FEKETE-SZEGÖ PROBLEM FOR A CERTAIN SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. Given a starlike function $g \in \mathcal{S}^*$, an analytic standardly normalized function f in the unit disk $\mathbb{D}$ is called close-to-convex with respect to g if there exists $\delta \in (-\pi/2, \pi/2)$ such that

$$\operatorname{Re}\left\{e^{i\delta}\frac{zf'(z)}{g(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

For the class $\mathcal{C}(h)$ of all close-to-convex functions with respect to $h(z) := z/(1-z)$, $z \in \mathbb{D}$, a Fekete-Szegö problem is examined.

1. Introduction

A classical problem settled by Fekete and Szegö [9] is to find for each $\lambda \in [0,1]$ the maximum value of the coefficient functional

$$\Phi_\lambda(f) := \left|a_3 - \lambda a_2^2\right|$$

over the class $\mathcal{S}$ of univalent functions f in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

By applying the Loewner method they proved that

$$\max_{f\in\mathcal{S}} \Phi_\lambda(f) = \begin{cases} 1 + 2\exp\left(-2\lambda/(1-\lambda)\right), & \lambda \in [0,1), \\ 1, & \lambda = 1. \end{cases}$$

The problem of calculating $\max_{f\in\mathcal{F}} \Phi_\lambda(f)$ for various compact subclasses $\mathcal{F}$ of the class of all normalized analytic functions f in $\mathbb{D}$ of the form (1.1), as well as for λ being an arbitrary real or complex number, was considered by many authors (see e.g. [12], [15], [26], [17], [13], [6], [2]).

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Let $\mathcal{S}^*$ denote the class of starlike functions, i.e., $f \in \mathcal{S}^*$ if f is of the form (1.1) and

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{D}.$$

Given $\delta \in (-\pi/2, \pi/2)$ and $g \in \mathcal{S}^*$, a function f of the form (1.1) is called close-to-convex with argument δ with respect to g if

$$\operatorname{Re} \left\{ e^{i\delta} \frac{zf'(z)}{g(z)} \right\} > 0, \quad z \in \mathbb{D}. \tag{1.2}$$

Let $\mathcal{C}_\delta(g)$ denote the class of all such functions. Let

$$\mathcal{C}(g) := \bigcup_{\delta \in (-\pi/2, \pi/2)} \mathcal{C}_\delta(g), \quad \mathcal{C}_\delta := \bigcup_{g \in \mathcal{S}^*} \mathcal{C}_\delta(g)$$

be the classes of functions called close-to-convex with respect to g and close-to-convex with argument δ, respectively (see [25, pp. 184-185], [11]). At the end let

$$\mathcal{C} := \bigcup_{\delta \in (-\pi/2, \pi/2)} \mathcal{C}_\delta = \bigcup_{\delta \in (-\pi/2, \pi/2)} \bigcup_{g \in \mathcal{S}^*} \mathcal{C}_\delta(g)$$

denote the class of close-to-convex functions (see [25], [14]). It is well known that $\mathcal{S}^*$ and $\mathcal{C}$ are the subclasses of $\mathcal{S}$.

Using a specific starlike function from $\mathcal{S}^*$ the inequality (1.2) defines related subclass of close-to-convex functions, namely, $\mathcal{C}_\delta(g)$. Two important ones are given by the Koebe function

$$k(z) := \frac{z}{(1-z)^2}, \quad z \in \mathbb{D}.$$

and by the convex function

$$h(z) := \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n, \quad z \in \mathbb{D}, \tag{1.3}$$

i.e., the classes of analytic functions f of the form (1.1) are defined, respectively, by the following conditions:

$$\operatorname{Re} \left\{ e^{i\delta} (1-z)^2 f'(z) \right\} > 0, \quad z \in \mathbb{D}, \tag{1.4}$$

and

$$\operatorname{Re} \left\{ e^{i\delta} (1-z) f'(z) \right\} > 0, \quad z \in \mathbb{D}, \tag{1.5}$$

where $\delta \in (-\pi/2, \pi/2)$.

For the first time the inequalities (1.4) and (1.5), treated as the univalence criteria, were distinguished explicitly, probably, in [25, p. 185], where some coefficients results for both classes were shown, as well. Clearly, h as in (1.3), and k have integer coefficients in their power series in $\mathbb{D}$. It is known that there are only nine such starlike functions (see e.g. [10], [24]). Therefore starlike functions with integer coefficients, and the corresponding classes $\mathcal{C}(g)$ of close-to-convex functions with respect to g, as well as their generalizations, are the subject of studies by many authors with using various techniques (some recent results see e.g. [3], [20], [21], [5], [7], [28], [23], [4]).

Since the Koebe function k and the function h are extremal for various computational problems in the class of starlike and convex univalent functions, respectively, it is interesting to examine the Fekete-Szegö functional for the classes $\mathcal{C}(k)$ and $\mathcal{C}(h)$. For the whole class $\mathcal{C}$ of close-to-convex functions, the sharp bound of the Fekete-Szegö functional was calculated by Koepf in [17] who extended the earlier result for the class $\mathcal{C}_0$ due to Keogh and Merkes [15], namely, it was proved that

$$\max_{f\in\mathcal{C}} \Phi_\lambda(f) = \max_{f\in\mathcal{C}_0} \Phi_\lambda(f)$$

$$= \begin{cases} |3-4\lambda|, & \lambda\in(-\infty,1/3]\cup[1,+\infty), \\ 1/3+4/(9\lambda), & \lambda\in[1/3,2/3], \\ 1, & \lambda\in[2/3,1]. \end{cases}$$

For further results on the Fekete-Szegö functional for classes of close-to-convex functions, particularly, for strongly close-to-convex functions see [18], [1], [22], [8] and [16].

In [19] the authors considered the Fekete and Szegö problem for the class $\mathcal{C}(k)$. It was shown that

$$\max_{f\in\mathcal{C}(k)} \Phi_\lambda(f)$$

$$\leq \begin{cases} |3-4\lambda|, & \lambda\in(-\infty,1/3]\cup[1,+\infty), \\ \dfrac{1}{3}\cdot\dfrac{(2-3\lambda)^2}{2-|2-3\lambda|}+|1-\lambda|+\dfrac{2}{3}, & \lambda\in[1/3,1], \end{cases}$$

with sharpness of the result when $\lambda\in\mathbb{R}\setminus(2/3,1)$.

In this paper we examine the Fekete and Szegö problem for the class $\mathcal{C}(h)$. We show that

$$\max_{f\in\mathcal{C}(h)} \Phi_\lambda(f)$$

$$\leq \begin{cases} \left|\dfrac{1}{3}-\dfrac{1}{4}\lambda\right|+\dfrac{2}{3}|2-3\lambda|, & \lambda\in(-\infty,2/9]\cup[10/9,+\infty), \\ \dfrac{1}{12}\cdot\dfrac{(2-3\lambda)^2}{2-|2-3\lambda|}+\left|\dfrac{1}{3}-\dfrac{1}{4}\lambda\right|+\dfrac{2}{3}, & \lambda\in[2/9,10/9], \end{cases}$$

with sharpness of the result when $\lambda\in\mathbb{R}\setminus(2/3,4/3)$.

2. Main result

By $\mathcal{P}$ we denote the class of all analytic functions p in $\mathbb{D}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{2.1}$$

having a positive real part in $\mathbb{D}$. Let

$$L(z) := \frac{1+z}{1-z}, \quad z \in \mathbb{C} \setminus \{1\}.$$

For each $\varepsilon \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ let

$$p_\varepsilon(z) := L(\varepsilon z), \quad z \in \mathbb{D}.$$

Clearly $p_\varepsilon \in \mathcal{P}$ for every $\varepsilon \in \mathbb{T}$.

The inequalities (2.2) and (2.3) below are well known. They can be found in [27, pp. 41 and 166].

Lemma 2.1. *If $p \in \mathcal{P}$ is of the form* (2.1)*, then*

$$|c_n| \le 2, \quad n \in \mathbb{N}, \tag{2.2}$$

and

$$\left| c_2 - \frac{c_1^2}{2} \right| \le 2 - \frac{|c_1|^2}{2}. \tag{2.3}$$

Both inequalities are sharp. The equality in (2.2) *holds for every function $p_\varepsilon \in \mathcal{P}$, $\varepsilon \in \mathbb{T}$. The equality in* (2.3) *holds for every function*

$$\begin{aligned} p_{t,\theta}(z) &:= tL\left(e^{i\theta}z\right) + (1-t)L\left(e^{2i\theta}z^2\right) \\ &= 1 + 2te^{i\theta}z + 2e^{2i\theta}z^2 + \cdots, \quad z \in \mathbb{D}, \end{aligned} \tag{2.4}$$

where $t \in [0,1]$ and $\theta \in \mathbb{R}$.

Now we prove the main theorem of this paper. The source of the method of proof is in Koepf's work [17], where the upper bound of Φ_λ for close-to-convex functions with λ restricted to the interval $(1/2, 2/3)$ was calculated. However we use the technique homogenously for the class $\mathcal{C}(h)$ for all real λ, analogously as in [19] for the class $\mathcal{C}(k)$.

Theorem 2.2.

$$\max_{f \in \mathcal{C}(h)} \Phi_\lambda(f) \tag{2.5}$$

$$\le \begin{cases} \left| \dfrac{1}{3} - \dfrac{1}{4}\lambda \right| + \dfrac{2}{3}|2 - 3\lambda|, & \lambda \in (-\infty, 2/9] \cup [10/9, +\infty), \\ \dfrac{1}{12} \cdot \dfrac{(2-3\lambda)^2}{2 - |2 - 3\lambda|} + \left| \dfrac{1}{3} - \dfrac{1}{4}\lambda \right| + \dfrac{2}{3}, & \lambda \in [2/9, 10/9]. \end{cases}$$

For each $\lambda \in \mathbb{R}\setminus(2/3,4/3)$, *the inequality is sharp and the equality is attained by a function in* $\mathcal{C}_0(h)$. *In particular, for each* $\lambda \in [2/9,2/3]$ *the second equality is attained by the function* f_λ *given by the differential equation*

$$f_\lambda'(z) = \frac{1}{1-z} p_{t_\lambda,0}(z), \quad f_\lambda(0)=0,\ z\in\mathbb{D}, \tag{2.6}$$

where $t_\lambda := 1/(3\lambda) - 1/2$. *For each* $\lambda \in (-\infty,2/9]\cup[4/3,+\infty)$, *the first equality is attained by the function*

$$f_{2/9}(z) := \log(1-z) + \frac{2z}{1-z}, \quad \log 1 = 0,\ z\in\mathbb{D}. \tag{2.7}$$

Proof. Observe that $f \in \mathcal{C}(h)$ if and only if

$$e^{i\delta}(1-z)f'(z) = p(z)\cos\delta + i\sin\delta, \quad z\in\mathbb{D}, \tag{2.8}$$

for some $\delta \in (-\pi/2,\pi/2)$ and $p\in\mathcal{P}$. Thus

$$zf'(z) = e^{-i\delta}h(z)\left(p(z)\cos\delta + i\sin\delta\right), \quad z\in\mathbb{D}. \tag{2.9}$$

Setting the series (1.1), (1.3) and (2.1) into (2.9), by comparing coefficients, we get

$$\begin{aligned} a_2 &= \frac{1}{2}\left(c_1 e^{-i\delta}\cos\delta + 1\right), \\ a_3 &= \frac{1}{3}\left(c_2 e^{-i\delta}\cos\delta + c_1 e^{-i\delta}\cos\delta + 1\right). \end{aligned} \tag{2.10}$$

Let $\lambda\in\mathbb{R}$. Using (2.3) from the above we have

$$\begin{aligned} \Phi_\lambda(f) &= \left|a_3 - \lambda a_2^2\right| \\ &= \left|\frac{1}{3}c_2 e^{-i\delta}\cos\delta + \frac{1}{3}c_1 e^{-i\delta}\cos\delta + \frac{1}{3}\right. \\ &\qquad \left. -\frac{1}{4}\lambda\left(c_1^2 e^{-2i\delta}\cos^2\delta + 2c_1 e^{-i\delta}\cos\delta + 1\right)\right| \\ &= \left|\frac{1}{3} - \frac{1}{4}\lambda + \frac{1}{3}\left(c_2 - \frac{c_1^2}{2}\right)e^{-i\delta}\cos\delta + \frac{1}{6}c_1^2\left(1 - \frac{3}{2}\lambda e^{-i\delta}\cos\delta\right)e^{-i\delta}\cos\delta \right. \\ &\qquad \left. + \left(\frac{1}{3} - \frac{1}{2}\lambda\right)c_1 e^{-i\delta}\cos\delta\right| \\ &\le \left|\frac{1}{3} - \frac{1}{4}\lambda\right| + \frac{1}{3}\left(2 - \frac{|c_1|^2}{2}\right)\cos\delta + \frac{|c_1|^2}{6}\left|1 - \frac{3}{2}\lambda e^{-i\delta}\cos\delta\right|\cos\delta \\ &\qquad + \left|\frac{1}{3} - \frac{1}{2}\lambda\right||c_1|\cos\delta \end{aligned} \tag{2.11}$$

$$= \left|\frac{1}{3} - \frac{1}{4}\lambda\right|$$

$$+ \left(\frac{2}{3} + \frac{|c_1|^2}{6}\left(\sqrt{1 - \left(3\lambda - \frac{9}{4}\lambda^2\right)\cos^2\delta} - 1\right) + \frac{1}{6}\,|2-3\lambda|\,|c_1|\right)\cos\delta.$$

Set $x := |c_1|$ and $y := \cos\delta$. Clearly, $y \in (0,1]$ and, in view of (2.2), $x \in [0,2]$. Set $R := [0,2] \times [0,1]$. It is convenient to use in further computation $\gamma := 2 - 3\lambda$ instead of λ. For $(x,y) \in R$ and $\gamma \in \mathbb{R}$ define

$$F_\gamma(x,y)$$

$$:= \frac{1}{12}|2+\gamma| + \frac{1}{3}\left(2 + \frac{x^2}{2}\left(\sqrt{1 - \left(1 - \frac{1}{4}\gamma^2\right)y^2} - 1\right) + \frac{1}{2}|\gamma|x\right)y.$$

Consequently, in view of (2.11) we have

$$\max_{f\in\mathcal{C}(h)} \Phi_\lambda(f) \le \max_{(x,y)\in R} F_\gamma(x,y).$$

Now for each $\gamma \in \mathbb{R}$ we find the maximum value of F_γ on the rectangle R.

1. In the corners of R we have

$$\begin{aligned} &F_\gamma(0,0) = F_\gamma(2,0) = \frac{1}{12}|2+\gamma|,\\ &F_\gamma(0,1) = \frac{1}{12}|2+\gamma| + \frac{2}{3}, \quad F_\gamma(2,1) = \frac{1}{12}|2+\gamma| + \frac{2}{3}|\gamma|. \end{aligned} \tag{2.12}$$

2. $x = 0,\ y \in (0,1)$.
Then a linear function

$$(0,1) \ni y \mapsto F_\gamma(0,y) = \frac{1}{12}|2+\gamma| + \frac{2}{3}y$$

has no critical point in $(0,1)$, evidently.

3. $x \in (0,2),\ y = 0$.
Then we have a constant function

$$(0,2) \ni x \mapsto F_\gamma(x,0) = \frac{1}{12}|2+\gamma|. \tag{2.13}$$

4. $x \in (0,2),\ y = 1$.
Let

$$\begin{aligned} G_\gamma(x) &:= F_\gamma(x,1)\\ &= \frac{1}{12}\left(|\gamma|-2\right)x^2 + \frac{1}{6}|\gamma|x + \frac{1}{12}|2+\gamma| + \frac{2}{3}. \end{aligned} \tag{2.14}$$

(a) For $|\gamma| = 2$ we get the linear functions G_{-2} and G_2 which have no critical points in $(0,2)$.

(b) Let $|\gamma| \neq 2$. Then $G'_\gamma(x) = 0$ if and only if

$$x = \frac{|\gamma|}{2-|\gamma|} =: x_\gamma. \tag{2.15}$$

Hence $x_\gamma \in (0,2)$ if and only if

$$0 < \frac{|\gamma|}{2-|\gamma|} < 2.$$

The left-hand inequality holds when

$$\gamma \neq 0 \quad \text{and} \quad |\gamma| < 2. \tag{2.16}$$

We can write the right-hand inequality as

$$\frac{3|\gamma|-4}{2-|\gamma|} < 0$$

and, in view of (2.16), it holds when $|\gamma| < 4/3$. This with (2.16) yield that $x_\gamma \in (0,2)$ when $\gamma \in (-4/3, 4/3) \setminus \{0\}$.

Thus, taking into account of part (a), we have that the function G_γ has a critical point in $(0,2)$, namely, x_γ as the unique one, if and only if $\gamma \in (-4/3, 4/3) \setminus \{0\}$.

Moreover we have

$$\begin{aligned} F_\gamma(x_\gamma, 1) &= G_\gamma(x_\gamma) \\ &= \frac{1}{12} \cdot \frac{(|\gamma|-2)\,|\gamma|^2}{(2-|\gamma|)^2} + \frac{1}{6} \cdot \frac{|\gamma|^2}{2-|\gamma|} + \frac{1}{12}|2+\gamma| + \frac{2}{3} \\ &= \frac{1}{12} \cdot \frac{\gamma^2}{2-|\gamma|} + \frac{1}{12}|2+\gamma| + \frac{2}{3} \\ &= \frac{1}{12} \cdot \frac{(|\gamma|-4)^2}{2-|\gamma|} + \frac{1}{12}|2+\gamma|. \end{aligned} \tag{2.17}$$

5. $x = 2,\ y \in (0,1)$.

Let

$$\begin{aligned} H_\gamma(y) &:= F_\gamma(2,y) \\ &= \frac{1}{12}|2+\gamma| + \frac{2}{3}y\sqrt{1-\left(1-\frac{1}{4}\gamma^2\right)y^2} + \frac{1}{3}|\gamma|y. \end{aligned}$$

(a) For $|\gamma| = 2$ we get the linear functions H_{-2} and H_2 which have no critical points in $(0,1)$.

(b) Let $|\gamma| \neq 2$. Note first that

$$\sqrt{1-\left(1-\frac{1}{4}\gamma^2\right)y^2} > 0, \quad y \in (0,1). \tag{2.18}$$

Indeed, equating the left-hand side of (2.18) to zero, we get the equation equivalently written as

$$(4-\gamma^2)y^2 = 4, \quad y \in (0,1). \tag{2.19}$$

Since $y^2 > 0$, we have $|\gamma| < 2$. But then from (2.19) we obtain

$$y^2 = \frac{4}{4-\gamma^2} > 1,$$

which is a contradiction. Thus the equation (2.19) has no solution, so (2.18) holds. Now we have

$$H'_\gamma(y) = 0 \tag{2.20}$$

if and only if

$$\sqrt{1-\left(1-\frac{1}{4}\gamma^2\right)y^2} + \frac{-\left(1-\frac{1}{4}\gamma^2\right)y^2}{\sqrt{1-\left(1-\frac{1}{4}\gamma^2\right)y^2}} + \frac{1}{2}|\gamma| = 0.$$

Setting

$$s := \sqrt{1-\left(1-\frac{1}{4}\gamma^2\right)y^2},$$

the above equality is equivalent to

$$s + \frac{s^2-1}{s} + \frac{1}{2}|\gamma| = 0,$$

i.e.,

$$4s^2 + |\gamma|s - 2 = 0. \tag{2.21}$$

By (2.18) we have $s > 0$, so we conclude that

$$s = \frac{\sqrt{\gamma^2+32}-|\gamma|}{8}$$

is the unique solution of (2.21). Thus

$$\sqrt{1-\left(1-\frac{1}{4}\gamma^2\right)y^2} = \frac{\sqrt{\gamma^2+32}-|\gamma|}{8}.$$

As $|\gamma| \neq 2$, simple calculations yield

$$8\left(4-\gamma^2\right)y^2 = 16-\gamma^2+|\gamma|\sqrt{\gamma^2+32}.$$

Hence, obviously, $|\gamma| < 2$ and

$$y^2 = \frac{16-\gamma^2+|\gamma|\sqrt{\gamma^2+32}}{8(4-\gamma^2)}. \tag{2.22}$$

In consequence, the solution in $(0,1)$ of the equation (2.22), and hence (2.20), exists if and only if

$$0 < \frac{16-\gamma^2+|\gamma|\sqrt{\gamma^2+32}}{8(4-\gamma^2)} < 1. \tag{2.23}$$

The left-hand inequality in (2.23) is clearly true since $|\gamma|<2$ and

$$16-\gamma^2+|\gamma|\sqrt{\gamma^2+32} > 0.$$

Write the right-hand inequality as

$$16-\gamma^2+|\gamma|\sqrt{\gamma^2+32} < 8(4-\gamma^2).$$

Equivalently, we have

$$|\gamma|\sqrt{\gamma^2+32} < 16-7\gamma^2. \tag{2.24}$$

Since both sides of (2.24) are positive, squaring them we obtain

$$3\gamma^4-16\gamma^2+16>0.$$

Taking into account that $|\gamma|<2$ and solving the last inequality we obtain that $|\gamma|<2/\sqrt{3}$. Thus (2.23) holds for $|\gamma|<2/\sqrt{3}$, and it is false for $\gamma\in(-2,-2/\sqrt{3}]\cup[2/\sqrt{3},2)$.

Summarizing, we proved that (2.23) holds, and hence the solution in $(0,1)$ of (2.22) and further of (2.20) exists, if and only if $|\gamma|<2/\sqrt{3}$. Consequently, we can conclude that the function H_γ has a critical point in $(0,1)$, namely,

$$y=\sqrt{\frac{16-\gamma^2+|\gamma|\sqrt{\gamma^2+32}}{8(4-\gamma^2)}} =: y_\gamma$$

as the unique solution of (2.22) if and only if $|\gamma|<2/\sqrt{3}$.

Moreover

$$\begin{aligned} F_\gamma(2,y_\gamma) &= H_\gamma(y_\gamma) \\ &= \frac{1}{12}|2+\gamma| + \frac{2}{3}y_\gamma\left(\frac{\sqrt{\gamma^2+32}-|\gamma|}{8}+\frac{1}{2}|\gamma|\right) \\ &= \frac{1}{12}|2+\gamma| + \frac{1}{12}\sqrt{\frac{16-\gamma^2+|\gamma|\sqrt{\gamma^2+32}}{8(4-\gamma^2)}}\left(\sqrt{\gamma^2+32}+3|\gamma|\right). \end{aligned} \tag{2.25}$$

6. $x\in(0,2),\ y\in(0,1)$.

We will prove that for each $\gamma\in\mathbb{R}$, the function F_γ has no critical point in $(0,2)\times(0,1)$.

Observe first that

$$\frac{\partial F_\gamma}{\partial x}=0$$

if and only if

$$y\left(\frac{1}{3}x\left(\sqrt{1-\left(1-\frac{1}{4}\gamma^2\right)y^2}-1\right)+\frac{1}{6}|\gamma|\right)=0,$$

and since $y \neq 0$ and $x \neq 0$, if and only if

$$\sqrt{1-\left(1-\frac{1}{4}\gamma^2\right)y^2}=1-\frac{|\gamma|}{2x}. \tag{2.26}$$

Note that $\gamma \neq 0$ because if $\gamma = 0$, then $y = 0$ in (2.26) which contradicts the assumption. Moreover, by (2.18) the left-hand side of (2.26) is positive. Thus the solution of (2.26) can exist only when $\gamma \neq 0$ and $x > |\gamma|/2$. For $x > |\gamma|/2$, since $x \in (0, 2)$, we have

$$0 < |\gamma| < 2x < 4. \tag{2.27}$$

By squaring (2.26), we get

$$-\left(1-\frac{1}{4}\gamma^2\right)y^2=-\frac{|\gamma|}{x}+\frac{\gamma^2}{4x^2}. \tag{2.28}$$

On the other hand, we have

$$\frac{\partial F_\gamma}{\partial y}=0 \tag{2.29}$$

if and only if

$$\frac{2}{3}+\frac{x^2}{6}\left(\sqrt{1-\left(1-\frac{1}{4}\gamma^2\right)y^2}-1\right)$$
$$+\frac{1}{6}|\gamma|x+\frac{x^2}{6}\cdot\frac{-\left(1-\frac{1}{4}\gamma^2\right)y^2}{\sqrt{1-\left(1-\frac{1}{4}\gamma^2\right)y^2}}=0.$$

Then from (2.28), we equivalently have

$$4+x^2\left(-\frac{|\gamma|}{2x}\right)+|\gamma|x+\frac{\left(-\frac{|\gamma|}{x}+\frac{\gamma^2}{4x^2}\right)x^2}{1-\frac{|\gamma|}{2x}}=0,$$

and after simplifying,

$$4+\frac{1}{2}|\gamma|x+\frac{-2|\gamma|x^2+\frac{1}{2}\gamma^2x}{2x-|\gamma|}=0.$$

Thus

$$|\gamma|x^2 - 8x + 4|\gamma| = 0, \quad x \in (0,2). \tag{2.30}$$

By (2.27), $\gamma \neq 0$. Then the discriminant $\Delta = 16(4-\gamma^2) \geq 0$ if only if $\gamma \in [-2,2] \setminus \{0\}$. If $\Delta = 0$, then $|\gamma| = 2$ which implies $x = 2$, a contradiction. Thus the equation (2.30) has no root when $|\gamma| \geq 2$. Consequently, for $|\gamma| \geq 2$ the function F_γ has no critical point in $(0,2) \times (0,1)$.

Now consider $\gamma \in (-2,2) \setminus \{0\}$. The roots of (2.30) are the following:

$$x_1 = \frac{4 - 2\sqrt{4-\gamma^2}}{|\gamma|}, \quad x_2 = \frac{4 + 2\sqrt{4-\gamma^2}}{|\gamma|}.$$

Since $x_2 > 0$, $\gamma \neq 0$ and $x_1 x_2 = 4$, we immediately see that $0 < x_1 < 2 < x_2$. Thus $x_2 \notin (0,2)$ and it remains to consider x_1.

Observe that $x_1 > |\gamma|/2$. This follows from the fact that the inequality

$$\frac{4 - 2\sqrt{4-\gamma^2}}{|\gamma|} > \frac{1}{2}|\gamma|$$

is equivalent to

$$8 - \gamma^2 > 4\sqrt{4-\gamma^2},$$

which is evidently true for $\gamma \in (-2,2) \setminus \{0\}$.

Setting x_1 to the equation (2.28), we have

$$y^2 = \frac{\dfrac{|\gamma|}{x_1} - \dfrac{\gamma^2}{4x_1^2}}{1 - \dfrac{1}{4}\gamma^2} = \frac{4|\gamma|x_1 - \gamma^2}{x_1^2(4-\gamma^2)} \tag{2.31}$$

$$= \frac{\left(16 - \gamma^2 - 8\sqrt{4-\gamma^2}\right)\gamma^2}{\left(4 - 2\sqrt{4-\gamma^2}\right)^2 (4-\gamma^2)}.$$

A solution in $(0,1)$ of the above equation exists if and only if, for $|\gamma| < 2$ and $\gamma \neq 0$,

$$0 < \frac{\left(16 - \gamma^2 - 8\sqrt{4-\gamma^2}\right)\gamma^2}{\left(4 - \sqrt{4-\gamma^2}\right)^2 (4-\gamma^2)} < 1. \tag{2.32}$$

Since

$$16 - \gamma^2 - 8\sqrt{4-\gamma^2} > 0 \Leftrightarrow \gamma^4 + 32\gamma^2 > 0$$

and the last inequality is true, so the left-hand inequality in (2.32) holds. Write the right-hand inequality in (2.32) as

$$\left(16-\gamma^2-8\sqrt{4-\gamma^2}\right)\gamma^2<\left(4-2\sqrt{4-\gamma^2}\right)^2(4-\gamma^2)$$

which, after a simple computation, will give

$$8(8-3\gamma^2)\sqrt{4-\gamma^2}<5\gamma^4-64\gamma^2+128. \tag{2.33}$$

The left-hand side of (2.33) is nonnegative if and only if

$$\gamma\neq 0 \quad \text{and} \quad -\sqrt{8/3}\leq\gamma\leq\sqrt{8/3}\approx 1.633.$$

On the other hand, the right-hand side of (2.33) is nonnegative if and only if $\gamma\neq 0$ and $|\gamma|\leq\gamma_1$, where

$$\gamma_1=\sqrt{\frac{8\left(4-\sqrt{6}\right)}{5}}\approx 1.575.$$

Thus for $|\gamma|\leq\gamma_1$ and $\gamma\neq 0$, by squaring both sides of (2.33) and simplifying, we get

$$\gamma^6(25\gamma^2-64)>0.$$

Note that the above inequality holds if and only if $|\gamma|>8/5$. But $8/5>\gamma_1$, which yields a contradiction. For $\gamma_1<|\gamma|\leq\sqrt{8/3}$ the inequality (2.33) is evidently false. The same holds for $\sqrt{8/3}<|\gamma|<2$. Indeed, in this case both sides of (2.33) are negative, so squaring them and simplifying, we get the inequality

$$\gamma^6(25\gamma^2-64)<0,$$

which, as easy to see, is false (because will imply $|\gamma|<8/5<\sqrt{8/3}$).

Thus we have proved that for $\gamma\in(-2,2)\setminus\{0\}$, the equation (2.31) has no solution in $(0,1)$, which implies that for such γ the function F_γ has no critical point in $(0,2)\times(0,1)$.

Therefore for each $\gamma\in\mathbb{R}$, the function F_γ has no critical point in $(0,2)\times(0,1)$.

7. Now we calculate the maximum value of F_γ in R, which is attained on parts of the boundary of R.

(a) $|\gamma|\geq 4/3$. Then, in view of Parts 4(b) and 5(b), the maximum value of F_γ is attained at the corner of R, so by (2.12) it suffices to compare the following values:

$$\frac{1}{12}|2+\gamma|,\quad \frac{1}{12}|2+\gamma|+\frac{2}{3},\quad \frac{1}{12}|2+\gamma|+\frac{2}{3}|\gamma|. \tag{2.34}$$

Since $|\gamma| \geq 4/3 > 1$, we see at once that

$$\max_{(x,y)\in R} F_\gamma(x,y) = F_\gamma(2,1) = \frac{1}{12}|2+\gamma| + \frac{2}{3}|\gamma|. \tag{2.35}$$

(b) $\gamma = 0$. Then, in view of Parts 4(b) and 5(b), the maximum value of F_0 is attained at the corner of R or at $y_0 = 1/\sqrt{2}$. Thus comparing all the values in (2.34) with $\gamma = 0$ and $F_0\,(2, y_0) = 1/2$, by (2.25), we have

$$\max_{(x,y)\in R} F_0(x,y) = F_0(0,1) = \frac{5}{6}. \tag{2.36}$$

(c) $2/\sqrt{3} \leq |\gamma| < 4/3$. Then, in view of Parts 4(b) and 5(b), the maximum value of F_γ is attained at the corner of R or at $x_\gamma = |\gamma|/(2-|\gamma|)$. Thus we compare all the values in (2.34) and $F_\gamma(x_\gamma, 1)$.

Since $F_\gamma(2,1)$ is the largest value among all the values in (2.34), it is enough to show that

$$F_\gamma\,(x_\gamma, 1) \geq F_\gamma(2,1) = \frac{1}{12}|2+\gamma| + \frac{2}{3}|\gamma|. \tag{2.37}$$

By (2.17), inequality (2.37) becomes

$$\frac{(|\gamma|-4)^2}{2-|\gamma|} \geq 8|\gamma|,$$

or, equivalently,

$$\frac{(3|\gamma|-4)^2}{2-|\gamma|} \geq 0, \tag{2.38}$$

which is obviously true since $|\gamma| < 4/3 < 2$.

(d) $|\gamma| < 2/\sqrt{3},\ \gamma \neq 0$. Then we compare all the values in (2.34) and, by (2.17) and (2.25), $F_\gamma\,(x_\gamma, 1)$ and $F_\gamma\,(2, y_\gamma)$. We will show that $F_\gamma\,(x_\gamma, 1)$ is the largest one.

Observe that for $|\gamma| \leq 1,\ \gamma \neq 0,\ F_\gamma(0,1)$ is the largest value among all the values in (2.34), and so is $F_\gamma(2,1)$ for $1 < |\gamma| < 2/\sqrt{3}$.

For $|\gamma| \leq 1,\ \gamma \neq 0$, we have $\gamma^2/(2-|\gamma|) > 0$. So in view of (2.17) we get at once that

$$F_\gamma\,(x_\gamma, 1) \geq F_\gamma(0,1) = \frac{1}{12}|2+\gamma| + \frac{2}{3}. \tag{2.39}$$

For $1 < |\gamma| < 2/\sqrt{3}$, the inequality (2.38) is true, so is (2.37).

It remains to prove that for $|\gamma| < 2/\sqrt{3},\ \gamma \neq 0$,

$$F_\gamma\,(x_\gamma, 1) \geq F_\gamma\,(2, y_\gamma). \tag{2.40}$$

In view of (2.17) and (2.25) we have

$$\frac{1}{12}\cdot\frac{(|\gamma|-4)^2}{2-|\gamma|}+\frac{1}{12}|2+\gamma| \tag{2.41}$$
$$\geq \frac{1}{12}|2+\gamma|+\frac{1}{12}\sqrt{\frac{16-\gamma^2+|\gamma|\sqrt{\gamma^2+32}}{8(4-\gamma^2)}}\left(\sqrt{\gamma^2+32}+3|\gamma|\right)$$

if and only if

$$\frac{(4-|\gamma|)^2}{2-|\gamma|}\geq\sqrt{\frac{16-\gamma^2+|\gamma|\sqrt{\gamma^2+32}}{8(4-\gamma^2)}}\left(\sqrt{\gamma^2+32}+3|\gamma|\right). \tag{2.42}$$

Since, for $|\gamma|<2/\sqrt{3}$, both sides of (2.42) are positive, by squaring them, we have

$$\frac{(4-|\gamma|)^4}{(2-|\gamma|)^2}\geq\frac{16-\gamma^2+|\gamma|\sqrt{\gamma^2+32}}{8(4-\gamma^2)}\left(10|\gamma|^2+32+6|\gamma|\sqrt{\gamma^2+32}\right).$$

Setting $u:=|\gamma|\in\left(0,2/\sqrt{3}\right)$, we can write the last inequality, equivalently, as

$$8(2+u)(4-u)^2$$
$$\geq(2-u)\left(16-u^2+u\sqrt{u^2+32}\right)\left(10u^2+32+6u\sqrt{u^2+32}\right)$$

which, after a straightforward computation, is equivalent to

$$u^5-26u^4+208u^3-288u^2-384u+768\geq(2-u)u\left(u^2+32\right)^{3/2}. \tag{2.43}$$

Clearly, the right-hand side of the above is positive. To verify this for the left-hand side, denote

$$Q_1(v):=v^5-26v^4+208v^3-288v^2-384v+768,\quad v\in[0,u_1],$$

where $u_1:=2/\sqrt{3}$. But writing Q_1 as

$$Q_1(v)=v^5+26(u_1-v)v^3+(64-26u_1)v^3+144v(v-1)^2$$
$$+528(u_1-v)+(768-528u_1),\quad v\in[0,u_1],$$

we see that the coefficients of the above expression are all positive. Therefore $Q_1(v)>0$ in $[0,u_1]$, so the left-hand side of (2.43) is positive in $(0,u_1)$.

Squaring now the inequality (2.43) we get

$$\left(u^5-26u^4+208u^3-288u^2-384u+768\right)^2\geq(2-u)^2u^2\left(u^2+32\right)^3.$$

After a straightforward computation we can write the last inequality as

$$3u^9-62u^8+688u^7-3376u^6+5376u^5+10112u^4 \tag{2.44}$$
$$-41984u^3+26624u^2+36864u-36864\leq 0,\quad u\in(0,u_1).$$

To verify (2.44), we will prove that

$$Q_2(v) \le 0, \quad v \in [0, u_1], \tag{2.45}$$

where

$$Q_2(v) := 3v^9 - 62v^8 + 688v^7 - 3376v^6 + 5376v^5 + 10112v^4$$
$$-41984v^3 + 26624v^2 + 36864v - 36864, \quad v \in [0, u_1].$$

Since

$$Q_2(u_1) = \frac{1}{81}\left(681472 - \frac{1282560}{\sqrt{3}}\right) < 0,$$

in order to prove that (2.45) holds, it is enough to show that $Q_2'(v) \ge 0$ for $0 \le v \le u_1$. By the change of variable $t := v/u_1$, we observe that

$$Q_2'(u_1 t) = 36864 + \frac{106496}{\sqrt{3}}t - 167936t^2 + \frac{323584}{3\sqrt{3}}t^3$$
$$+\frac{143360}{3}t^4 - \frac{216064}{3\sqrt{3}}t^5 + \frac{308224}{27}t^6 - \frac{63488}{27\sqrt{3}}t^7 + \frac{256}{3}t^8$$
$$\ge 36864 + 61485t - 167936t^2 + 62273t^3$$
$$+47786t^4 - 41582t^5 + 11415t^6 - 1358t^7 + 85t^8 =: S(t).$$

In order to show that $Q_2'(u_1 t) \ge 0$ for $0 \le t \le 1$, it suffices to see that $S(t) \ge 0$ for $0 \le t \le 1$. But, after computing we have

$$S(1-t) = 9032 + 44670t + 34866t^2 - 23127t^3 - 30479t^4 - 3150t^5$$
$$+4289t^6 + 678t^7 + 85t^8$$
$$\ge 9032 + 44670t + 34866t^2 - 23127t - 30479t^2 - 3150t$$
$$+4289t^6 + 678t^7 + 85t^8$$
$$= 9032 + 18393t + 4387t^2 + 4289t^6 + 678t^7 + 85t^8 > 0, \quad t \in [0, 1].$$

Consequently, the inequality (2.45), so (2.44) and further (2.43) hold which implies that so are (2.41) and (2.40).

Summarizing, taking into account (2.35), (2.36), (2.37), (2.39) and (2.40) we have proved that

$$\max_{(x,y)\in R} F_\gamma(x, y) = \begin{cases} \frac{1}{12}|2+\gamma| + \frac{2}{3}|\gamma|, & |\gamma| \ge 4/3, \\ \frac{1}{12} \cdot \frac{(|\gamma|-4)^2}{2-|\gamma|} + \frac{1}{12}|2+\gamma|, & |\gamma| \le 4/3. \end{cases}$$

Finally, recalling that $\gamma = 2 - 3\lambda$, the above yields the inequality (2.5).

Now we prove that for $\lambda \in \mathbb{R} \setminus (2/3, 4/3)$ the bounds (2.5) are sharp.

Let $\lambda \in [2/9, 2/3]$. Since

$$\frac{1}{12} \cdot \frac{(|\gamma|-4)^2}{2-|\gamma|} + \frac{1}{12}|2+\gamma|$$

$$= \frac{1}{12} \cdot \frac{(2-3\lambda)^2}{2-|2-3\lambda|} + \left|\frac{1}{3} - \frac{1}{4}\lambda\right| + \frac{2}{3}$$
$$= \frac{1}{12} \cdot \frac{(2-3\lambda)^2}{3\lambda} + 1 - \frac{1}{4}\lambda = \frac{2}{3} + \frac{1}{9\lambda},$$

the inequality in (2.5) can be written as

$$\max_{f\in\mathcal{C}(h)} \Phi_\lambda(f) \le \frac{2}{3} + \frac{1}{9\lambda}, \quad \lambda \in [2/9, 2/3]. \tag{2.46}$$

Let $t_\lambda := 1/(3\lambda) - 1/2$. Then $t_\lambda \in [0,1]$ and, in view of (2.4), $p_{t_\lambda,0} \in \mathcal{P}$ with $c_1 = 2t_\lambda$ and $c_2 = 2$. Setting $\delta := 0$ and $p := p_{t_\lambda,0}$ into (2.8), we get the function f_λ given by the equation (2.6) for which, in view of (2.10),

$$a_2 = t_\lambda + \frac{1}{2} = \frac{1}{3\lambda}$$

and

$$a_3 = \frac{1}{3}(3 + 2t_\lambda) = \frac{2}{3} + \frac{2}{9\lambda}.$$

Hence

$$\Phi_\lambda(f_\lambda) = \left|\frac{2}{3} + \frac{2}{9\lambda} - \lambda\left(\frac{1}{3\lambda}\right)^2\right| = \frac{2}{3} + \frac{1}{9\lambda},$$

which makes equality in (2.46), so in (2.5). Clearly, $f_\lambda \in \mathcal{C}(h)$ because (2.8) is satisfied for $\delta = 0$. So $f_\lambda \in \mathcal{C}_0(h)$.

Let $\lambda \in (-\infty, 2/9] \cup [4/3, +\infty)$. Since

$$\left|\frac{1}{3} - \frac{1}{4}\lambda\right| + \frac{2}{3}|2-3\lambda| = \left|\frac{5}{3} - \frac{9}{4}\lambda\right|,$$

the first inequality in (2.5) can be written as

$$\max_{f\in\mathcal{C}(h)} \Phi_\lambda(f) \le \left|\frac{5}{3} - \frac{9}{4}\lambda\right|, \quad \lambda \in (-\infty, 2/9] \cup [4/3, +\infty). \tag{2.47}$$

Set $\delta := 0$ and $p := L$ into (2.8). Then $f = f_{2/9}$, where $f_{2/9}$ is given by (2.6), i.e., it is of the form (2.7), with, by (2.10), $a_2 = 3/2$ and $a_3 = 5/3$. Since $\Phi_\lambda\left(f_{2/9}\right) = |5/3 - 9\lambda/4|$, it makes equality in (2.47), so in (2.5). Clearly, $f_{2/9} \in \mathcal{C}_0(h)$.

The sharp bound of Φ_λ for $\lambda \in (2/3, 4/3)$ remains an open question. □

***Remark* 2.3.** In the first version of this manuscript submitted to the journal, the authors applied Laguerre's rule of counting zeros of polynomials in an interval to prove that $Q_1(v) > 0$ and $Q_2(v) \le 0$ in $[0, u_1]$. However, the computation presented in the proof of the above theorem to show required inequalities for Q_1 and Q_2, without using Laguerre theorem, was proposed and done himself by one of the referees

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***Remark* 2.4.** Observe that we can rewrite Theorem 2.2 as

$$\max_{f\in\mathcal{C}(h)} \Phi_\lambda(f) = \begin{cases} |5/3 - 9\lambda/4|\,, & \lambda \in (-\infty, 2/9] \cup [4/3, +\infty), \\ 2/3 + 1/(9\lambda), & \lambda \in [2/9, 2/3], \end{cases}$$

and

$$\max_{f\in\mathcal{C}(h)} \Phi_\lambda(f) \le \begin{cases} \dfrac{9\lambda^2 - 30\lambda + 26}{6(4-3\lambda)}, & \lambda \in (2/3, 10/9], \\ -1 + 7\lambda/4, & \lambda \in [10/9, 4/3). \end{cases}$$

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