $E(M)$. Then M is essentially pseudo-injective if and only if M_1, M_2 are essentially pseudo-injective.*

Proof. ($\Rightarrow$) by Corollary 2.8.

($\Leftarrow$). It is well know that $E(M) = E(M_1) \oplus E(M_2)$. Let $\alpha : E(M) \to E(M)$ be a monomorphism. Then $\alpha|_{E(M_i)} : E(M_i) \to E(M_i)$ is monomorphism. Since M_1, M_2 are essentially pseudo-injective,

$$\begin{aligned}\alpha(M) &= \alpha(M_1 + M_2)\\ &= \alpha(M_1) + \alpha(M_2)\\ &= \alpha|_{E(M_1)}(M_1) + \alpha|_{E(M_2)}(M_2)\\ &\leq M_1 + M_2 = M\end{aligned}$$

.

Thus M is essentially pseudo-injective by Corollary 2.4. □

A ring R is called *right essentially pseudo-injective* if R_R is an essentially pseudo-injective module.

Recall that an R-module M is called *self-generator* if, for each submodule N of M, there exists an index set J and an epimorphism $\theta : M^{(J)} \to N$.

Theorem 2.10. *Let M be a self-generator module. If $End(M)$ is right essentially pseudo-injective, then M is essentially pseudo-injective.*

Proof. Let $S = End(M)$ and $f : A \to M$ be a monomorphism with $A \leq^e M$. Let $I = \{g \in S |\ g(M) \leq A\}$. Then I is a right ideal of S. We show that I is an essential right ideal of S. In fact, for all $0 \neq s \in S$, then $s(m_0) \neq 0$ for some $m_0 \in M$. Since $A \leq^e M$, there exists $r \in R$ such that $0 \neq s(m_0 r) \in A$ or $s(m_0 rR) \leq A$. On the other hand, since M is self-generator, we write $m_0 rR = \sum_{u \in K} u(M)$ for some $K \subset S$. But $m_0 rR \neq 0$ implies that there exists $u \in K$ such that $0 \neq su(M) \leq A$ or $0 \neq su \in I$. Define $\phi : I \to S_S$ by $\phi(g) = fg$. Since f is a monomorphism, ϕ is also an S-monomorphism. Since S is right essentially pseudo-injective, $\phi(g) = \bar{f}g$ for some $\bar{f} \in S$. It follows that $\bar{f}g = fg$ for all $g \in I$. For every $a \in A$, there exists $u_1, \dots, u_k \in I$, $m_1, \dots, m_k \in M$ such that $a = u_1(m_1) + \cdots + u_k(m_k)$. Hence we have

$$\begin{aligned}\bar{f}(a) &= \bar{f}u_1(m_1) + \cdots + \bar{f}u_k(m_k)\\ &= fu_1(m_1) + \cdots + fu_k(m_k)\\ &= f(a).\end{aligned}$$

Thus $\bar{f}$ is extension of f. □

Let R be a ring and Ω a class of R-modules, Ω is called *socle fine* whenever for any $M, N \in \Omega$, we have $Soc(M) \simeq Soc(N)$ if and only if $M \simeq N$ ([8]).

A module M is said to be *strongly essentially pseudo-injective* if, M is essentially pseudo N-injective for all right R-module N. We denote by $\mathcal{SE}$ the class of strongly essentially pseudo-injective right R-modules and $\mathcal{PR}$ the class of projective right R-modules.

Theorem 2.11. *The following conditions are equivalent for ring R.*
(1) R is quasi Frobenius.
(2) The class $\mathcal{PR} \cup \mathcal{SE}$ is socle fine.

Proof. (1) $\Rightarrow$ (2) If R is quasi-Frobenius, then projective R-modules are injective. Thus $\mathcal{PR} \cup \mathcal{SE} = \mathcal{SE}$. Let $M, N \in \mathcal{SE}$ with $Soc(M) \simeq Soc(N)$. It follows that $E(Soc(M)) \simeq E(Soc(N))$. Since R is right Artinian, $Soc(M) \leq^e M$ and $Soc(N) \leq^e N$. Hence $E(M) \simeq E(N)$. Then, by Proposition 2.6 (4), we can obtain that $M \simeq N$. It follows that the class $\mathcal{PR} \cup \mathcal{SE}$ is socle fine.
(2) $\Rightarrow$ (1) Let P be a projective right R-module. Then $P \in \mathcal{PR}, E(P) \in \mathcal{SE}$ and $Soc(P) = Soc(E(P))$. By (2), we get $P \simeq E(P)$ and hence P is injective. It follows that R is quasi-Frobenius. □

Theorem 2.12. *The following conditons are equivalent for ring R.*
(1) R is semisimple.
(2) The class of all essentially pseudo-injective modules is socle fine.
(3) The class $\mathcal{SE}$ is socle fine.

Proof. (1) $\Rightarrow$ (2) Since over every semisimple ring R then the class of all R-modules is socle fine.
(2) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (1) Clearly $Soc(E(R_R)) = Soc(Soc(R_R))$. Since $E(R_R)$ and $Soc(M)$ are essentially pseudo-injective, we can obtain that $E(R_R) \simeq Soc(R_R)$ by (3). It implies that $E(R_R)$ is semisimple and so R is semisimple. □

Let's continue to study on some well known rings.

Theorem 2.13. *Let R be right essentially pseudo-injective. If $e^2 = e \in R$ satisfies $ReR = R$, then $S = eRe$ is right essentially pseudo-injective.*

Proof. Let $\theta : T \to S_S$ be an essential S-monomorphism, with T is an essential right ideal of S. We define $h : TR \to R_R$ by $h(\sum_i t_i r_i) = \sum_i \theta(t_i) r_i$ for all $t_i \in T$ and $r_i \in R$. Assume that $\sum_i t_i r_i = 0$. Then, for all $r \in R$, $\sum_i t_i r_i re = 0$ or $\sum_i t_i(er_ire) = 0$. It follows that $\sum_i \theta(t_i(er_ire)) = 0$ or $\sum_i \theta(t_i)(er_ire) = 0$, which implies that $\sum_i \theta(t_i) r_i re = 0$. Thus $\sum_i \theta(t_i) r_i = 0$. That means θ is a well-defined right R-homomorphism. Repeating this process, h is also an R-monomorphism. Next, we claim that $TR \leq^e eR$ and $Im(h) \leq^e eR$. In fact, for all $ex \in eR$ with $ex \neq 0$, there exists $x_0 \in R$ such that $exx_0e \neq 0$. Since $T \leq^e S_S$, there exists $ex_1e \in S$ such that $0 \neq (exx_0e)ex_1e \in T$ or $0 \neq (ex)(x_0ex_1e) \in TR$. Thus $TR \leq^e eR$. Therefore $TR \oplus (1-e)R \leq^e R_R$ and $Im(h) \oplus (1-e)R \leq^e R_R$. They imply that there exists an essential R-monomorphism $g : TR \oplus (1-e)R \to R_R$ that extends h. Since R is right essentially pseudo-injective, g can be extended to a R-homomorphism $\phi : R_R \to R_R$. There exists $c \in R$ such that $\phi(x) = cr$ for all

$r \in R$. Then $\theta(t) = e\theta(t) = e\phi(t) = ect = ecet$. Let $\bar{\theta} : S_S \to S_S$ via $\bar{\theta}(s) = (ece)s$ for all $s \in S$. Then $\bar{\theta}$ is an S-homomorphism that extends θ. □

The following example shows that the assumption "$ReR = R$" is not superfluous in Theorem 2.13.

Example 2.14. *Let R be as in* [9, Example 9]. *That is, let R be the algebra of matrices, over a field K, of the form*

$$\begin{pmatrix} a & x & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & y & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & z \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}$$

Let $e = e_{11} + e_{22} + e_{33} + e_{44} + e_{55}$, where e_{ij} matrixes units. Then e is an idempotent of R and $ReR \neq R$. Moreover, R is right essentially pseudo-injective, but $S = eRe$ is not right essentially pseudo-injective.

M is called a *V-module* by Hirano in [6] (or *cosemisimple* by Fuller [5]) if every proper submodule of M is an intersection of maximal submodules. R is called a *V-ring* if the right module R_R is a V-module.

It is well know that M is a V-module if and only if every simple module is M-injective.

Theorem 2.15. *Let R be a ring.*

(1) Every direct sum of two essentially pseudo-injective module is essentially pseudo-injective if and only if every essentially pseudo-injective is injective. Furthermore, R is a right V-ring and right Noetherian.

(2) Essential extensions of semi-simple right R-modules are essentially pseudo-injective if and only if R is right V-ring and right Noetherian.

Proof. (1) Let M be an essentially pseudo-injective module. Then $M \oplus E(M)$ is an essentially pseudo-injective module and so M is injective. The converse is clear.

Since every semisimple right R-module is injective, R is a right V-ring and a right Noetherian ring.

(2) Let M be a semisimple module. Then $M \oplus E(M)$ is an essential extensions of a semi-simple module. It follows that $M \oplus E(M)$ is an essentially pseudo-injective module and so M is injective. Thus R is a right V-ring and a right Noetherian ring. The converse is clear because every semi-simple right R-module is injective. □

We finish this section with the following two ring extensions.

Theorem 2.16. *Let M be a $S-R$-bimodule. Assume that* $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$ *is right essentially pseudo-injective. Then*

(1) R is right essentially pseudo-injective

(2) If ${}_SM$ is faithful then M_R is essentially pseudo-injective.

Proof. (1) Let I be an essential right ideal of R and $f : I \to R$ a monomorphism. Let $\bar{I} = \begin{pmatrix} S & M \\ 0 & I \end{pmatrix}$. It is easy to see that $\bar{I}$ is an essential right ideal of T. We define $\theta : \bar{I} \to T$ via $\theta(\begin{pmatrix} s & m \\ 0 & r \end{pmatrix}) = \begin{pmatrix} s & m \\ 0 & f(r) \end{pmatrix}$. Then θ is a T-monomorphism. By the hypothesis, there exists $\begin{pmatrix} s_0 & m_0 \\ 0 & r_0 \end{pmatrix} \in T$ such that $\theta = \begin{pmatrix} s_0 & m_0 \\ 0 & r_0 \end{pmatrix}$. Thus $f(r) = r_0$.
(2) Assume that that ${}_SM$ is faithful, N is an essential submodule of M and $f : N \to M$ is a monomorphism. Let $\bar{N} = \begin{pmatrix} 0 & N \\ 0 & R \end{pmatrix}$. It is easy to see that $\bar{N}$ is an essential right ideal of T. We define $\theta : \bar{N} \to T$ via $\theta(\begin{pmatrix} 0 & n \\ 0 & r \end{pmatrix}) = \begin{pmatrix} 0 & f(n) \\ 0 & r \end{pmatrix}$. Then θ is a T-monomorphism. By the hypothesis, there exists a T-homomorphism $\phi : T \to T$ extension of θ. We can define an R-homomorphism $\bar{f} : M \to M$ as following: For every $x \in M$, $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in T$ and $\theta(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} s_x & m_x \\ 0 & r_x \end{pmatrix}$ for some $s_x \in S, r_x \in R$ and $m_x \in M$, $\bar{f}(x) = m_x$. Then $\bar{f}$ is well-defined and an extension of f. □

The converse of Theorem 2.16 is not true. In fact, let $S = M = R = K$ with a field K. Then R is right essentially pseudo-injective, ${}_SM$ is faithful and M_R is essentially pseudo-injective. But T is not right essentially pseudo-injective. Because, in case T is right essentially pseudo-injective, then T must be a right self-injective ring, which is a contradiction.

Theorem 2.17. *Let R be a ring. If polynomial ring $R[x]$ is right essentially pseudo-injective, then R is right essentially pseudo-injective.*

Proof. Let $\varphi : I \to R_R$ be a monomorphism with $I \leq^e R_R$. Then $\beta : I[x] \to R[x]$ with $\beta(\sum_n a_n x^n) = \sum_n f(a_n)x^n$ is an $R[x]$-monomorphism. It is easy to see that $Im(\beta) = Im(\varphi)[x]$. Since $R[x]$ is right essentially pseudo-injective, there exists $f_0 = \sum_{n=0}^{k} f(c_n)x^n \in R[x]$ such that $\beta = f_0 \cdot$. For all $u \in I$, we have $\alpha(u) = c_0 u$. Therefore α can be extended to an endomorphism of R_R. Thus R is a right essentially pseudo-injective ring. □

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