Ear Decomposition of Factor-critical Graphs and Number of Maximum Matchings *

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Abstract A connected graph G is said to be factor-critical if $G-v$ has a perfect matching for every vertex v of G. Lovász proved that every factor-critical graph has an ear decomposition. In this paper, the ear decomposition of the factor-critical graphs G satisfying that $G-v$ has a unique perfect matching for any vertex v of G with degree at least 3 is characterized. From this, the number of maximum matchings of factor-critical graphs with the special ear decomposition is obtained.

Key words maximum matching; factor-critical graph; ear decomposition

1 Introduction and terminology

First, we give some notation and definitions. For details, see [1] and [2]. Let G be a simple graph. An edge subset $M \subseteq E(G)$ is a *matching* of G if no two edges in M are incident with a common vertex. A matching M of G is a *perfect matching* if every vertex of G is incident with an edge in M. A matching M of G is a *maximum matching* if $|M'| \leq |M|$ for any matching M' of G. Let v be a vertex of G. The degree of v in G is denoted by $d_G(v)$ and $\delta(G)$ =min$\{d_G(v) \mid v \in V(G)\}$. Let $P = u_1u_2 \cdots u_k$ be a path and $1 \leq s \leq t \leq k$. Then $u_su_{s+1} \cdots u_t$ is said to be a *subpath* of P, denoted by $P(u_s, u_t)$. Let

*This work is supported by National Natural Science Foundation of China (No. 11061027) and Program on International Cooperation and Innovation, Department of Education, Guangdong Province (No. 2012gjhz0007). Correspondence should be addressed to Yan Liu(e-mail:liuyan@scnu.edu.cn)

$P = u_1 \cdots u_k$ and $Q = u_k u_{k+1} \cdots u_{k+s}$ be two paths of G such that $V(P) \cap V(Q) = \{u_k\}$. The path $u_1 \cdots u_k u_{k+1} \cdots u_{k+s}$ is denoted by $u_1 P u_k Q u_{k+s}$. Denote $u_k u_{k-1} \cdots u_1$ by P^{-1}.

Let $P = uv_1 \cdots v_k v$ be a path or a cycle (in this case, $u = v$) of G. We say that P is *odd* if k is even (i.e., P has an odd number of edges), otherwise, P is *even*. P is said to be *pending* if $d_G(u) \geq 3$, $d_G(v) \geq 3$, and either P has no interior vertices (i.e., $P = uv$) or each of its interior vertices has degree 2 (i.e., $d_G(v_i) = 2$ for $1 \leq i \leq k$). We say that a path (cycle) is *even pending* if it is both even and pending and a path (cycle) is *odd pending* if it is both odd and pending. Let P be a pending path or cycle of G. Denote the subgraph of G obtained from G by either deleting the edge if P has only one edge or deleting all interior vertices of P by $G - P$.

We say that a connected graph G is *factor-critical* if $G - v$ has a perfect matching for every vertex $v \in V(G)$. Let G be a factor-critical graph. Then G has the odd number of vertices, has no cut edges and all pending cycles of G are odd.

The problem of finding the number of maximum matchings of a graph plays an important role in graph theory and combinatorial optimization since it has a wide range of applications. For example, in the chemical context, the number of perfect matchings of graphs is referred to as Kekulé structure count [3]. In physical field, the Dimer problem is essentially equal to the number of perfect matchings of a graph [4]. The number of perfect matchings is an important topological index which had been applied for estimation of the resonant energy, total π-electron energy and calculation of Pauling bond order [5] and [6]. But the enumeration problem for perfect matchings in general graphs (even in bipartite graphs) is NP-complete [2]. Hence, it makes sense that the enumeration problem for maximum matchings is a difficult one. In this paper, we study the number of maximum matchings in factor-critical graphs. The reasons to be interested in factor-critical graphs are following.

According to **Gallai-Edmonds' Decomposition Theorem**, any graph can be constructed by three types of graphs which are graphs with a perfect matching, bipartite graphs and factor-critical graphs. So the factor-critical graphs in Matching Theory are important.

In the following, we introduce the ear decomposition of factor-critical graphs.

Let G be a graph and G' a subgraph of G. An *ear* of G relative to G' is either an odd path or an odd cycle of G and having both ends (two ends are the same in the case that the ear is a cycle)–but no interior vertices–in G'. An ear is said to be *open* if it is a path, otherwise, *closed*. An *ear decomposition* of G starting with G' is a representation of G in the form: $G = G' + P_1 + \cdots + P_k$, where P_i is an ear of G relative to G_{i-1}, where $G_0 = G'$, $G_{i-1} = G' + P_1 + \cdots + P_{i-1}$ for $1 \leq i \leq k$ (see Figure 1).

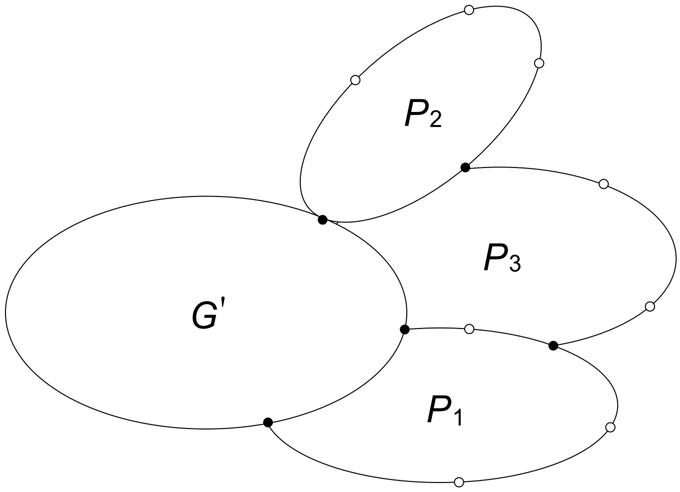

Figure 1. An ear decomposition of a graph G with three ears.

Proposition 1.1. [2] *Let G be a connected graph. Then G is factor-critical if and only if G has an ear decomposition $G = C + P_1 + \cdots + P_k$ starting with an odd cycle C, where $k = |E(G)| - |V(G)|$.*

Let $G = C + P_1 + \cdots + P_k$ be an ear decomposition of a factor-critical graph G starting with an odd cycle C. Then by Proposition 1.1, G_i is also a factor-critical graph for any $0 \leq i \leq k-1$, where $G_0 = C$ and $G_i = C + P_1 + \cdots + P_i$. If some ear P_i is pending, then $G = C + P_1 + \cdots + P_{i-1} + P_{i+1} + \cdots + P_k + P_i$ is also an ear decomposition of G and $G - P_i = C + P_1 + \cdots + P_{i-1} + P_{i+1} + \cdots + P_k$ is an ear decomposition of $G - P_i$. Then by Proposition 1.1, $G - P_i$ is also a factor-critical graph. Let $S_G = \{w \in V(G) | d_G(w) \geq 3\}$. Then $x \in S_G$ if and only if x is an end of some ear P_i. We say a subgraph G' of a graph G is *nice* if either $G - V(G') = \emptyset$ or $G - V(G')$ has a perfect matching. Then C and $G_i = C + P_1 + \cdots + P_i$ are nice subgraphs of G for $1 \leq i \leq k-1$.

Let Q be a path of G, w_1 and w_s the end vertices of Q. Q is said to be a *quasi-even-pending path* if $w_1, w_s \in S_G$, and either Q is even pending or $Q(w_i, w_j)$ is even for any two vertices $w_i, w_j \in S_G \cap V(Q)$ such that the subpath $Q(w_i, w_j)$ is pending, that is, Q can be written as $Q = w_1 Q_1 w_2 Q_2 \cdots w_{s-1} Q_{s-1} w_s$, where $w_i \in S_G$ for $1 \leq i \leq s$ and Q_j is an even pending path of G joining w_j and w_{j+1} for $1 \leq j \leq s-1$ (see Figure 2). Further, let $Q_j = w_j u_{j1} u_{j2} \cdots u_{j(2k_j-1)} w_{j+1}$ for $1 \leq j \leq s-1$. Then we say that $u_{j(2l-1)}$ is an *odd vertex* of Q for $1 \leq l \leq k_j$.

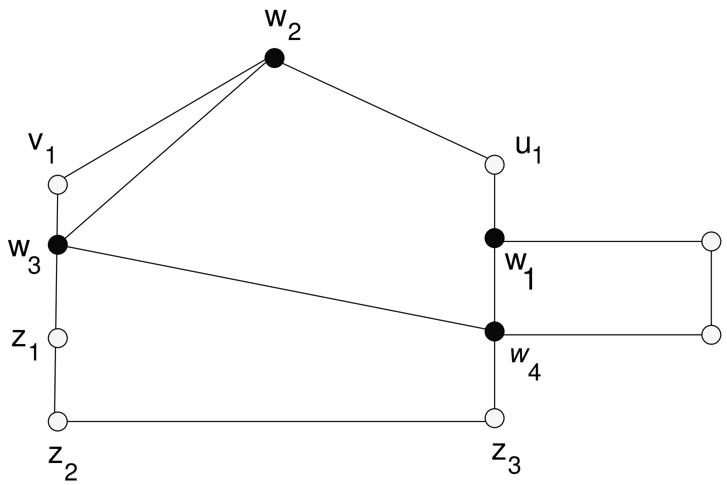

Figure 2. A quasi-even-pending path $Q = w_1 u_1 w_2 v_1 w_3 z_1 z_2 z_3 w_4$, where u_1, v_1, z_1, z_3 are odd vertices of Q.

Let G be a graph and H a subgraph of G. Then a quasi-even-pending path Q of H is also quasi-even-pending path of G if any odd vertex of Q has degree 2 in G (i.e., the degrees of all odd vertices are unchanged in G). Conversely, a quasi-even-pending path Q of G is also quasi-even-pending path of H if Q is a path of H and two ends of Q have degree at least 3 in H. Then according to the definitions of quasi-even-pending paths and ears, it is easy to obtain the following.

Proposition 1.2. *Let H be a subgraph of a graph G, P an ear of G relative to H and $G = H + P$. Then Q_1 is also a quasi-even-pending path of H for any quasi-even-pending path Q_1 of G such that two ends of Q_1 are in S_H and Q_2 is also a quasi-even-pending path of G for any quasi-even-pending path Q_2 of H such that no ends of P are odd vertices of Q_2.*

Let $Q = xQ_1w_2Q_2\cdots w_{s-1}Q_{s-1}y$ be a quasi-even-pending paths of G, where Q_i is an even pending path of G for $1 \leq i \leq s-1$. Let $S = \{w_2, \cdots, w_{s-1}\}$. Then $|S| = s-2$ and $Q - x - y - S$ has $s-1$ odd components which are also components of $G - x - y - S$. Hence $G - x - y$ has no perfect matchings. For any $S_1 \subseteq S_G$, $G - x - y - S_1 - S$ has at least $s-1$ odd components. Hence $G - x - y - S_1$ has no perfect matchings. So we have the following.

Proposition 1.3. *Let G be a factor-critical graph and Q a quasi-even-pending path of G joining x and y. Then $G - x - y$ and $G - x - y - S_1$ have no perfect matchings, where $S_1 \subseteq S_G$.*

Let $Q = xQ_1w_2Q_2\cdots w_{s-1}Q_{s-1}y$ and $Q' = xQ'_1z_2Q'_2\cdots z_{t-1}Q'_{t-1}y$ be two interior disjoint quasi-even-pending paths of G joining x and y, where Q_i and Q'_j are an even pending path of G, respectively, for $2 \leq i \leq s-1$ and $2 \leq j \leq t-1$. Let $S_1 = \{w_2, \cdots, w_{s-1}\}$, $S_2 = \{z_2, \cdots, z_{t-1}\}$ and $S = S_1 \cup S_2 \cup \{y\}$. Then $|S| = s+t-3$ and $G - x - S$ has at least $s+t-2$ odd components. Hence $G - x$ has no perfect matchings. So we have the following.

Proposition 1.4. *Let G be a factor-critical graph and S_G defined as above. Then for any two vertices $x, y \in S_G$, there exists at most one quasi-even-pending path of G joining x and y.*

Proof Suppose, to the contrary, that P and Q are two quasi-even-pending paths of G joining x and y. Clearly, we can find two vertices x_1 and y_1 in $S_G \cap V(P) \cap V(Q)$ such that the subpath $P(x_1, y_1)$ of P and the subpath $Q(x_1, y_1)$ of Q are interior-disjoint. Then $G - x_1$ has no perfect matchings, which contradicts with that G is factor-critical.

Proposition 1.5. *Let G be a factor-critical graph, P an odd pending path of G, Q a quasi-even-pending path of G, and P and Q share the same end vertices. Then $P \cup Q$ is a nice cycle of G.*

Proof Let $P = xu_1\cdots u_{2k}y$ be an odd pending path of G and $Q = xQ_1z_2Q_2\cdots z_{t-1}Q_{t-1}y$ be a quasi-even-pending path of G, where Q_i is an even pending path of G for $1 \leq i \leq t-1$. Then $Q - x$ has a perfect matching, say M_1, and P and Q are interior disjoint. Hence

$P \cup Q = xPyQ^{-1}x$ is an odd cycle of G, say C. Since G is factor-critical, $G - x$ has a perfect matching. Let $M_0 = \{u_1u_2, u_3u_4, \cdots, u_{2k-1}u_{2k}\}$. Then $M_0 \cup M_1 \subseteq M$ for any perfect matching M of $G - x$. Hence $M - M_0 - M_1$ is a perfect matching of $G - V(C)$. So, C is a nice cycle of G.

Pulleyblank proved [2] that a 2-connected factor-critical graph G has at least $|E(G)|$ maximum matchings. Liu and Hao proved [7] that G has exactly $|E(G)|$ maximum matchings if and only if G has an ear decomposition $G = C + P_1 + \cdots + P_k$ such that two ends of P_i are joined in G_{i-1} by a pending path of length 2 of G, and if G has a such ear decomposition, then $G - w$ has a unique perfect matching for any $w \in S_G$. But it is not vice versa. In this paper, we study the ear decomposition of the factor-critical graph G with the property that $G - w$ has a unique perfect matching for any $w \in S_G$ and from this, the enumeration problem for maximum matchings of factor-critical graphs with the special ear decomposition is solved.

2 Results and proofs

For convenience, we say that an ear decomposition $G = C + P_1 + \cdots + P_k$ starting from an odd cycle C of a factor-critical graph G has *Property A* if for any open ear P_i, two ends of P_i are joined by a path Q_i of G_{i-1} which is a quasi-even-pending path of G. Then we have the following.

Proposition 2.1. *Let $G = C + P_1 + \cdots + P_k$ be an ear decomposition of a factor-critical graph G having Property A. Then $G_{k-1} = C + P_1 + \cdots + P_{k-1}$ has also Property A.*

Proof Then $G = G_{k-1} + P_k$. Since $G = C + P_1 + \cdots + P_k$ has Property A, we can assume that Q_i is a path of G_{i-1} joining two ends of P_i which is a quasi-even-pending path of G for any open ear P_i, where $1 \leq i \leq k-1$. Since G_{i-1} is factor-critical, two ends of Q_i have degree at least 2 in G_{i-1}. Then the ends of Q_i have degree at least 3 in G_i. So, by Proposition 1.2, Q_i is also quasi-even-pending of G_{k-1}. Then $G_{k-1} = C + P_1 + \cdots + P_{k-1}$ also has Property A.

By the similar reasons as above, we have the following.

Proposition 2.2. *Let $G = C + P_1 + \cdots + P_k$ be an ear decomposition of a factor-critical graph G having Property A and P_i be pending. Then $G = C + P_1 + \cdots P_{i-1} + P_{i+1} + \cdots + P_{k-1} + P_k + P_i$ and $G - P_i = C + P_1 + \cdots P_{i-1} + P_{i+1} + \cdots + P_{k-1} + P_k$ also have Property A.*

We say that a factor-critical graph has *Property A* if there exists an ear decomposition of G having Property A. We say that a factor-critical graph G has *Property B* if $G - v$ has a unique perfect matching for any $v \in S_G$ (see Figure 3).

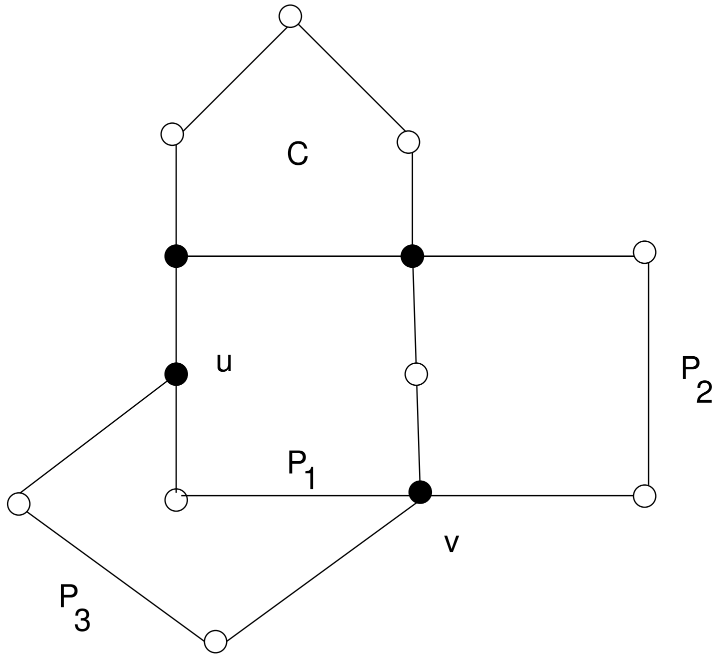

Figure 3. A factor-critical graph G with Property B.

Theorem 1. *Let G be a factor-critical graph and the ear decomposition $G = C + P_1 + \cdots + P_k$ of G have Property A. Then G has Property B.*

Proof. Then for $1 \le i \le k-1$, $G_i = C + P_1 + \cdots + P_i$ has also Property A by Proposition 2.2. We prove the statement by induction on k. When $k = 0$, $G = C$. When $k = 1$, $G = C + P_1$. Clearly, C and $C + P_1$ have Property B. Suppose that $k = m \ge 2$ and it holds for $k < m$. Let $u \in S_G$. Now we prove that $G - u$ has a unique perfect matching. We distinguish the following cases.

Case 1 $d_{G_{k-1}}(u) \geq 3$.

Let $P_k = xu_1 \cdots u_{2l}y$. Then $G_{k-1} = G - \{u_1, \cdots, u_{2l}\}$. In the case that P_k has only one edge (i.e., $P_k = xy$), $G_{k-1} = G - xy$. Then G_{k-1} is factor-critical and $G_{k-1} = C + P_1 + \cdots + P_{k-1}$ has Property A by Proposition 2.1. So, by the induction hypothesis, $G_{k-1} - u$ has a unique perfect matching.

Case 1.1 P_k is open.

Let m_1 be the number of perfect matchings of $G - u$ containing $\{u_{2j-1}u_{2j} | 1 \leq j \leq l\}$ and m_2 the one of $G - u$ containing $\{xu_1, yu_{2l}\} \cup \{u_{2j}u_{2j+1} | 1 \leq j \leq l-1\}$. Since P_k is pending, $G - u$ has exactly $m_1 + m_2$ perfect matchings. Clearly, the number of perfect matchings of $G_{k-1} - u$ and $G_{k-1} - \{u, x, y\}$ are m_1 and m_2, respectively. Then $m_1 = 1$. Since G has Property A, there exists a quasi-even-pending path of G joining x and y. By Proposition 1.3, $G_{k-1} - \{x, y, u\}$ has no perfect matchings. Then $m_2 = 0$. It follows that $G - u$ has a unique perfect matching.

Case 1.2 P_k is closed.

In this case, $x = y$. Clearly, every perfect matching of $G - u$ contains all edges $u_{2j-1}u_{2j}$ for $1 \leq j \leq l$. Then the number of perfect matchings of $G - u$ is equal to one of $G_{k-1} - u$. It follows that $G - u$ has a unique perfect matching.

Case 2 $d_{G_{k-1}}(u) = 2$.

Then u is an end of P_k (see the vertex u in Figure 3). Without loss of generality, $u = x$, where x and y are two ends of P_k. In the following, we prove that $G - x$ has a unique perfect matching. We can assume that all other ears P_i are not pending, $1 \leq i \leq k-1$. Otherwise, suppose that some P_i is pending, where $i \leq k-1$. Then $u \notin V(P_i)$. Hence x has degree at least 3 in $G - P_i = C + P_1 + \cdots P_{i-1} + P_{i+1} + \cdots + P_{k-1} + P_k$. By Proposition 2.2, G can be rewritten as $G = C + P_1 + \cdots P_{i-1} + P_{i+1} + \cdots + P_{k-1} + P_k + P_i$ which also has Property A . Then it belongs to Case 1. So, we only consider the case that P_k is a unique pending ear. Then some end of P_k must be an interior vertex of P_{k-1}. Now we distinguish the following cases.

Case 2.1 x and y are not on the same pending path or pending cycle of G_{k-1}.

Then P_k is open. Since G has property A, x and y are joined in G_{k-1} by a quasi-even-pending path Q of G. Let $Q_1 = xx_1 \cdots x_{2s+1}w_1$ be the pending subpath of Q starting from x to the second vertex w_1 of degree at least 3 (where, x is the first vertex of degree at least 3) on Q. Then $w_1 \neq y$. So $d_{G_{k-1}}(w_1) = d_G(w_1) \geq 3$. It is easy to prove that the number of perfect matchings of $G - x$ is equal to the number of perfect matchings of $G_{k-1} - w_1$ since P_k is odd pending and Q_1 is even pending. By the induction hypothesis, $G_{k-1} - w_1$ has a unique perfect matching. It follows that $G - x$ has a unique perfect matching.

Case 2.2 x and y are vertices of a pending path or pending cycle of G_{k-1}.
It follows that $x, y \in V(P_{k-1})$ and x is an interior vertex of P_{k-1} since $d_{G_{k-1}}(u) = 2$. Let $P_{k-1} = wv_1 \cdots v_{2t}z$. Then we can assume that $x = v_j$, where $1 \leq j \leq 2t$. First, we consider the case that P_k is closed. Then $x = y$. It is easy to prove that the number of perfect matchings of $G - x$ is equal to the number of perfect matchings of $G_{k-1} - x$, and the number of perfect matchings of $G_{k-1} - x$ is equal to the number of perfect matchings of $G_{k-1} - z$ if j is odd, otherwise, the number of perfect matchings of $G_{k-1} - x$ is equal to the number of perfect matchings of $G_{k-1} - w$. By the induction hypothesis, both $G_{k-1} - w$ and $G_{k-1} - z$ have a unique perfect matching, respectively. Then $G - x$ has a unique perfect matching. Thus we can assume that P_k is open. Without loss of generality, suppose that y is on the subpath of P_{k-1} from x to z. Since G has Property A, x and y are joined in G_{k-1} by a quasi-even-pending path Q of G. Hence either $P_{k-1}(x, y) = Q$ or $(P_{k-1}(w, x))^{-1}$ and $P_{k-1}(y, z)$ are two subpaths of Q. If $P_{k-1}(x, y) = Q$, then either $P_{k-1}(w, x)$ or $P_{k-1}(y, z)$ is even since P_{k-1} is odd. If $(P_{k-1}(w, x))^{-1}$ and $P_{k-1}(y, z)$ are two subpaths of Q, then both $P_{k-1}(w, x)$ and $P_{k-1}(y, z)$ are even since Q is a quasi-even-pending path of G. Hence $P_{k-1}(w, x)$ or $P_{k-1}(y, z)$ is even in any case. First, suppose that $P_{k-1}(w, x)$ is even. Then the number of perfect matchings of $G - x$ is equal to the number of perfect matchings of $G_{k-1} - w$. By the induction hypothesis, $G_{k-1} - w$ has a unique perfect matching. It follows that $G - x$ has a unique perfect matching. Suppose that $P_{k-1}(w, x)$ is odd and $P_{k-1}(y, z)$ is even. Then $P_{k-1}(x, y)$ is even since P_{k-1} is odd. It follows that the number of perfect matchings of $G - x$ is equal to the number of perfect matchings of $G_{k-1} - z$. By the induction hypothesis, $G_{k-1} - z$ has a unique perfect matching. It follows that $G - x$ has a unique perfect matching. The proof is completed. ∎

Lemma 2.1. *Let G be a factor-critical graph and $G = C + P_1 + \cdots + P_k$ have Property A. Then $G - u$ has at least two perfect matchings for any quasi-even-pending path Q of G and any odd vertex u of Q.*

Proof. It suffices to prove that $G-u$ has at least two perfect matchings for any even pending path Q of G and any odd vertex u of Q. We prove the lemma by induction on $k=|E(G)|-|V(G)|$. When $k=0$, G is an odd cycle, say C. When $k=1$, $G=C+P_1$. Clearly, the statement is true. Suppose that $k=m\geq 2$ and the statement is true for $k\leq m-1$. Since $G=C+P_1+\cdots+P_k$ has Property A, $G_{k-1}=C+P_1+\cdots+P_{k-1}$ has Property A by Proposition 2.1. Let $P_k=xu_1\cdots u_{2l}y$ and $M_0=\{u_1u_2,u_3u_4,\cdots,u_{2l-1}u_{2l}\}$. Then $x,y\in S_G$, $S_G-\{x,y\}\subseteq S_{G_{k-1}}\subseteq S_G$ and $d_G(u_i)=2$ for $1\leq i\leq 2l$. Let Q be an even pending path of G and u an odd vertex of Q. Then $d_G(u)=2$ and two ends of Q are in S_G. If u is an odd vertex of an even pending path of G_{k-1}, then by the induction hypothesis, $G_{k-1}-u$ has at least two perfect matchings, say M_1 and M_2. Then $M_1\cup M_0$ and $M_2\cup M_0$ are two perfect matchings of $G-u$. So we can assume that u is not odd vertex of any even pending path of G_{k-1}. Since P_k is odd pending and Q is even pending, Q is a path of G_{k-1}. It follows that $d_{G_{k-1}}(u)=2$ and there exists at least one end of Q not in $S_{G_{k-1}}$. Then x or y is an end of Q. Without loss of generality, suppose that x is an end of Q. Then $d_{G_{k-1}}(x)=2$. Since $k\geq 2$, G_{k-1} is not a cycle. Then u is an interior vertex of a pending path or pending cycle of G_{k-1}. Since G has Property A, there exists a path P of G_{k-1} joining x and y which is a quasi-even-pending path of G.

Claim Q is a subpath of P.

We distinguish two cases to prove the claim.

Case 1 u is an interior vertex of a pending cycle of G_{k-1}, say C'.

Then C' is odd and Q is a part of C'. Let $S_{G_{k-1}}\cap V(C')=\{w\}$. If two ends of Q are interior vertices of C', then it follows that two ends of Q are x and y since two ends of Q are in S_G. Then by Proposition 1.4, $Q=P$. So we can assume that w is the other end of Q. Let $P^*=C'-V(Q)$. Then P^* is odd since C' is odd and Q is even. Hence $P^*\cap V(P)=\emptyset$. It follows that Q is a subpath of P.

Case 2 u is an interior vertex of a pending path of G_{k-1}, say P'.

Let w and z be two ends of P'. Then $w,z\in S_{G_{k-1}}$ and Q is a subpath of P'. If two ends of Q are interior vertices of P', then we can deduce that $Q=P$ by the similar method as above. So we can assume that Q and P' share a common end, say w. Then P' is odd. (Otherwise, u is an odd vertices of P' since u is odd vertex of Q, a contradiction.) It follows that Q is a subpath of P.

By Proposition 1.5, $P_k \cup P$ is a nice cycle of G. Then $G - V(P_k \cup P)$ has a perfect matching, say M'. Since u is odd vertex of Q, u is odd vertex of P by Claim. Let M be the perfect matching of $P_k \cup P - u$. Then M contains $\{xu_1, u_2u_3, \cdots, u_{2l-2}u_{2l-1}, u_{2l}y\}$. It follows that $M \cup M'$ is a perfect matching of $G-u$. Since G_{k-1} is factor-critical, $G_{k-1}-u$ has a perfect matching, say M_1. Then $M_1 \cup M_0$ is a perfect matching of $G-u$. Hence $G-u$ has at least two perfect matchings. ∎

Lemma 2.2. *Let G be a factor-critical graph having Property A. Then there exists a unique quasi-even-pending path of G joining u and v for any two vertices $u, v \in S_G$.*

Proof. By Proposition 1.4, it suffices to prove that there exists a quasi-even-pending path of G joining u and v for any two vertices $u, v \in S_G$. Let $G = C + P_1 + \cdots + P_k$ be an ear decomposition of G having Property A. We prove the lemma by induction on k. When $k \le 1$, $G = C$ or $G = C + P_1$. Clearly, the statement is true. Suppose that $k = m \ge 2$ and the statement is true for $k \le m-1$. Let $P_k = xu_1 \cdots u_{2l}y$, $H = G_{k-1}$ and $S_H = \{w \in V(H) | d_H(w) \ge 3\}$. Then $x, y \in S_G$ and $S_G - \{x, y\} \subseteq S_H \subseteq S_G$. According to Property A, there exists a path Q of H joining x and y which is a quasi-even-pending path of G. Let $Q = xQ_1z_2Q_2 \cdots z_{t-1}Q_{t-1}y$ be a quasi-even-pending path of G, where Q_i is an even pending path of G for $1 \le i \le t-1$. Let $u, v \in S_G$. Then we can assume that $\{u, v\} \ne \{x, y\}$. Then $\{u, v\} \cap S_H \ne \emptyset$. By Proposition 2.1, $H = C + P_1 + \cdots + P_{k-1}$ has Property A. We distinguish the following cases.

Case 1 $u, v \in S_H$.

By the induction hypothesis, there exists a quasi-even-pending path P of H joining u and v. Since G has Property A, G has Property B. Hence $G - x$ and $G - y$ have a unique perfect matching, respectively. Then by Lemma 2.1, x and y are not odd vertices of P. By Proposition 1.2, P is also a quasi-even-pending path of G.

Case 2 Either $u \notin S_H$ and $v \in S_H$ or $v \notin S_H$ and $u \in S_H$.

Without loss of generality, suppose that $u \notin S_H$ and $v \in S_H$. Since H is factor-critical, $d_H(u) = 2$. It follows that u is an end of P_k. Without loss of generality, suppose that $u = x$. Then $v \ne y$. In the following, we prove that there exists a quasi-even-pending path of G joining x and v. If v is on Q, then $Q(x, v)$ is a quasi-even-pending path of G joining x and v. Suppose that $v \notin V(Q)$.

Case 2.1 $V(Q) \cap S_H \neq \emptyset$.

Then $z_2 \in S_H$ and $z_2 = y$ if Q is pending. By the induction hypothesis, there exists a quasi-even-pending path P of H joining z_2 and v. If x is a vertex of P, then Q_1^{-1} is the subpath of P since $d_H(x) = 2$. Then $P(x, v)$ is a quasi-even-pending path of G joining x and v. Suppose that $x \notin V(P)$. Then $V(Q_1) \cap V(P) = \{z_2\}$. Hence xQ_1z_2Pv is a quasi-even-pending path of G joining x and v.

Case 2.2 $V(Q) \cap S_H = \emptyset$.

Then Q is a pending path of G and $d_H(x) = d_H(y) = 2$. Since $k \geq 2$, H is not a cycle. Then there exists a pending path or a pending cycle of H containing Q.

Case 2.2.1 C_1 is a pending cycle of H containing Q.

Since H is factor-critical, C_1 is odd. Let $V(C_1) \cap S_H = \{w\}$. Then x, y, w partite C_1 into three paths, say $C_1(w, x)$, $C_1(x, y)$ and $C_1(y, w)$, where $C_1(x, y) = Q$. Then either $C_1(w, x)$ or $C_1(y, w)$ is even. Without loss of generality, suppose that $C_1(w, x)$ is even. By the induction hypothesis, there exists a quasi-even-pending path P of H joining w and v(in the case that $w = v$, P is a vertex). Then $xC_1(w, x)^{-1}wPv$ is a quasi-even-pending path of G joining x and v.

Case 2.2.2 P' is a pending path of H containing Q.

Let $V(P') \cap S_H = \{w, z\}$. Then x, y partite P' into three paths, one of which is Q. Suppose that other two paths are $P'(w, x)$ and $P'(y, z)$, respectively. By Lemma 2.1, $P'(w, x)$ is even or $P'(y, z)$ is even.(Otherwise, P' is even. Then P' is a quasi-even-pending path of H and x is an odd vertex of P'. Then $H - x$ has at least two perfect matchings by Lemma 2.1. Hence $G - x$ has at least two perfect matchings, which contradicts with that G has Property B.) Suppose that $P'(w, x)$ is even. By the induction hypothesis, there exists a quasi-even-pending path of H joining w and v(in the case that $w = v$, the path is a vertex). Then we can find a quasi-even-pending path of G joining x and v by the similar method as one in Case 2.1. Similarly, if $P'(y, z)$ is even, then we can also find a quasi-even-pending path of G joining x and v. ∎

Theorem 2. *Let G be a factor-critical graph with Property B. Then G has Property A.*

Proof. Since G is factor-critical, G has an ear decomposition $G = C + P_1 + \cdots + P_k$. We prove the theorem by induction on $k = |E(G)| - |V(G)|$. When $k = 0$, $G = C$.

When $k = 1$, $G = C + P_1$. Clearly, the statement is true. Suppose that $k = m \geq 2$ and the statement is true for $k \leq m - 1$. Let $P_k = xu_1 \cdots u_{2l}y$. Then $x, y \in S_G$ and $d_G(u_i) = 2$ for $1 \leq i \leq 2l$. Since G has Property B, G_{k-1} has Property B. (Otherwise, suppose $v \in S_{G_{k-1}}$ such that $G_{k-1} - v$ has two perfect matchings, say M_1 and M_2. Then $v \in S_G$, $M_1 \cup \{u_1u_2, u_3u_4 \cdots u_{2l-1}u_{2l}\}$ and $M_2 \cup \{u_1u_2, u_3u_4 \cdots u_{2l-1}u_{2l}\}$ are two perfect matchings of $G - v$, a contradiction.) Then by the induction hypothesis, there exists an ear decomposition $G_{k-1} = C' + Q_1 + \cdots + Q_{k-1}$ having Property A. Then by Lemma 2.1, $G_{k-1} - u$ has at least two perfect matchings for any odd vertex u of a quasi-even-pending path of G_{k-1}. It follows that x and y are not odd vertices of a quasi-even-pending path of G_{k-1} since $G - x$ and $G - y$ have a unique perfect matching, respectively. Then any quasi-even-pending path of G_{k-1} is also quasi-even-pending path of G by Proposition 1.2. In the following, we prove that $G = C' + Q_1 + \cdots + Q_{k-1} + P_k$ is an ear decomposition of G having Property A. If P_k is closed, then $G = C' + Q_1 + \cdots + Q_{k-1} + P_k$ is an ear decomposition of G having Property A. We can assume that P_k is open (i.e., $x \neq y$). Then it suffices to prove that there exists a quasi-even-pending path of G joining x and y. We distinguish the following cases.

Case 1 $x, y \in S_{G_{k-1}}$.

Since $G_{k-1} = C' + Q_1 + \cdots + Q_{k-1}$ has Property A, there exists a quasi-even-pending path Q of G_{k-1} joining x and y by Lemma 2.2. Clearly, Q is a quasi-even-pending path of G.

Case 2 $d_{G_{k-1}}(x) = 2$ and $y \in S_{G_{k-1}}$ or $d_{G_{k-1}}(y) = 2$ and $x \in S_{G_{k-1}}$.

Without loss of generality, suppose that $d_{G_{k-1}}(x) = 2$ and $y \in S_{G_{k-1}}$. Then x is a vertex of a pending path or a pending cycle of G_{k-1}.

Case 2.1 x is a vertex of an even pending path P of G_{k-1}.

Then x is not odd vertex of P. Let w, z be two ends of P. Then $w, z \in S_{G_{k-1}}$, both $P(w, x)$ and $P(x, z)$ are even. By Lemma 2.2, there exists a quasi-even-pending path of G_{k-1} joining w and y. Then we can find a quasi-even-pending path of G joining x and y.

Case 2.2 x is a vertex of an odd pending path P of G_{k-1}.

Let w, z be two ends of P. Then P is interior disjoint with any quasi-even-pending path of G_{k-1}, and either $P(w, x)$ is even or $P(x, z)$ is even. Without loss of generality, suppose that $P(w, x)$ is even. Since G_{k-1} has Property A and $w, y \in S_{G_{k-1}}$, there exists a quasi-even-pending path Q of G_{k-1} joining w and y by Lemma 2.2. Then $xP^{-1}(w, x)wQy$ is a quasi-even-pending path of G.

Case 2.3 x is a vertex of a pending cycle C_1 of G_{k-1}.

Let $V(C_1) \cap S_{G_{k-1}} = \{w\}$. Since G_{k-1} is factor-critical, C_1 is odd. Then $C_1 - w - x$ consists of two paths one of which is even and the other is odd, say the odd one is P. Let $P' = C_1 - V(P)$. Then P' is an even pending path of G joining x and w. By the similar reasons, there exists a quasi-even-pending path Q of G_{k-1} joining w and y. Then $xP'wQy$ is a quasi-even-pending path of G.

Case 3 $d_{G_{k-1}}(x) = 2$ and $d_{G_{k-1}}(y) = 2$.

Then x and y are an interior vertex of a pending path or a pending cycle of G_{k-1}, respectively.

Case 3.1 x and y are vertices of a pending cycle C_1 of G_{k-1}.

Let $V(C_1) \cap S_{G_{k-1}} = \{w\}$. Then w, x and y partite C_1 into three paths, say $C_1(w, x)$, $C_1(x, y)$, and $C_1(y, w)$. If $C_1(x, y)$ is even, then $C_1(x, y)$ is a quasi-even-pending path of G. We can assume that $C_1(x, y)$ is odd. Then either $C_1(w, x)$ and $C_1(y, w)$ are odd or $C_1(w, x)$ and $C_1(y, w)$ are even. Since G has Property B, $G - w$ has a unique perfect matching. Then $C_1(w, x)$ and $C_1(y, w)$ are even. (Otherwise, we can find two perfect matchings in the subgraph $C_1 + P_k - w$. Clearly, $C_1 + P_k$ is a nice subgraph of G. Then $G - w$ has at least two perfect matchings, a contradiction.) So, $xC_1(w, x)^{-1}wC_1(y, w)^{-1}y$ is a quasi-even-pending path of G.

Case 3.2 x and y are vertices of an odd pending path P of G_{k-1}.

Let w, z be two ends of P. Since G_{k-1} has Property A, there exists a quasi-even-pending path Q of G_{k-1} joining w and z by Lemma 2.2. Then $P \cup Q$ is a nice subgraph of G_{k-1} by Proposition 1.5. Hence $P \cup Q + P_k$ is a nice subgraph of G. Clearly, x and y partite P into three paths, say $P(w, x)$, $P(x, y)$ and $P(y, z)$. If $P(x, y)$ is even, then $P(x, y)$ is a quasi-even-pending path of G. So, suppose that $P(x, y)$ is odd. Then either $P(w, x)$ and $P(y, z)$ are odd or $P(w, x)$ and $P(y, z)$ are even. Since G has Property B, $G - w$ has a unique perfect matching. Then $P(w, x)$ and $P(y, z)$ are even. (Otherwise, we can find two perfect matchings in the subgraph $P \cup Q + P_k - w$. Then $G - w$ has at least two perfect matchings, a contradiction.) So, $xP(w, x)^{-1}wQzP(y, z)^{-1}y$ is a quasi-even-pending path of G.

Case 3.3 x and y are vertices of an even pending path P of G_{k-1}.

Let w, z be two ends of P. Clearly, x and y partite P into three paths, say $P(w,x)$, $P(x,y)$ and $P(y,z)$. Since G has Property B, $P(x,y)$ is even. (Otherwise, suppose that $P(x,y)$ is odd. Then either $P(w,x)$ is even or $P(y,z)$ is even. Without loss of generality, suppose that $P(w,x)$ is even and $P(y,z)$ is odd. It is easy to check that $P+P_k$ is a nice subgraph of G and $P+P_k-z$ has two perfect matchings. Then $G-z$ has at least two perfect matchings, a contradiction.) Then $P(x,y)$ is a quasi-even-pending path of G.

Case 3.4 x and y are on the different pending paths or cycles of G_{k-1}.
Let P and P' be two pending paths or cycles of G_{k-1} containing x and y, respectively, w_1, z_1 be two ends of P and w_2, z_2 be two ends of P' (in the case that P and P' are cycles, $w_1 = z_1$ and $w_2 = z_2$). Since x and y are not odd vertices of a quasi-even-pending path of G_{k-1}, $P(w_1,x)$ is even or $P(x,z_1)$ is even, and $P'(w_2,y)$ or $P'(y,z_2)$ is even. Without loss of generality, suppose that $P(w_1,x)$ and $P'(w_2,y)$ are even. Since G_{k-1} has Property A, there exists a quasi-even-pending path Q of G_{k-1} joining w_1 and w_2 by Lemma 2.2. Then by the similar method, we can find a quasi-even-pending path of G joining x and y in $P \cup P' \cup Q$. ∎

By Theorem 1 and Theorem 2, we have the following.

Theorem 3. *Let G be a factor-critical graph G. Then G has Property A if and only if Property B.*

Now we study the number of maximum matchings of a factor-critical graph having Property A. The following lemma is useful.

Lemma 2.3. *Let G be a factor-critical graph and $G = C + P_1 + \cdots + P_k$ an ear decomposition having Property A. Then for any open ear P_j, $G - x - y$ has exactly $\frac{l_j}{2}$ maximum matchings, where x and y are the ends of P_j and l_j is the length of the quasi-even-pending path Q_j of G joining x and y in G_{j-1}, where $G_{j-1} = C + P_1 + \cdots + P_{j-1}$.*

Proof. Since G is factor-critical, a maximum matching of $G-x-y$ covers all but one vertex. Since Q_j is a quasi-even-pending path of G, the vertex uncovered by M must be on Q_j and all other vertices on Q_j are matched with vertices on Q_j by M for any maximum matching M of $G-x-y$, that is, every maximum matching of $G-x-y$ consists of a maximum matching of $Q_j - \{x,y\}$ and a perfect matching of $G - V(Q_j)$.

We can assume that $Q_j = xu_1 \cdots u_{2k_1-1}w_1u_{2k_1} \cdots u_{2k_1+2k_2-2}w_2 \cdots w_{s-1}u_{n+1} \cdots u_{n+2k_s-1}y$, where $n = (2k_1 - 1) + (2k_2 - 1) + \cdots + (2k_{s-1} - 1)$, $l_j = 2k_1 + \cdots + 2k_s$, $d_G(w_i) \geq 3$ and $d_G(u_j) = 2$ for all w_i and u_j on Q. By Theorem 1, $G - x$ has a unique perfect matching, say M_0. Then $\{u_1u_2, \cdots, u_{2k_1-1}w_1, u_{2k_1}u_{2k_1+1}, \cdots, u_{n+2k_s-1}y\} \subset M_0$. It follows that $G - V(Q_j)$ has a unique perfect matching. So, the number of maximum matchings of $G - x - y$ equals to that of $Q_j - \{x, y\}$. It is easy to check that $Q_j - \{x, y\}$ has exactly $\frac{l_j}{2}$ maximum matchings. So, $G - x - y$ has precisely $\frac{l_j}{2}$ maximum matchings. The proof is completed. ∎

Theorem 4. *Let G be a factor-critical graph and the ear decomposition $G = C + P_1 + \cdots + P_k$ have Property A. Then G has precisely $|E(G)| + \frac{l_1}{2} + \cdots + \frac{l_k}{2} - k$ maximum matchings, where l_i is the length of the quasi-even-pending path Q_i of G joining two ends of P_i in G_{i-1} and $l_i = 0$ in the case that P_i is closed for $1 \leq i \leq k$.*

Proof. By induction on k. When $k = 0$, $G = C$. Then G has exactly $|E(G)|$ maximum matchings. Suppose that it holds for $k < m$ and consider the case for $k = m \geq 1$. Let $P_k = xu_1 \cdots u_{2l}y$, $Q_k = xv_1 \cdots v_{l_k-1}y$ and $G_{k-1} = C + P_1 + \cdots + P_{k-1}$. By the induction hypothesis, G_{k-1} has exactly $|E(G_{k-1})| + \frac{l_1}{2} + \cdots + \frac{l_{k-1}}{2} - k + 1$ maximum matchings. Let

$$\mathcal{M} = \{M | M \text{ is a maximum matching of } G\},$$
$$\mathcal{M}' = \{M \in \mathcal{M} | \{u_1u_2, \cdots, u_{2l-1}u_{2l} \subset M\},$$
$$\mathcal{M}^* = \{M \in \mathcal{M} | xu_1, u_{2l}y, u_{2i}u_{2i+1} \in M, 1 \leq i \leq l-1\},$$
$$\mathcal{M}_i = \{M \in \mathcal{M} | M \text{ misses } u_i\}, \text{ for } 1 \leq i \leq 2l.$$

Then $(\mathcal{M}', \mathcal{M}^*, \mathcal{M}_1, \cdots, \mathcal{M}_{2l})$ is a partition of $\mathcal{M}$. Clearly, $|\mathcal{M}'| = |E(G_{k-1})| + \frac{l_1}{2} + \cdots + \frac{l_{k-1}}{2} - k + 1$. By Lemma 2.3, $G - x - y$ has $\frac{l_k}{2}$ maximum matchings. Simple checks show that $|\mathcal{M}^*| = \frac{l_k}{2}$. By Theorem 1, $G - x$ and $G - y$ have a unique perfect matching, respectively. It follows that $|\mathcal{M}_i| = 1$ for $1 \leq i \leq 2l$. Thus $|\mathcal{M}| = |E(G_{k-1})| + \frac{l_1}{2} + \cdots + \frac{l_{k-1}}{2} - k + 1 + 2l + \frac{l_k}{2} = |E(G)| + \frac{l_1}{2} + \cdots + \frac{l_k}{2} - k$. The proof is completed. ∎

According to Theorem 4, we can easily obtain a sufficient condition that a factor-critical graph G has precisely $|E(G)| - c + 1$ maximum matchings shown as the corollary in the following, where c is the number of blocks of G. In fact, the condition in the corollary is a sufficient and necessary condition that a factor-critical graph G has precisely $|E(G)| - c + 1$ maximum matchings as shown in Theorem 9 in [7].

Corollary 2.1. *Let G be a factor-critical graph and the ear decomposition $G = C + P_1 + \cdots + P_k$ satisfy that for any open ear P_i, two ends of P_i are joined in G_{i-1} by a pending path of G with length 2. Then G has precisely $|E(G)| - c + 1$ maximum matchings, where c is the number of blocks of G.*

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