Linesearch methods for equilibrium problems and an infinite family of nonexpansive mappings

P.N. Anh*and D.D. Thanh †

Abstract. In this paper, we propose new projection methods for finding a common element of the set of fixed points of an infinite family of nonexpansive mappings and the solution set of a pseudomonotone equilibrium problem. The iterative schemes are based on the extended extragradient methods, fixed point methods and Armijo-type linesearch techniques. We show that all of the iterative sequences generated by the scheme converge to the common element under mild assumptions on parameters.

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1 Introduction

Let C be a nonempty, closed and convex subset of $\mathcal{R}^s$ and f be a bifunction from $C \times C$ to $\mathcal{R}$ such that $f(x, x) = 0$ for all $x \in C$. Then f is called *strongly monotone* on C iff there exists a positive constant α such that

$$f(x, y) + f(y, x) \leq -\alpha\|x - y\|^2, \quad \forall x, y \in C;$$

monotone on C iff

$$f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in C;$$

pseudomonotone on C iff

$$f(x, y) \geq 0 \text{ implies } f(y, x) \leq 0, \quad \forall x, y \in C;$$

Lipschitz-type continuous on C if there exists positive constants $c_1 > 0, c_2 > 0$ such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2, \quad \forall x, y, z \in C.$$

The equilibrium problem in the sense of Blum and Oettli [8] is presented as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \quad \forall y \in C. \qquad EP(f, C)$$

Denote the set of solutions of Problem $EP(f, C)$ by $Sol(f, C)$. Problem $EP(f, C)$ contains optimization problems, variational inequalities, minimax problems, fixed point problems,

*Department of Scientific Fundamentals, Posts and Telecommunications Institute of Technology, Hanoi, Vietnam (anhpn@ptit.edu.vn).

†Department of Mathematics, Haiphong university, Vietnam.

Nash equilibrium as special cases, and many methods have been proposed to solve this problem (see [4, 8, 15, 19]). Recall that a mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|S(x) - S(y)\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Denote the set of fixed points of S by $Fix(S)$.

In this paper, we are interested in the problem of find a common element of the solution set of Problem $EP(f, C)$ and the set of fixed points $\cap_{k=1}^{\infty} Fix(S_k)$ of an infinite family of nonexpansive mappings $\{S_k\}$, namely:

$$\text{Find } x^* \in \cap_{i=1}^{\infty} Fix(S_i) \cap Sol(f, C), \tag{1.1}$$

where the function f and the mappings $S_k (k = 1, 2, \cdots)$ satisfy the following conditions:

A_1. f is pseudomonotone and continuous on C,

A_2. $\cap_{i=1}^{\infty} Fix(S_i) \cap Sol(f, C) \neq \emptyset$,

A_3. For each $x \in C, f(x, \cdot)$ is convex and subdifferentiable on C,

A_4. If the sequence $\{t^k\}$ is bounded then $\{v^k\}$ is also bounded, where $v^k \in \partial_2 f(t^k, t^k)$,

A_5. S_k is nonexpansive on C for all $k = 1, 2, \cdots$.

An important special case of Problem (1.1) is that $f(x, y) = \langle F(x), y - x\rangle$, where $F : C \to R^s$ and this problem is reduced to find a common element of the solution set of variational inequalities and the set of fixed points of an infinite family of nonexpansive mappings (see [1, 5, 9, 10, 12, 16, 21]).

The solution methods for finding a common element of the set of solutions of Problem (1.1) have been recently well-studied in many research papers, for instance (see [2, 6, 11, 13, 17, 18, 20, 22]). Most of these algorithms are based on solving the underlying approximation equilibrium problem, namely that the sequence $\{x^n\}$ is generated by $x^0 \in C$ and, for each $n \geq 0$, by

$$x^{n+1} \in C : \; f(x^{n+1}, y) + \frac{1}{r_n} \langle y - x^{n+1}, x^{n+1} - x^n\rangle \geq 0, \quad \forall y \in C,$$

where $\{r_n\} \subset (0, \infty)$ satisfies the condition $\liminf_{n\to\infty} r_n > 0$.

Motivated by the viscosity method in [7] via an improvement set of iteration methods in [3] under modified Armijo linesearch techniques, we introduce a new iteration scheme for finding a common element of Problem (1.1). At each iteration n, the main work is to solve a strongly convex program with a *pseudomonotone* bifunction without Lipschitz-type continuous conditions and compute the projection of a point on a closed convex set. We show that the iterative sequences generated by this algorithm converge to the common element.

The paper is organized as follows. Section 2 recalls some concepts in equilibrium problems and fixed point problems that will be used in the sequel and an iterative algorithm for solving Problem (1.1). Section 3 investigates the convergence of the algorithms presented in Section 2 as the main results of our paper. Application to find a common element of the solution set of variational inequalities and the set of fixed points of an infinite family of nonexpansive mappings is presented in Section 4.

2 Preliminaries

For finding a common fixed point of an infinite family of nonexpansive mappings $\{S_k\}$, Aoyama et al. in [7] introduced an iterative sequence $\{x^k\}$ of C defined by $x^0 \in C$ and

$$x^{k+1} = \alpha_k x^k + (1-\alpha_k)S_k(x^k), \quad \forall k \geq 0, \tag{2.1}$$

where C is a closed convex subset of a real Hilbert space, $\{\alpha_k\} \subset [0,1]$ and $\cap_{i=1}^{\infty} Fix(S_i) \neq \emptyset$. The authors proved that the sequence $\{x^k\}$ converges strongly to $x^* \in \cap_{i=1}^{\infty} Fix(S_i)$. Recently, this scheme has been extended by Jao et al. (see [19]) to find a common element of the set of solutions of Problem (1.1) in a real Hilbert space:

$$\begin{cases} f(y^k, x) + \frac{1}{r_k}\langle x - y^k, y^k - x^k\rangle \geq 0, \quad \forall x \in C, \\ x^{k+1} := \alpha_k f(x^k) + \beta_k x^k + \gamma_k W_k(y^k), \end{cases}$$

where $f : C \to C$ is contractive and W_k is W-mapping of $\{S_k\}$. Under mild assumptions on parameters, the authors proved that the sequences $\{x^k\}$ and $\{y^k\}$ converge strongly to x^*, where

$$x^* = Pr_{\cap_{i=1}^{\infty} Fix(S_i) \cap Sol(f,C)}\big(f(x^*)\big).$$

In the following algorithm, the main steps are to solve a strongly convex problem, and compute the next iteration point by fixed point and modified Armijo linesearch techniques.

Algorithm 2.1 *Choose* $x^0 \in C, \gamma \in (0,1), 0 < \sigma < \frac{\beta}{2}$ *and* $\{\alpha_k\} \subset [a,b] \subset (0,1)$.

Step 1. *Solve the strongly convex problem*

$$y^k = argmin\{f(x^k, y) + \frac{\beta}{2}\|y - x^k\|^2 : \; y \in C\} \quad \textit{and set} \quad r(x^k) = x^k - y^k.$$

If $\|r(x^k)\| \neq 0$ *then go to Step* 2. *Otherwise, set* $w^k = x^k$ *and go to Step* 3.

Step 2. *(Armijo-type linesearch technique) Find the smallest positive integer number* m_k *such that*

$$f\big(x^k - \gamma^{m_k} r(x^k), y^k\big) \leq -\sigma ||r(x^k)||^2. \tag{2.2}$$

Compute

$$w^k = Pr_{C \cap H_k}(x^k),$$

where $z^k = x^k - \gamma^{m_k} r(x^k), v^k \in \partial_2 f(z^k, z^k)$ *and* $H_k = \{x \in \mathcal{R}^s : \; \langle v^k, x - z^k\rangle \leq 0\}$, *and go to Step* 3.

Step 3. *Compute* $x^{k+1} = \alpha_k w^k + (1-\alpha_k)S_k(w^k)$. *Increase* k *by* 1 *and go back to Step* 1.

To investigate the convergence of Algorithm 2.1, we recall the following technical lemma which will be used in the sequel.

Lemma 2.2 *(see [14]) Let* $\mathcal{H}$ *be a real Hilbert space,* $\{\alpha_n\}$ *be a sequences of real numbers such that* $0 < a \leq \alpha_n \leq b < 1$ *for all* $n = 0, 1, \cdots$, $\{v^n\}$ *and* $\{w^n\}$ *be sequences in* $\mathcal{H}$ *such that*

$$\limsup_{n\to\infty} \|v^n\| \leq c,$$
$$\limsup_{n\to\infty} \|w^n\| \leq c,$$
$$\lim_{n\to\infty} \|\alpha_n v^n + (1-\alpha_n)w^n\| = c,$$

for some $c \geq 0$. Then,

$$\lim_{n\to\infty} \|v^n - w^n\| = 0.$$

Lemma 2.3 *(see [15]) Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Suppose that, for all $u \in C$, the sequence $\{x^n\}$ satisfies*

$$\|x^{n+1} - u\| \leq \|x^n - u\|, \quad \forall n \geq 0.$$

Then, the sequence $\{Pr_C(x^n)\}$ converges strongly to $\bar{x} \in C$.

3 Convergence Results

In this section, we show the convergence of the sequences $\{x^n\}, \{y^n\}$ and $\{w^n\}$ defined by Algorithm 2.1 which solves the problem of finding a common element of two sets $\cap_{i=1}^p Fix(S_i)$ and $Sol(f, C)$ for a pseudomonotone bifunction f without Lipschitz-type continuity. We now state and prove the convergence of the proposed iteration algorithm.

Theorem 3.1 *Let C be a nonempty closed convex subset of $\mathcal{R}^s$, $S_i : C \to C$ for all $i \geq 1$ and $f : C \times C \to \mathcal{R}$ satisfy Assumptions A_1-A_5. Then the sequences $\{x^k\}, \{y^k\}$ and $\{w^k\}$ generated by Algorithm 2.1 converge to the same point $x^* \in \cap_{i=1}^\infty Fix(S_i) \cap Sol(f, C)$, where*

$$x^* = \lim_{k\to\infty} Pr_{\cap_{i=1}^\infty Fix(S_i) \cap Sol(f,C)}(x^k).$$

Proof. We divide the proof into several steps.

Step 1. If there exists k_0 such that $x^k = y^k$ for all $k \geq k_0$ then the sequence $\{x^k\}$ converges $x^* \in \cap_{i=1}^\infty Fix(S_i) \cap Sol(f, C)$.
Proof of Step 1. If $\|r(x^k)\| = 0$, then $x^k \in Sol(f, C)$ (see [3]). Then, we have

$$x^{k+1} = \alpha_k x^k + (1 - \alpha_k) S_k(x^k), \quad \forall k \geq k_0.$$

By (2.1), the sequence $\{x^k\}$ converge to a point $x^* \in \cap_{i=1}^\infty Fix(S_i)$. So, $x^* \in \cap_{i=1}^\infty Fix(S_i) \cap Sol(f, C)$.

Step 2. If $\|r(x^k)\| \neq 0$ then we show that there exists the smallest nonnegative integer m_k such that

$$f(x^k - \gamma^{m_k} r(x^k), y^k) \leq -\sigma \|r(x^k)\|^2.$$

Proof of Step 2. For $\|r(x^k)\| \neq 0$ and $\gamma \in (0, 1)$, we suppose for contradiction that for every nonnegative integer m, we have

$$f\big(x^k - \gamma^m r(x^k), y^k\big) + \sigma ||r(x^k)||^2 > 0.$$

Passing to the limit above inequality as $m \to \infty$, by continuity of f, we obtain

$$f\big(x^k, y^k\big) + \sigma ||r(x^k)||^2 \geq 0. \tag{3.1}$$

On the other hand, since y^k is the unique solution of the strongly convex problem

$$\min\{f(x^k, y) + \frac{\beta}{2} ||y - x^k||^2 : \ y \in C\},$$

we have

$$f(x^k, y) + \frac{\beta}{2}||y - x^k||^2 \geq f(x^k, y^k) + \frac{\beta}{2}||y^k - x^k||^2, \quad \forall y \in C.$$

With $y = x^k$, the last inequality implies

$$f(x^k, y^k) + \frac{\beta}{2}||r(x^k)||^2 \leq 0. \tag{3.2}$$

Combining (3.1) with (3.2), we obtain

$$\sigma||r(x^k)||^2 \geq \frac{\beta}{2}||r(x^k)||^2.$$

Hence it must be either $\|r(x^k)\| = 0$ or $\sigma \geq \frac{\beta}{2}$. The first case contradicts to $\|r(x^k)\| \neq 0$, while the second one contradicts to the fact $\sigma < \frac{\beta}{2}$.

Step 3. For $\|r(x^k)\| \neq 0$, we show that $x^k \notin H_k$.
Proof of Step 3. From $z^k = x^k - \gamma^{m_k} r(x^k)$, it follows that

$$y^k - z^k = \frac{1 - \gamma^{m_k}}{\gamma^{m_k}}(z^k - x^k).$$

Then using (2.2) and the assumption $f(x, x) = 0$ for all $x \in C$, we have

$$\begin{aligned}
0 &> -\sigma\|r(x^k)\|^2 \\
&\geq f(z^k, y^k) \\
&= f(z^k, y^k) - f(z^k, z^k) \\
&\geq \langle v^k, y^k - z^k \rangle \\
&= \frac{1 - \gamma^{m_k}}{\gamma^{m_k}} \langle z^k - x^k, v^k \rangle.
\end{aligned}$$

Hence

$$\langle x^k - z^k, v^k \rangle > 0.$$

This implies that $x^k \notin H_k$.

Step 4. For $\|r(x^k)\| \neq 0$, we show that $w^k = Pr_{C \cap H_k}(\bar{y}^k)$, where $\bar{y}^k = Pr_{H_k}(x^k)$.
Proof of Step 4. For $K = \{x \in \mathcal{R}^s : \ \langle w, x - x^0 \rangle \leq 0\}$ and $\|w\| \neq 0$, we know that

$$Pr_K(y) = y - \frac{\langle w, y - x^0 \rangle}{\|w\|^2} w.$$

Hence,

$$\begin{aligned}
\bar{y}^k &= Pr_{H_k}(x^k) \\
&= x^k - \frac{\langle v^k, x^k - z^k \rangle}{\|v^k\|^2} v^k \\
&= x^k - \frac{\gamma^{m_k} \langle v^k, r(x^k) \rangle}{\|v^k\|^2} v^k.
\end{aligned}$$

Otherwise, for every $y \in C \cap H_k$ there exists $\lambda \in (0,1)$ such that

$$\hat{x} = \lambda x^k + (1-\lambda)y \in C \cap \partial H_k,$$

where

$$\partial H_k = \{x \in \mathcal{R}^s : \langle v^k, x - z^k \rangle = 0\}.$$

From Step 2, it follows that $x^k \in C$ but $x^k \notin H_k$. Therefore, we have

$$\begin{aligned}
\|y - \bar{y}^k\|^2 &\geq (1-\lambda)^2 \|y - \bar{y}^k\|^2 \\
&= \|\hat{x} - \lambda x^k - (1-\lambda)\bar{y}^k\|^2 \\
&= \|(\hat{x} - \bar{y}^k) - \lambda(x^k - \bar{y}^k)\|^2 \\
&= \|\hat{x} - \bar{y}^k\|^2 + \lambda^2 \|x^k - \bar{y}^k\|^2 - 2\lambda \langle \hat{x} - \bar{y}^k, x^k - \bar{y}^k \rangle \\
&= \|\hat{x} - \bar{y}^k\|^2 + \lambda^2 \|x^k - \bar{y}^k\|^2 \\
&\geq \|\hat{x} - \bar{y}^k\|^2, \qquad (3.3)
\end{aligned}$$

because $\bar{y}^k = Pr_{H_k}(x^k)$. Also we have

$$\begin{aligned}
\|\hat{x} - x^k\|^2 &= \|\hat{x} - \bar{y}^k + \bar{y}^k - x^k\|^2 \\
&= \|\hat{x} - \bar{y}^k\|^2 - 2\langle \hat{x} - \bar{y}^k, x^k - \bar{y}^k \rangle + \|\bar{y}^k - x^k\|^2 \\
&= \|\hat{x} - \bar{y}^k\|^2 + \|\bar{y}^k - x^k\|^2.
\end{aligned}$$

Using $w^k = Pr_{C \cap H_k}(x^k)$ and the Pythagorean theorem, we can reduce that

$$\begin{aligned}
\|\hat{x} - \bar{y}^k\|^2 &= \|\hat{x} - x^k\|^2 - \|\bar{y}^k - x^k\|^2 \\
&\geq \|w^k - x^k\|^2 - \|\bar{y}^k - x^k\|^2 \\
&= \|w^k - \bar{y}^k\|^2. \qquad (3.4)
\end{aligned}$$

From (3.3) and (3.4), we have

$$\|w^k - \bar{y}^k\| \leq \|y - \bar{y}^k\|, \quad \forall y \in C \cap H_k,$$

which means

$$w^k = Pr_{C \cap H_k}(\bar{y}^k).$$

Step 5. Claim that if $\|r(x^k)\| > 0$ and the sequence $\{v^k\}$ is uniformly bounded by $M > 0$ then the sequence $\{\|x^k - x^*\|\}$ is nonincreasing and hence convergent. Moreover, we have

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - (1-b)\|w^k - \bar{y}^k\|^2 - (1-b)\left(\frac{\gamma^{m_k}\sigma}{M(1-\gamma^{m_k})}\right)^2 \|r(x^k)\|^4. \quad (3.5)$$

where $\bar{y}^k = Pr_{H_k}(x^k)$ and $x^* \in \cap_{i=1}^{\infty} Fix(S_i) \cap Sol(f, C)$.

Proof of Step 5. By Step 4, we have $w^k = Pr_{C \cap H_k}(\bar{y}^k)$, i.e.,

$$\langle \bar{y}^k - w^k, z - w^k \rangle \leq 0, \quad \forall z \in C \cap H_k,$$

where $\bar{y}^k = Pr_{H_k}(x^k)$. Substituting $z = x^* \in C \cap H_k$, then we have

$$\langle \bar{y}^k - w^k, x^* - w^k \rangle \leq 0,$$

which implies that

$$\|w^k - \bar{y}^k\|^2 \le \langle w^k - \bar{y}^k, x^* - \bar{y}^k\rangle.$$

Hence

$$\begin{aligned}
\|w^k - x^*\|^2 &= \|w^k - \bar{y}^k + \bar{y}^k - x^*\|^2 \\
&= \|w^k - \bar{y}^k\|^2 + \|\bar{y}^k - x^*\|^2 + 2\langle w^k - \bar{y}^k, \bar{y}^k - x^*\rangle \\
&\le \langle x^* - \bar{y}^k, w^k - \bar{y}^k\rangle + \|\bar{y}^k - x^*\|^2 + 2\langle w^k - \bar{y}^k, \bar{y}^k - x^*\rangle \\
&= \|\bar{y}^k - x^*\|^2 + \langle w^k - \bar{y}^k, \bar{y}^k - x^*\rangle \\
&\le \|\bar{y}^k - x^*\|^2 - \|w^k - \bar{y}^k\|^2. \qquad (3.6)
\end{aligned}$$

Since $z^k = x^k - \gamma^{m_k} r(x^k)$ and

$$\bar{y}^k = Pr_{H_k}(x^k) = x^k - \frac{\langle v^k, x^k - z^k\rangle}{\|v^k\|^2} v^k,$$

we have

$$\begin{aligned}
&\|\bar{y}^k - x^*\|^2 \\
&= \|x^k - x^*\|^2 + \frac{\langle v^k, x^k - z^k\rangle^2}{\|v^k\|^4}\|v^k\|^2 - \frac{2\langle v^k, x^k - z^k\rangle}{\|v^k\|^2}\langle v^k, x^k - x^*\rangle \\
&= \|x^k - x^*\|^2 + \left(\frac{\gamma^{m_k}\langle v^k, r(x^k)\rangle}{\|v^k\|}\right)^2 - \frac{2\gamma^{m_k}\langle v^k, r(x^k)\rangle}{\|v^k\|^2}\langle v^k, x^k - x^*\rangle \\
&= \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k}\langle v^k, r(x^k)\rangle}{\|v^k\|}\right)^2 \\
&\quad -2\left[\frac{\gamma^{m_k}\langle v^k, r(x^k)\rangle}{\|v^k\|^2}\langle v^k, x^k - x^*\rangle - \left(\frac{\gamma^{m_k}\langle v^k, r(x^k)\rangle}{\|v^k\|}\right)^2\right] \\
&= \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k}\langle v^k, r(x^k)\rangle}{\|v^k\|}\right)^2 \\
&\quad -\frac{2\gamma^{m_k}\langle v^k, r(x^k)\rangle}{\|v^k\|^2}\left[\langle v^k, x^k - x^*\rangle - \gamma^{m_k}\langle v^k, r(x^k)\rangle\right] \\
&= \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k}\langle v^k, r(x^k)\rangle}{\|v^k\|}\right)^2 \\
&\quad -\frac{2\gamma^{m_k}\langle v^k, r(x^k)\rangle}{\|v^k\|^2}\langle v^k, x^k - x^* - \gamma^{m_k} r(x^k)\rangle \\
&= \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k}\langle v^k, r(x^k)\rangle}{\|v^k\|}\right)^2 - \frac{2\gamma^{m_k}\langle v^k, r(x^k)\rangle}{\|v^k\|^2}\langle v^k, z^k - x^*\rangle. \qquad (3.7)
\end{aligned}$$

It follows from $v^k \in \partial_2 f(z^k, z^k)$ that

$$f(z^k, y) - f(z^k, z^k) \ge \langle v^k, y - z^k\rangle, \quad \forall y \in C. \qquad (3.8)$$

Replacing y by y^k and combining with assumptions $f(z^k, z^k) = 0$ and $z^k = x^k - \gamma^{m_k} r(x^k)$, we have

$$\begin{aligned}
f(z^k, y^k) &\ge \langle v^k, y^k - z^k\rangle \\
&= -(1 - \gamma^{m_k})\langle v^k, r(x^k)\rangle.
\end{aligned}$$

Combining this inequality with (2.2) and assumption $\gamma \in (0,1)$, we obtain

$$\langle v^k, r(x^k)\rangle \geq \frac{\sigma}{1-\gamma^{m_k}}\|r(x^k)\|^2. \tag{3.9}$$

Substituting $y = x^*$ into (3.8) and using $f(z^k, z^k) = 0$, we have

$$f(z^k, x^*) \geq \langle v^k, x^* - z^k\rangle. \tag{3.10}$$

Since f is pseudomonotone on C and $f(x^*, x) \geq 0, \ \ \forall x \in C$, we have

$$f(z^k, x^*) \leq 0.$$

Combining this with (3.10), we get

$$0 \geq \langle v^k, x^* - z^k\rangle. \tag{3.11}$$

Using (3.7), (3.9) and (3.11), we have

$$\begin{aligned} \|\bar{y}^k - x^*\|^2 &\leq \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k}\langle v^k, r(x^k)\rangle}{\|v^k\|}\right)^2 \\ &\leq \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k}\sigma}{\|v^k\|(1-\gamma^{m_k})}\right)^2 \|r(x^k)\|^4. \end{aligned} \tag{3.12}$$

Combining (3.6) with (3.12), we obtain

$$\|w^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|w^k - \bar{y}^k\|^2 - \left(\frac{\gamma^{m_k}\sigma}{\|v^k\|(1-\gamma^{m_k})}\right)^2 \|r(x^k)\|^4. \tag{3.13}$$

Using (3.13) and $x^{k+1} = \alpha_k w^k + (1-\alpha_k)S_k(w^k)$, we have

$$\begin{aligned} \|x^{k+1} - x^*\|^2 =& \|\alpha_k(w^k - x^*) + (1-\alpha_k)\big(S_k(w^k) - x^*\big)\|^2 \\ \leq& \alpha_k\|w^k - x^*\|^2 + (1-\alpha_k)\|S_k(w^k) - S_k(x^*)\|^2 \\ \leq& \alpha_k\|w^k - x^*\|^2 + (1-\alpha_k)\|w^k - x^*\|^2 \\ \leq& \|x^k - x^*\|^2 - (1-\alpha_k)\|w^k - \bar{y}^k\|^2 - (1-\alpha_k)\left(\frac{\gamma^{m_k}\sigma}{\|v^k\|(1-\gamma^{m_k})}\right)^2 \|r(x^k)\|^4 \\ \leq& \|x^k - x^*\|^2 - (1-b)\|w^k - \bar{y}^k\|^2 - (1-b)\left(\frac{\gamma^{m_k}\sigma}{\|v^k\|(1-\gamma^{m_k})}\right)^2 \|r(x^k)\|^4. \end{aligned} \tag{3.14}$$

In the case $\|r(x^k)\| = 0$, by Algorithm 2.1, we have $w^k = x^k$ and

$$\begin{aligned} \|x^{k+1} - x^*\| =& \|\alpha_k x^k + (1-\alpha_k)S_k(x^k) - x^*\|^2 \\ =& \|\alpha_k(x^k - x^*) + (1-\alpha_k)\big(S_k(x^k) - x^*\big)\|^2 \\ \leq& \|x^k - x^*\|^2. \end{aligned} \tag{3.15}$$

Using (3.15) and (3.14), we have

$$\|x^{k+1} - x^*\| \leq \|x^k - x^*\|, \ \ \forall k \geq 0.$$

So the sequence $\{\|x^k - x^*\|\}$ is nonincreasing and hence convergent. Since (3.14) and the sequence $\{v^k\}$ is uniformly bounded by $M > 0$, i.e.,

$$\|v^k\| \leq M, \quad \forall k \geq 0,$$

we obtain (3.5).

Now we denote $\{x^{k_j}\}$ which is a subsequence of $\{x^k\}$ such that

$$\|r(x^{k_j-1})\| \neq 0, \quad \forall j \geq 0.$$

Step 6. Suppose that $x^* \in \cap_{i=1}^{\infty} Fix(S_i) \cap Sol(f, C)$ and the sequence $\{v^k\}$ is uniformly bounded by $M > 0$, we show that

$$\begin{aligned} \|x^{k_{j+1}} - x^*\|^2 \leq \|x^{k_j} - x^*\|^2 - (1-b)\|w^{k_j+p} - \bar{y}^{k_j+p}\|^2 \\ - (1-b)\left(\frac{\gamma^{m_{k_j+p}}\sigma}{M(1-\gamma^{m_{k_j+p}})}\right)^2 \|r(x^{k_j+p})\|^4. \end{aligned} \tag{3.16}$$

where $p = k_{j+1} - k_j - 1, \bar{y}^{k_j+p} = Pr_{H_{k_j+p}}(x^{k_j+p})$.
Proof of Step 6. If $k_{j+1} = k_j + 1$ then it is clear from Step 5. Otherwise, we suppose that there exists a positive integer p such that $k_j + p + 1 = k_{j+1}$. Note that $\|r(x^{k_j+i})\| = 0$ for all $i = 0, 1, \cdots, p-1$. By $r(x^{k_j+p}) \neq 0$, (3.15) and Step 4, we have

$$\begin{aligned} \|x^{k_{j+1}} - x^*\|^2 =& \|x^{k_j+p+1} - x^*\|^2 \\ \leq& \|x^{k_j+p} - x^*\|^2 - (1-\alpha_{k_j+p})\|w^{k_j+p} - \bar{y}^{k_j+p}\|^2 \\ &- (1-\alpha_{k_j+p})\left(\frac{\gamma^{m_{k_j+p}}\sigma}{M(1-\gamma^{m_{k_j+p}})}\right)^2 \|r(x^{k_j+p})\|^4 \\ \leq& \|\alpha_{k_j+p-1}x^{k_j+p-1} + (1-\alpha_{k_j+p-1})S_{k_j+p-1}(x^{k_j+p-1}) - x^*\|^2 \\ &- (1-b)\|w^{k_j+p} - \bar{y}^{k_j+p}\|^2 - (1-b)\left(\frac{\gamma^{m_{k_j+p}}\sigma}{M(1-\gamma^{m_{k_j+p}})}\right)^2 \|r(x^{k_j+p})\|^4 \\ \leq& \|x^{k_j+p-1} - x^*\|^2 - (1-b)\|w^{k_j+p} - \bar{y}^{k_j+p}\|^2 \\ &- (1-b)\left(\frac{\gamma^{m_{k_j+p}}\sigma}{M(1-\gamma^{m_{k_j+p}})}\right)^2 \|r(x^{k_j+p})\|^4 \\ \leq& \cdots \\ \leq& \|x^{k_j} - x^*\|^2 - (1-b)\|w^{k_j+p} - \bar{y}^{k_j+p}\|^2 \\ &- (1-b)\left(\frac{\gamma^{m_{k_j+p}}\sigma}{M(1-\gamma^{m_{k_j+p}})}\right)^2 \|r(x^{k_j+p})\|^4. \end{aligned}$$

This implies (3.16).

Step 7. We claim that if $\|r(x^k)\| \neq 0$ then $Sol(f, C) \subseteq C \cap H_k$.
Proof of Step 7. Suppose $x^* \in Sol(f, C)$. Using the definition of x^*, $f(x^*, x) \geq 0$ for all $x \in C$ and f is pseudomonotone on C, we get

$$f(z^k, x^*) \leq 0. \tag{3.17}$$

It follows from $v^k \in \partial_2 f(z^k, z^k)$ that

$$\begin{aligned} f(z^k, x^*) =& f(z^k, x^*) - f(z^k, z^k) \\ \geq& \langle v^k, x^* - z^k \rangle. \end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18), we have

$$\langle v^k, x^* - z^k \rangle \leq 0.$$

By the definition of H_k, we have $x^* \in H_k$. Thus $Sol(f, C) \subseteq C \cap H_k$.

Step 8. We claim that there exists $c = \lim\limits_{k\to\infty} \|x^k - x^*\| = \lim\limits_{k\to\infty} \|w^k - x^*\|$, where $x^* \in \cap_{i=1}^{\infty} Fix(S_i) \cap Sol(f, C)$. Consequently, the sequences $\{x^k\}, \{y^k\}, \{z^k\}, \{v^k\}$ and $\{w^k\}$ are bounded.

Proof of Step 8. By Step 6, we denote

$$c = \lim_{k\to\infty} \|x^k - x^*\|. \tag{3.19}$$

From $w^k = x^k$ if $\|r(x^k)\| = 0$, $w^k = Pr_{C\cap H_k}(x^k)$ if $\|r(x^k)\| \neq 0$ and Step 7, it follows that

$$\|w^k - x^*\| \leq \|x^k - x^*\|, \quad \forall k \geq 0.$$

Hence

$$\lim_{k\to\infty} \|w^k - x^*\| \leq \lim_{k\to\infty} \|x^k - x^*\| = c. \tag{3.20}$$

By the similar way as (3.15), for $x^{k+1} = \alpha_k w^k + (1-\alpha_k)\bar{S}_k(w^k)$, we have

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|\alpha_k w^k + (1-\alpha_k) S_k(w^k) - x^*\|^2 \\ &= \|\alpha_k (w^k - x^*) + (1-\alpha_k)\big(S_k(w^k) - x^*\big)\|^2 \\ &\leq \alpha_k \|w^k - x^*\|^2 + (1-\alpha_k)\|S_k(w^k) - S_k(x^*)\|^2 \\ &\leq \|w^k - x^*\|^2. \end{aligned}$$

Hence

$$c \leq \lim_{k\to\infty} \|w^k - x^*\|. \tag{3.21}$$

From (3.21) and (3.20), it follows that

$$c = \lim_{k\to\infty} \|w^k - x^*\|.$$

Since y^k is the unique solution to

$$\min\{f(x^k, y) + \frac{\beta}{2}\|y - x^k\|^2 : \ y \in C\},$$

we have

$$f(x^k, y) + \frac{\beta}{2}\|y - x^k\|^2 \geq f(x^k, y^k) + \frac{\beta}{2}\|y^k - x^k\|^2, \quad \forall y \in C.$$

With $y = x^k \in C$ and $f(x^k, x^k) = 0$, we have

$$0 \geq f(x^k, y^k) + \frac{\beta}{2}\|y^k - x^k\|^2. \tag{3.22}$$

Since $f(x^k, \cdot)$ is convex and subdifferentiable on C, i.e.,

$$f(x^k, y) - f(x^k, x^k) \geq \langle s^k, y - x^k \rangle, \quad \forall y \in C,$$

where $s^k \in \partial_2 f(x^k, x^k)$. Using $y = y^k$, we have

$$f(x^k, y^k) \geq \langle s^k, y^k - x^k \rangle.$$

Combining this and (3.22), we obtain

$$\langle s^k, y^k - x^k \rangle + \frac{\beta}{2}\|x^k - y^k\|^2 \leq 0.$$

Hence

$$\|x^k - y^k + \frac{1}{\beta}s^k\| \leq \frac{1}{\beta}\|s^k\|. \tag{3.23}$$

From Assumption A_4 and (3.19), it implies that the sequence $\{s^k\}$ is bounded. Then, it follows from (3.23) that $\{y^k\}$ is bounded and hence the sequences $\{z^k = x^k - \gamma^{m_k}(x^k - y^k)\}, \{v^k\}$ are bounded.

Step 9. We show that the sequences $\{x^k\}, \{y^k\}$ and $\{w^k\}$ converge to $\bar{x} \in \cap_{i=1}^{\infty} Fix(S_i) \cap Sol(f, C)$.

Proof of Step 9. Using Step 6 and Step 8, we obtain that the sequence $\{v^k\}$ is bounded by $M > 0$ and

$$\|x^{n_{j+1}} - x^*\|^2 \leq \|x^{k_j} - x^*\|^2 - (1-b)\left(\frac{\gamma^{m_{k_j+p}}\sigma}{M(1-\gamma^{m_{k_j+p}})}\right)^2 \|r(x^{k_j+p})\|^4,$$

with $p = k_{j+1} - k_j - 1$. Since $\{\|x^k - x^*\|\}$ is convergent, it is easy to see that

$$\lim_{i\to\infty} \gamma^{m_{k_j+p}}\|r(x^{k_j+p})\| = 0.$$

The cases remaining to consider are the following.

Case 1. $\limsup_{k\to\infty} \gamma^{m_{k_j+p}} > 0$. This case must follow that $\liminf_{k\to\infty} \|r(x^{k_j+p})\| = 0$. Since $\{x^{k_j+p}\}$ is bounded, there exists an accumulation point $\bar{x}$ of $\{x^{k_j+p}\}$. In other words, a subsequence $\{x^{k_{j_i}}\}$ converges to some $\bar{x}$ as $i \to \infty$ such that $r(\bar{x}) = 0$. Then by Step 1, we have $\bar{x} \in Sol(f, C)$.

Case 2. $\lim_{k\to\infty} \gamma^{m_{k_j+p}} = 0$. Since $\{\|x^{k_j+p} - x^*\|\}$ is convergent, there is the subsequence $\{x^{k_{j_i}}\}$ of $\{x^{k_j+p}\}$ which converges to $\bar{x}$ as $i \to \infty$. Since m_{k_j+p} is the smallest nonnegative integer, $m_{k_j+p} - 1$ does not satisfy (2.2). Hence, we have

$$f\big(x^{k_{j_i}} - \gamma^{m_{k_{j_i}}-1} r(x^{k_{j_i}}), y^{k_{j_i}}\big) > -\sigma||r(x^{k_{j_i}})||^2.$$

Passing onto the limit as $i \to \infty$ and using the continuity of f, we have $y^{k_{j_i}} \to \bar{y}$ and

$$f(\bar{x}, \bar{y}) \geq -\sigma||r(\bar{x})||^2, \tag{3.24}$$

where $r(\bar{x}) = \bar{x} - \bar{y}$. It follows from (3.2) that

$$f(x^{k_{j_i}-1} - \gamma^{m_{k_{j_i}}} r(x^{k_{j_i}}), y^{k_{j_i}-1}) + \frac{\beta}{2}||r(x^{k_{j_i}-1})||^2 \leq 0.$$

Since f is continuous and passing onto the limit as $i \to \infty$, we obtain

$$f(\bar{x}, \bar{y}) + \frac{\beta}{2}||r(\bar{x})||^2 \leq 0.$$

Combining this with (3.24), we have

$$\sigma\|r(\bar{x})\|^2 \geq -f(\bar{x}, \bar{y}) \geq \frac{\beta}{2}\|r(\bar{x})\|^2.$$

which implies $r(\bar{x}) = 0$, and hence $\bar{x} = \bar{y} \in Sol(f, C)$. Thus every cluster point of the sequence $\{x^{k_j+p}\}$ is solutions of Problem $EP(f, C)$.

Let S be a nonexpansive mapping of C into itself defined by

$$S(x) = \lim_{k\to\infty} S_k(x) \tag{3.25}$$

for all $x \in C$ and suppose that $Fix(S) = \cap_{i=1}^{\infty} Fix(S_i)$.

Now we show that every cluster point of $\{x^{k_j+p}\}$ is fixed points of the nonexpansive mapping S. Suppose that there exists a subsequence $\{x^{k_{j_i}}\}$ of $\{x^{k_j+p}\}$ which converges to $\bar{x}$, as $i \to \infty$. By the above proof, we have $\bar{x} \in Sol(f, C)$. Then $\{y^{k_{j_i}}\}$ and $\{w^{k_{j_i}}\}$ converge also to $\bar{x}$, as $i \to \infty$. For each $x^* \in Sol(f, C) \cap Fix(S)$,

$$\begin{aligned}\|S(w^{k_{j_i}}) - x^*\| &= \|S(w^{k_{j_i}}) - S(x^*)\| \\ &\leq \|w^{k_{j_i}} - x^*\|,\end{aligned}$$

and using Step 7, we have

$$\limsup_{i\to\infty} \|S(w^{k_{j_i}}) - x^*\| \leq \limsup_{i\to\infty} \|w^{k_{j_i}} - x^*\| = c.$$

Further, it follows from (3.19) and Step 7 that

$$\lim_{i\to\infty} \|\alpha_{k_{j_i}}(x^{k_{j_i}} - x^*) + (1 - \alpha_{k_{j_i}})\big(S(w^{k_{j_i}}) - x^*\big)\| = \lim_{i\to\infty} \|x^{k_{j_i}} - x^*\| = c.$$

By Lemma 2.2, we obtain

$$\lim_{i\to\infty} \|S(w^{k_{j_i}}) - x^{k_{j_i}}\| = 0.$$

Then,

$$\begin{aligned}\|S(x^{k_{j_i}}) - x^{k_{j_i}}\| &\leq \|S(x^{k_{j_i}}) - S(w^{k_{j_i}})\| + \|S(w^{k_{j_i}}) - x^{k_{j_i}}\| \\ &\leq \|w^{k_{j_i}} - x^{k_{j_i}}\| + \|S(w^{k_{j_i}}) - x^{k_{j_i}}\|. \end{aligned} \tag{3.26}$$

Applying the property of the projection Pr_C

$$\|Pr_C(x) - x\|^2 \leq \|x - y\|^2 - \|Pr_C(x) - y\|^2, \quad \forall y \in C, x \in \mathcal{R}^s,$$

we have

$$\begin{aligned}\|w^{k_{j_i}} - x^{k_{j_i}}\|^2 &= \|Pr_{C\cap H_{k_{j_i}}}(x^{k_{j_i}}) - x^{k_{j_i}}\|^2\\ &\le \|x^{k_{j_i}} - x^*\|^2 - \|Pr_{C\cap H_{k_{j_i}}}(x^{k_{j_i}}) - x^*\|^2\\ &= \|x^{k_{j_i}} - x^*\|^2 - \|w^{k_{j_i}} - x^*\|^2\\ &\to 0 \ \text{ as } i\to\infty, \end{aligned} \tag{3.27}$$

for every $x^* \in Sol(f,C)\cap Fix(S)$. Combining (3.26) and (3.27), we have

$$\lim_{i\to\infty} \|S(x^{k_{j_i}}) - x^{k_{j_i}}\| = 0.$$

Using $x^{k_{j_i}} \to \bar{x}$ as $i\to\infty$ and $\lim_{i\to\infty}\|S(x^{k_{j_i}}) - x^{k_{j_i}}\| = 0$, we have $\bar{x}\in Fix(S)$.

Thus, $\bar{x}\in Sol(f,C)\cap Fix(S)$, letting $x^*=\bar{x}$ and using Step 7, we have

$$c = \lim_{k\to\infty}\|x^k - \bar{x}\| = \lim_{i\to\infty}\|x^{k_{j_i}} - \bar{x}\| = 0.$$

We conclude that the whole sequence $\{x^k\}$ converges to $\bar{x}\in \cap_{i=1}^{\infty} Fix(S_i)\cap Sol(f,C)$. Consequently, the sequences $\{y^k\}$ and $\{w^k\}$ also converge to $\bar{x}$.

Step 10. We claim that the sequences $\{x^k\},\{y^k\}$ and $\{w^k\}$ converge to $\bar{x}$, where

$$\bar{x} = \lim_{k\to\infty} Pr_{Fix(S)\cap Sol(f,C)}(x^k),$$

where S is defined by (3.25).

Proof of Step 10. Suppose that $t^k := Pr_{Fix(S)\cap \mathrm{Sol}(f,C)}(x^k)$ and $x^k\to\bar{x}$ as $k\to\infty$. By the definition of $Pr_C(\cdot)$, we have

$$\langle t^k - x^k, t^k - x\rangle \le 0, \quad \forall x\in Fix(S)\cap \mathrm{Sol}(f,C). \tag{3.28}$$

It follows from Step 8 that

$$\|x^{k+1} - x^*\| \le \|x^k - x^*\|, \quad \forall k\ge 0, x^*\in Fix(S)\cap\mathrm{Sol}(f,C).$$

Then, by Lemma 2.3, we have

$$t^k = Pr_{Fix(S)\cap\mathrm{Sol}(f,C)}(x^k) \to \hat{x}\in Fix(S)\cap \mathrm{Sol}(f,C) \ \text{ as } \ k\to\infty. \tag{3.29}$$

Pass the limit in (3.28) and combining this with (3.29), we have

$$\langle \hat{x} - \bar{x}, \hat{x} - x\rangle \le 0, \quad \forall x\in Fix(S)\cap\mathrm{Sol}(f,C).$$

This means that $\bar{x}=\hat{x}$ and

$$\bar{x} = \lim_{k\to\infty} Pr_{Fix(S)\cap\mathrm{Sol}(f,C)}(x^k).$$

It follows from Step 8 that the sequences $\{x^k\},\{y^k\}$ and $\{w^k\}$ converge to $\bar{x}$, where

$$\bar{x} = \lim_{k\to\infty} Pr_{\cap_{i=1}^{\infty} Fix(S_i)\cap\mathrm{Sol}(f,C)}(x^k).$$

The proof is completed.

4 Application to Variational Inequalities

Let C be a nonempty, closed and convex subset of $\mathcal{R}^s$ and F be a function from C into $\mathcal{R}^s$. In this section, we consider the variational inequality $VI(F,C)$. The set of solutions of $VI(F,C)$ is denoted by $Sol(F,C)$. Recall that the function F is called

- *strongly monotone* on C with $\beta > 0$ iff
$$\langle F(x) - F(y), x - y\rangle \geq \beta\|x-y\|^2, \quad \forall x, y \in C;$$
- *monotone* on C iff
$$\langle F(x) - F(y), x - y\rangle \geq 0, \quad \forall x, y \in C;$$
- *pseudomonotone* on C iff
$$\langle F(y), x - y\rangle \geq 0 \Rightarrow \langle F(x), x - y\rangle \geq 0, \quad \forall x, y \in C;$$
- *Lipschitz continuous* on C with constants $L > 0$ (shortly, L-Lipschitz continuous) iff
$$\|F(x) - F(y)\| \leq L\|x-y\|, \quad \forall x, y \in C.$$

Since

$$\begin{aligned} y^k &= \mathrm{argmin}\{\lambda f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : \ y \in C\} \\ &= \mathrm{argmin}\{\lambda\langle F(x^k), y - x^k\rangle + \frac{1}{2}\|y - x^k\|^2 : \ y \in C\} \\ &= Pr_C\big(x^k - \lambda F(x^k)\big), \end{aligned}$$

and Theorem 3.1, the convergent theorem for finding a common element of the set of fixed points of an infinite family of nonexpansive mappings $\{S_k\}$ and the solution set $Sol(F,C)$ is presented as follows:

Theorem 4.1 *Let C be a nonempty, closed and convex subset of $\mathcal{R}^s$. Let $F : C \to \mathcal{R}^s$ be pseudomonotone and continuous, $\{S_k\}$ be an infinite family of nonexpansive mappings of C into itself, $\cap_{i=1}^{\infty} Fix(S_i) \cap Sol(F,C) \neq \emptyset, \gamma \in (0,1), 0 < \sigma < \frac{\beta}{2}$, and $\{\alpha_k\} \subset [a,b] \subset (0,1)$. Suppose that the sequences $\{x^k\}, \{y^k\}$, and $\{t^k\}$ are given as the following iteration steps:*

Step 1. *Solve the strongly convex problem*
$$y^k = Pr_C\big(x^k - \frac{1}{\beta}F(x^k)\big) \quad \text{and set} \quad r(x^k) = x^k - y^k.$$
If $\|r(x^k)\| \neq 0$ then go to Step 2. Otherwise, set $w^k = x^k$ and go to Step 3.

Step 2. *(Armijo-type linesearch technique) Find the smallest positive integer number m_k such that*
$$(1 - \gamma^{m_k})\langle F(x^k - \gamma^{m_k} r(x^k)), r(x^k)\rangle \geq \sigma\|r(x^k)\|^2.$$
Compute
$$w^k = Pr_{C\cap H_k}(x^k),$$
where $z^k = x^k - \gamma^{m_k} r(x^k), v^k \in \partial_2 f(z^k, z^k)$ and $H_k = \{x \in \mathcal{R}^s : \ \langle v^k, x - z^k\rangle \leq 0\}$, and go to Step 3.

Step 3. *Compute* $x^{k+1} = \alpha_k w^k + (1-\alpha_k)S_k(w^k)$.

Then, the sequences $\{x^k\}, \{y^k\}$ *and* $\{w^k\}$ *converge to the same point* $x^* \in \cap_{i=1}^{\infty} Fix(S_i) \cap Sol(F,C)$, *where*

$$x^* = \lim_{k\to\infty} Pr_{\cap_{i=1}^{\infty} Fix(S_i)\cap Sol(F,C)}(x^k).$$

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