THE INDICES OF SUBGROUPS OF FINITE GROUPS IN THE JOIN OF THEIR CONJUGATE PAIRS

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Abstract. Let G be a finite group and $H \leq G$. In this paper, we investigate the influence of the index of H in $\langle H, H^g \rangle$ on the structure of G.

Key words: conjugate subgroup pair, index, solvable group, supersolvable group.
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1. Introduction and Preliminaries

All groups in this paper are finite. Let $\pi(G)$ stand for the set of all prime divisors of the order of a group G, $|\pi(G)|$ the number of the elements of $\pi(G)$, $S(G)$ and $Soc(G)$ the largest normal solvable subgroup and the socle of G, respectively. Let $\mathcal{F}$ denote a formation, $\mathcal{U}$ the class of supersolvable groups.

Let Σ be an abstract group theoretical property, for example, solvability, nilpotency, supersolvability, p-closed, etc. Following Chen in [5], if all proper subgroups or all proper quotient groups of a group G have the property Σ but G does not have the property Σ, we say that G is an inner-Σ-group or an outer-Σ-group, respectively. If G is both an inner-Σ-group and an outer-Σ-group, then G is called a minimal non-Σ-group. The inner-Σ-groups here are also called minimal non-Σ-groups in [16, p.258] or [2], or critical groups or $\mathtt{S}$-critical groups for the class of Σ-groups in [7, VII, 6.1] or [1, p.252], respectively. The outer-Σ-groups here just are the groups in the boundary or $\mathtt{Q}$-boundary of Σ in [7, III, 2.1] or [1, 2.3.6]. In [13], we introduced the concepts of inner-Σ-Ω-groups and outer-Σ-Ω-groups by replacing all proper subgroups and all proper quotient groups of a group G with subsets Ω of proper subgroups and proper quotient groups of a group. The other notations and terminologies in this paper are standard (see [12]).

Let $H \leq G$. We have $H \leq \langle H, H^g \rangle \leq \langle H, g \rangle$ for any $g \in G$. It is clear that $H = \langle H, H^g \rangle$ for all $g \in G$ if and only if $H \trianglelefteq G$. In [7], H is called abnormal in G if $\langle H, H^g \rangle = \langle H, g \rangle$ for all $g \in G$. The famous Wielandt theorem shows that $H \lhd \lhd \langle H, H^g \rangle$ for all $g \in G$ if and only if $H \lhd \lhd G$. In [16] H is called pronormal in G if H is conjugate to H^g in $\langle H, H^g \rangle$ for all $g \in G$. Those show that the normality of a subgroup H in G may be detected from the normality of H in $\langle H, H^g \rangle$. Loosely speaking, the more $\langle H, H^g \rangle$ is near $\langle H, g \rangle$, the more H is not normal; the more $\langle H, H^g \rangle$ is near H, the more H is normal. The size of $\langle H, H^g \rangle$ is a measurement of normalities of H in G, or identity the size of $\langle H, H^g \rangle$ by the kinds of generalized normalities of H in G. This leads us to investigate properties of G from the size of H in $\langle H, H^g \rangle$. In this paper, we investigate the influence of the index of the subgroup H in $\langle H, H^g \rangle$ on the structure of G for cyclic subgroups H of G and $g \in G$ and get some results.

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2. Preliminaries

As introduced in [2], the inner-nilpotent groups and minimal non-supersolvable groups are classified completely by Redei in [15] and Nagebeckiĭ in [14] respectively. For the convenience of future reference, we collect the results given by Chen in [5] in Lemma 2.1 and Lemma 2.2.

Lemma 2.1. ([5, Theorem 1.1]) Suppose that G is an inner-nilpotent group. Then

(1) There are primes p and q such that $|G| = p^a q^b$.

(2) G has a normal Sylow q-subgroup Q; if $q > 2$, then $exp(Q) = q$ and if $q = 2$, then $exp(Q) \leq 4$; G has a cyclic Sylow p-subgroup $P = \langle a \rangle$.

(3) Let $c \in Q$, then c is a generator if and only if $[c, a] \neq 1$.

(4) If c is a generator of Q, then $[c, a] = c^{-1}c^a$ is also a generator of Q.

(5) If c is a generator of Q, then $Q = \langle c, c^a, \cdots, c^{a^{p-2}}, c^{a^{p-1}} \rangle$, namely, $Q = \langle [c, a], [c, a]^a, \cdots, [c, a]^{a^{p-1}} \rangle$.

Minimal inner-supersolvale groups have six classes. We fix the notations G_t standing for a group in the t-th classes and G_t may be described in the following Lemma 2.2 in this paper.

Lemma 2.2. ([5, p.49-51, Theorem 7.3]) Suppose that a group G is minimal inner-supersolvable. Then $G \cong G_t$ and G_t satisfies one of the following, where $1 \leq t \leq 6$.

(I) G_1 is a minimal nonabelian group and $|G_1| = pq^\beta$, where $p \nmid q - 1$, $\beta \geq 2$.

(II) $G_2 = \langle a, c_1 \rangle$ and $|G_2| = p^\alpha r^p$ and $p^{\alpha-1} \| r - 1$, where $\alpha \geq 2$,
$a^{p^\alpha} = c_1^r = c_2^r = \cdots = c_p^r = 1$; $c_i c_j = c_j c_i$; $c_i^a = c_{i+1}$, $i = 1, 2, \cdots, p-1$; $c_p^a = c_1^t$, where $t(mod\ r)'s$ exponent is $p^{\alpha-1}$.

(III) $G_3 = \langle a, b, c_1 \rangle$ and $|G_3| = 8r^2$ and $4 \mid r - 1$.
$a^4 = c_1^r = c_2^r = 1$, $a^2 = b^2$, $ba = a^{-1}b$, $c_1^a = c_2$, $c_2^a = c_1^{-1}$, $c_1^b = c_1^s$, $c_2^b = c_2^s$, where $s(mod\ r)'s$ exponent is 4.

(IV) $G_4 = \langle a, b, c_1 \rangle$ and $|G_4| = p^{\alpha+\beta} r^p$ and $p^{max\{\alpha,\beta\}} \mid r - 1$, where $\beta \geq 2$.
$a^{p^\alpha} = b^{p^\beta} = c_1^r = c_2^r = \cdots = c_p^r = 1$; $c_i c_j = c_j c_i$, $ab = b^{1+p^{\beta-1}}a$; $c_i^a = c_{i+1}$, $i = 1, 2, \cdots, p-1$; $c_p^a = c_1^t$, $c_i^b = c_i^{u^{1+ip^{\beta-1}}}$, $i = 1, 2, \cdots, p$; where t and $u(mod\ r)'s$ exponent are $p^{\alpha-1}$ and p^β, respectively.

(V) $G_5 = \langle a, b, c, c_1 \rangle$ and $|G_5| = p^{\alpha+\beta+1} r^p$ and $p^{max\{\alpha,\beta\}} \mid r - 1$.
$a^{p^\alpha} = b^{p^\beta} = c^p = c_1^r = c_2^r = \cdots = c_p^r = 1$; $c_i c_j = c_j c_i$, $ba = abc$, $ca = ac$, $cb = bc$, $c_i^a = c_{i+1}$, $i = 1, 2, \cdots, p-1$; $c_p^a = c_1^t$, $c_i^c = c_i^u$, $c_i^b = c_i^{vu^{p-i+1}}$, where t, v and $u(mod\ r)'s$ exponents are $p^{\alpha-1}$, p^β and p, respectively.

(VI) $G_6 = \langle a, b, c_1 \rangle$, $|G_6| = p^\alpha q r^p$ and $p^\alpha q \mid r - 1$, $p \mid q - 1$, $\alpha \geq 1$.
$a^{p^\alpha} = b^q = c_1^r = c_2^r = \cdots = c_p^r = 1$; $c_i c_j = c_j c_i$, $i, j = 1, 2, \cdots, p$; $c_i^a = c_{i+1}$, $i = 1, 2, \cdots, p-1$; $c_p^a = c_1^t$; $b^a = b^u$, $c_i^b = c_i^{vu^{i-1}}$, $i = 1, 2, \cdots, p$; where t and $v(mod\ r)'s$ exponents are $p^{\alpha-1}$ and q, respectively; $u(mod\ q)'s$ exponent is p.

A finite group G is called a simple K_n-group if G is a simple group with $|\pi(G)| = n$ (see [4]).

Lemma 2.3. ([11]) A K_3-simple group is isomorphic to one of the groups $L_2(5)$, $L_2(9)$, $L_2(7)$, $L_2(8))$, $L_2(17)$, $L_3(3)$, $U_3(3)$, $U_4(2)$.

Lemma 2.4. Let G be a simple group with non-solvable proper subgroups. If G is not isomorphic to $L_2(q)$ and the non-solvable composition factors of all non-solvable maximal subgroups of G are isomorphic to A_5, then $G \in \{L_3(5), U_3(4)\}$.

Proof (1) Suppose that $G \cong A_n$. If $n > 6$, then A_n has a simple maximal subgroup A_{n-1} such that A_{n-1} is not isomorphic to A_5. Hence $n \leq 6$. Since $A_6 \cong L_2(9)$ and all maximal subgroup of A_5 are solvable by [6], a contradiction.

(2) Suppose that G is a sporadic simple group. Since J_3 has a maximal subgroup $L_2(19)$, we have $G \ncong J_3$. But if $G \ncong J_3$, by [6], G has a maximal subgroup M containing a section $L_2(11)$ or $L_2(7)$, so the non-solvable composition factors of M are not isomorphic to A_5, a contradiction.

(3) Let G be a Lie type simple group over $GF(q)$, where $q = p^f$ and p is a prime. Checking the subgroup listed in Lemma [19, Lemma 2.5], the possibility of G is one of groups $L_n(q)$, $D_4(q)$, $U_n(q)$. Since $D_4(q)$ has a section $D_3(q)$, we have $G \ncong D_4(q)$. Assume that $G \cong L_n(q)$ with $n \geq 3$. Note that G has a section $L_{n-1}(q)$. We have $n = 3$ and $q \leq 5$. Since $L_3(4)$ has a maximal subgroup $L_2(7)$ and $L_3(3)$ is a minimal nonsolvable simple group, we have $G \cong L_3(5)$. Suppose that $G \cong U_n(q)$ with $n \geq 3$. Since $U_{n-1}(q)$ is a section of $U_n(q)$, we have $n = 3$ and $q \leq 5$. Note that $U_3(3)$ and $U_3(5)$ have a section $L_2(7)$ and A_7, respectively. So we are left with $G \cong U_3(4)$, and hence the result is true. □

Lemma 2.5. ([18]) Let G be a minimal simple group, then G is isomorphic to one of following groups.

(1) $G \cong L_2(q)$ with $q = p^f$, $|G| = \frac{1}{d}q(q^2-1)$ with $d = (2, q-1)$.

(2) $G \cong L_2(2^q)$, q is an odd prime, $|G| = 2^q(2^{2q}-1)$.

(3) $G \cong L_2(3^q)$, q is an odd prime, $|G| = \frac{1}{2}3^q(3^{2q}-1)$.

(4) $G \cong Sz(2^q)$, q is an odd prime, $|G| = (2^{2q}+1)2^{2q}(2^q-1)$.

(5) $G \cong L_3(3)$, $|G| = 2^4 \cdot 3^3 \cdot 13$.

3. Main results and their proofs

Theorem 3.1. Let G be a non-solvable group. If $|\pi(|\langle H, H^g\rangle : H|)| \leq 2$ for $g \in G$ and any cyclic subgroup H of G of prime power order, then $G/S(G)$ is isomorphic to one of A_5 or S_5.

Proof Clearly, the condition holds in every subgroup of G. For any $N \lhd G$, we consider $\overline{G} = G/N$. Let $C/N = \overline{C}$ be a cyclic subgroup of $\overline{G}$ such that the order of C/N is a power of a prime. Then there exists a cyclic subgroup H of G such that $C/N = HN/N$ and the order of H is a power of a prime. For any $g \in G$, we have $\langle HN/N, (HN/N)^g\rangle = \langle HN/N, H^gN/N\rangle = \langle H, H^g\rangle N/N$. It is easy to see that

$$|\langle HN/N, (HN/N)^g\rangle : HN/N| = |\langle H, H^g\rangle : H|/|\langle H, H^g\rangle \cap N : H \cap N|.$$

By the hypothesis, $|\pi(|\langle C/N, (C/N)^g\rangle : C/N)|)| \leq |\pi(|\langle H, H^g\rangle : H|)| \leq 2$, so the condition holds in every epimorphic image of G.

Now suppose that G is a minimal counterexample to our theorem. Then G is non-solvable and $S(G) = 1$. Since every proper subgroup of G satisfies the condition of Theorem 3.1, by the minimality of G, every proper subgroup H of G is either solvable or $Soc(H/S(H)) = A_5$. If $Soc(G) \neq G$, since the condition is subgroup heredity, we have $Soc(G) \cong A_5$ and the result is true. If $Soc(G) = G$, since $S(G) = 1$, G must be a simple group. If G has non-solvable maximal subgroups, since the non-solvable composition factors of all non-solvable maximal subgroups of G are isomorphic to A_5, by Lemma 2.4, G is one of $L_2(q)$, $L_3(5)$ and $U_3(4)$. If all maximal subgroups of G are solvable, then G is the minimal simple subgroup listed in Lemma 2.5. Hence we can divide the proof into the following cases.

(1) $G \cong L_2(q)$ with $q = p^f$, $|G| = \frac{1}{d}q(q^2-1)$ with $d = (2, q-1)$.

Suppose that $f \neq 1$, let $f_1 \mid f$ such that $f/f_1 = s$, a prime. Then G has a section $L_2(p^{f_1})$, hence $f_1 = 1$ and $p = 5$. Let r be a largest primitive prime divisor of $p^{2s}-1$. By [9], $r > 2s+1$ or $r^2 \mid p^s+1$ and $r = 2s+1$. By Dickson's Theorem [12], the group G has a dihedral maximal subgroup T of order $q+1$. Let H be a Sylow r-subgroup of T, then $T = N_G(H)$. Let $g \in G \setminus N_G(H)$. Then $H^g \nleq T$ and so $H < \langle H, H^g \rangle \leq G$. If $\langle H, H^g \rangle \neq G$, then there exists a maximal subgroup S of G such that $H < \langle H, H^g \rangle \leq S$. Since $|H| \nmid |A_5|$, by Dickson's Theorem and $r \nmid q-1$, it is easy to see that S is a dihedral subgroup of G of order $q+1$ and so $S = N_G(H) = T$, this is impossible. This implies that $G = \langle H, H^g \rangle$. It is clear that $|\pi(|\langle H, H^g \rangle : H|)| \geq 3$, a contradiction. Hence we may assume that $f = 1$ and $q = p$, a prime. We choose $H = G_p \in Syl_p(G)$ and $g \in G \setminus N_G(H)$. If $\langle H, H^g \rangle \neq G$, then there exists a maximal subgroup T of G such that $H < \langle H, H^g \rangle \leq T$. By Dickson's Theorem [12], if T contains a cyclic subgroup of order p, then T is a normalizer of some Sylow p-subgroup S of G, that is, $T = N_G(S)$ and $S \in Syl_p(G)$. By the uniqueness of Sylow p-subgroup of $T = N_G(S)$, we have $H^g = H$, thus $g \in N_G(H)$, a contradiction. This implies that $|\pi(|\langle H, H^g \rangle : H|)| = |\pi(|G : H|)| \geq |\pi(G)| - 1$. If $|\pi(G)| \geq 4$, then $|\pi(|\langle H, H^g \rangle : H|)| \geq 3$, a contradiction. So $|\pi(G)| = 3$. By Lemma 2.3, G is isomorphic to one of simple groups $L_2(5)$, $L_2(7)$, $L_2(17)$.

Suppose that $G \cong L_2(7)$, then G has two maximal subgroups classes S_4 and $7:3$, which have different orders. Let $a \in S_4$ and $g \in G$ such that $|a| = 4$ and $|g| = 7$. Then $D_8 = N_G(\langle a \rangle)$. If $\langle a, a^g \rangle \leq S_4$, then there exists $x \in S_4$ such that $a^g = a^x$, thus $(gx^{-1})a = a(gx^{-1})$. Since $C_G(a) = \langle a \rangle$, $gx^{-1} \in \langle a \rangle$. Hence $g \in S_4$, this is impossible. So $\langle a, a^g \rangle \nleq S_4$, and $\langle a, a^g \rangle = G$, $|\pi(|\langle H, H^g \rangle : H|)| = 3$, a contradiction.

Suppose that $G \cong L_2(17)$, then D_{16} is a maximal subgroup of G. Let $a \in D_{16}$ and $g \in G$ such that $|a| = 8$ and $|g| = 17$. Then $D_{16} = N_G(\langle a \rangle)$. If $\langle \langle a \rangle, \langle a \rangle^g \rangle \leq D_{16}$, then $\langle a \rangle = \langle a \rangle^g$, $g \in D_{16}$, a contradiction. Hence $\langle \langle a \rangle, \langle a \rangle^g \rangle \nleq D_{16}$. From all maximal subgroup of G, it is easy to see that $\langle \langle a \rangle, \langle a \rangle^g \rangle = G$. We have $|\pi(|\langle \langle a \rangle, \langle a \rangle^g \rangle : \langle a \rangle|)| = 3$, a contradiction.

If $G \cong L_2(5)$, then $G \cong A_5$ and the result is true.

(2) $G \cong L_2(2^q)$, q is an odd prime, $|G| = 2^q(2^{2q}-1)$.

Suppose that G is isomorphic to $L_2(8)$, then G has three maximal subgroups classes $2^3:7$, D_{18} and D_{14}, which have different orders. Let L be a subgroup of D_{18} of order 3, then $D_{18} = N_G(L)$. Let $g \in G$ with $|g| = 7$. Since the subgroup of D_{18} of order 3 is unique, if $\langle L, L^g \rangle \leq D_{18}$, then $L = L^g$, this is impossible. Hence $\langle L, L^g \rangle \nleq D_{18}$ and $\langle L, L^g \rangle = G$, $|\pi(\langle L, L^g \rangle : L)| = 3$, a contradiction. Hence we assume that $q > 3$. By Dickson's Theorem, G has a maximal subgroup T such that T is a dihedral group of order $2(2^q+1)$. Let T_1 be a maximal cyclic subgroup of T with order 2^q+1. We choose $H = R \in Syl_r(T_1)$ and $g \in G \setminus N_G(H)$, where $r = max(\pi(T_1))$ and $|H| = |R| = r^i$, then $T = N_G(H)$ and $H < \langle H, H^g \rangle \leq G$. If $\langle H, H^g \rangle \neq G$, then there exists a maximal subgroup S of G such that $H < \langle H, H^g \rangle \leq S$. By Dickson's Theorem [12], if S contains a cyclic subgroup of order r^i, then S is a dihedral subgroup of G of order $2(2^q+1)$. Since there is a unique cyclic subgroup of order r^i in a dihedral group of order $2(2^q+1)$, we have $H^g = H$, thus $g \in N_G(H)$, a contradiction. Hence $\langle H, H^g \rangle = G$. This implies that $|\pi(|\langle H, H^g \rangle : H|)| = |\pi(|G : H|)| \geq |\pi(G)| - 1$. By Lemma 2.3, $|\pi(G)| \geq 4$, hence $|\pi(\langle H, H^g \rangle : H)| \geq 3$, a contradiction.

(3) $G \cong L_2(3^q)$, q is an odd prime, $|G| = \frac{1}{2}3^q(3^{2q} - 1)$.

By Dickson's Theorem, G has a dihedral maximal subgroup T of order $3^q + 1$. Let T_1 be a maximal cyclic subgroup of T with order $\frac{1}{2}(3^q + 1)$. We choose $H = R \in Syl_r(T_1)$ and $g \in G \setminus N_G(H)$, where $r = max(\pi(T_1))$ and $|H| = |R| = r^i$, then $T = N_G(H)$ and $H < \langle H, H^g\rangle \leq G$. Since q is an odd prime, we have $|\pi(L_2(3^q))| \geq 4$. Using the same method as (2), we still get a contradiction.

(4) $G \cong Sz(2^q)$, q is an odd prime, $|G| = (2^{2q} + 1)2^{2q}(2^q - 1)$.

By [17], G has a maximal subgroup T of order $2^2(2^q + 2^{\frac{q+1}{2}} + 1)$. Let T_1 be a maximal cyclic subgroup of T with order $2^q + 2^{\frac{q+1}{2}} + 1$. We choose $H = R \in Syl_r(T_1)$ and $g \in G \setminus N_G(H)$, where $r = max(\pi(T_1))$ and $|H| = |R| = r^i$, then $T = N_G(H)$ and $H < \langle H, H^g\rangle \leq G$. Since q is an odd prime, we have $|\pi(Sz(2^q))| \geq 4$. Using the same method as (2), we also get a contradiction.

(5) $G \cong L_3(3)$, $|G| = 2^4 \cdot 3^3 \cdot 13$.

By [6], the maximal subgroup of G containing a Sylow 2-subgroup of G is isomorphic to $M = 3^2 : 2S_4$. Let L be a cyclic subgroup of M of order 8 and $g \in G$ with $|g| = 13$, then $C_G(L) = L$. Since $|Aut(L)| = 4$, we have $|N_G(L)| = 2^4$ and $N_G(L) \in Syl_2(G)$. Without loss of generality, we may assume that $N_G(L) \leq M$. If $\langle L, L^g\rangle \leq M$, then there exist $m \in M$ such that $N_G(L) = N_G(L)^{gm}$. Since $N_G(N_G(L)) = N_G(L)$, $g = am^{-1}$, where $a \in N_G(L)$. Thus $g \in M$, a contradiction. Hence $\langle L, L^g\rangle = G$, we have $|\pi(\langle L, L^g\rangle : L)| = 3$, a contradiction.

(6) $G \cong L_3(5)$, $|G| = 2^5 \cdot 3 \cdot 5^3 \cdot 31$.

By [6], the maximal subgroup M of G containing a Sylow 31-subgroup of G is isomorphic to $31 : 3$. Let L be a cyclic subgroup of M of order 31 and $g \in G$ with $|g| = 5$. Then $M = N_G(L)$ and $g \notin M$. If $\langle L, L^g\rangle < G$, then $\langle L, L^g\rangle$ is contained in M, as M is the unique maximal subgroup containing L, hence $L^g = L$ and so $g \in M$, a contradiction. Thus $\langle L, L^g\rangle = G$ and we have $|\pi(\langle L, L^g\rangle : L)| = 3$, a contradiction.

(7) $G \cong U_3(4)$, $|G| = 2^6 \cdot 3 \cdot 5^2 \cdot 13$.

By [6], the maximal subgroup M of G containing a Sylow 13-subgroup of G is isomorphic to $13 : 3$. The proof is similar to the case $G \cong L_3(5)$.

These contradictions complete the proof of this theorem. □

Theorem 3.2. Let G be a group. Assume that $|\langle H, H^g\rangle : H|$ is a prime power for $g \in G$ and any cyclic subgroup H of G of prime power order. Then G is solvable.

Proof By Theorem 3.1 and its proof of heredity, if we assume that G is a minimal counterexample to our theorem, then $G \cong A_5$. By [8, Proposition 4.21], if H and H^g are distinct Sylow 5-subgroups of A_5, then $A_5 = \langle H, H^g\rangle$, a contradiction. This contradiction completes the proof. □

Theorem 3.3. Let G be a group. Assume that $|\langle H, H^g\rangle : H|$ is square-free for $g \in G$ and any cyclic subgroup H of G of prime power order. Then G is supersolvable.

Proof Using the same proof as Theorem 3.1, we may obtain that the condition is subgroup and quotient group heredity. If we suppose that G is a minimal counterexample to our theorem, then G is a minimal inner-supersolvable group. By Lemma 2.2, we may divide the argument into the following cases. In the argument we shall use notations of generating elements of G_i in Lemma 2.2.

(1) G is isomorphic to G_1.

Then G is isomorphic to an inner-nilpotent group, $G = PQ$, where $P = \langle a\rangle$. We choose $H = P$ and $g = c$, where c is a generator of Q. Then $(a^{-1})^c \in \langle H, H^c\rangle$, so $[c, a] = (a^{-1})^c a \in \langle H, H^c\rangle$, thus $a, [c, a], [c, a]^a, \cdots, [c, a]^{a^{p-1}}$ belong to $\langle H, H^c\rangle$. By Lemma 2.1 (4) and (5), we have $\langle H, H^c\rangle = G$. Thus, by the hypothesis, $|\langle H, H^g\rangle : H| = |G : H| = r^p$ is square-free, a contradiction.

(2) G is isomorphic to G_2.

Let $P \in Syl_p(G)$. We choose $H = P = \langle a\rangle$ and $g = c_1$, then $(a^{-1}a^{c_1})^{-1} = c_1^{-1}c_2 \in \langle H, H^{c_1}\rangle$, so $(c_1^{-1}c_2)^a = c_2^{-1}c_3 \in \langle H, H^{c_1}\rangle$, thus $c_1^{-1}c_2c_2^{-1}c_3 = c_1^{-1}c_3$ belongs to $\langle H, H^{c_1}\rangle$. Similarly, we have $c_1^{-1}c_4, c_1^{-1}c_5, \cdots c_1^{-1}c_p$ belong to $\langle H, H^{c_1}\rangle$. Then $(c_1^{-1}c_p)^a = c_2^{-1}c_1^t \in \langle H, H^{c_1}\rangle$, so $c_1^{-1}c_2c_2^{-1}c_1^t = c_1^{t-1} \in \langle H, H^{c_1}\rangle$. Since $t(mod\ r)'s$ exponent is $p^{\alpha-1}$ and $\alpha \geq 2$, we get $(t-1, r) = 1$, thus $c_1 \in \langle H, H^{c_1}\rangle$. Hence $a, c_1, \cdots, c_p$ belong to $\langle H, H^{c_1}\rangle$, $\langle H, H^{c_1}\rangle = G$. By the hypothesis, $|\langle H, H^g\rangle : H| = |G : H| = r^p$ is square-free, a contradiction.

(3) G is isomorphic to G_3.

We choose $H = \langle a\rangle$ and $g = c_1$. Let $T = \langle a, a^{c_1}\rangle$. Then $a^{-1}a^{c_1} = (c_1^{-1})^a c_1 = c_2^{-1}c_1 \in T$, $(c_2^{-1}c_1)^a = c_1c_2 \in T$, thus $c_2^{-1}c_1c_1c_2 = (c_1)^2 \in T$. Since $(2, r) = 1$, we have $c_1 \in T$, then $\{a, c_1, c_2\} \subset T$. So $|T : H| = r^2$, contrary to the condition that $|\langle H, H^g\rangle : H| = |T : H|$ is square-free.

(4) G is isomorphic to G_t, where $t \in \{4, 5, 6\}$.

We choose $H = \langle a\rangle$ and $g = c_1$. Let $T = \langle a, a^{c_1}\rangle$. Assume that $p \geq 3$. Then $a^{-1}a^{c_1} = (c_1^{-1})^a c_1 = c_2^{-1}c_1 \in T$, $(c_2^{-1}c_1)^a = c_3^{-1}c_2 \in T$, thus $c_2^{-1}c_1c_3^{-1}c_2 = c_3^{-1}c_1 \in T$. So $\langle c_2^{-1}c_1, c_3^{-1}c_1\rangle \leq T$. Obviously, $\langle c_2^{-1}c_1\rangle \cap \langle c_3^{-1}c_1\rangle = 1$, then $r^2 \mid |\langle c_2^{-1}c_1, c_3^{-1}c_1\rangle|$, hence $r^2 \mid |T : H|$, contrary to the condition that $|\langle H, H^g\rangle : H| = |T : H|$ is square-free. Assume that $p = 2$ and $\alpha \geq 2$. We have $c_1c_2^{-1} \in T$ and $(c_1c_2^{-1})^a = c_2c_1^{-t} \in T$, so $c_1c_2^{-1}c_2c_1^{-t} = c_1^{1-t} \in T$. Since $t(mod\ r)'s$ exponent is $p^{\alpha-1}$ and $\alpha \geq 2$, we get $r \nmid t-1$, that is, $(1-t, r) = 1$, thus $c_1 \in T$. Hence a, c_1, c_2 belong to T. So $|T : H| = r^2$, contrary to the condition that $|\langle H, H^g\rangle : H| = |T : H|$ is square-free. Assume that $p = 2$ and $\alpha = 1$. We choose $H = \langle b\rangle$. Let $g = c_1^{-1}c_2^{-1}$ and $l = b^{-1}b^g$.

Assume that $G \cong G_4$. Then $l = c_1^{u^{1+p^{\beta-1}}-1}c_2^{u^{1+2p^{\beta-1}}-1}$, $l^b = c_1^{u^{1+p^{\beta-1}}(u^{1+p^{\beta-1}}-1)}c_2^{u^{1+2p^{\beta-1}}(u^{1+2p^{\beta-1}}-1)}$ and the order of l is r. Suppose that $l^b \in \langle l\rangle$. Then $l^b = l^m$, where $0 \leq m \leq r-1$, thus

$$c_1^{u^{1+p^{\beta-1}}(u^{1+p^{\beta-1}}-1)}c_2^{u^{1+2p^{\beta-1}}(u^{1+2p^{\beta-1}}-1)} = c_1^{(u^{1+p^{\beta-1}}-1)m}c_2^{(u^{1+2p^{\beta-1}}-1)m},$$

$$c_1^{(u^{1+p^{\beta-1}}-1)(u^{1+p^{\beta-1}}-m)} = c_2^{(u^{1+2p^{\beta-1}}-1)(m-u^{1+2p^{\beta-1}})} = 1.$$

Hence $r \mid (u^{1+p^{\beta-1}}-1)(u^{1+p^{\beta-1}}-m)$ and $r \mid (u^{1+2p^{\beta-1}}-1)(m-u^{1+2p^{\beta-1}})$. Since $u(mod\ r)'s$ exponent is p^β, we have $u^{p^\beta} \equiv 1(\ mod\ r)$ and $r \nmid u$. Since $\beta \geq 2$, we have $1+p^{\beta-1} < p^\beta$, so $r \nmid u^{1+p^{\beta-1}}-1$, thus $r \mid u^{1+p^{\beta-1}}-m$. If $r \mid u^{1+2p^{\beta-1}}-1$, then $r \mid u^{1+p^\beta}-u^{p^\beta}$, so $r \mid u-1$, contrary to the condition that $u(mod\ r)'s$ exponent is p^β. Thus $r \mid m-u^{1+2p^{\beta-1}}$, hence $r \mid u^{1+p^\beta}-u^{1+p^{\beta-1}}$, so $r \mid u^{p^{\beta-1}(p-1)}-1$, contrary to the condition that $u(mod\ r)'s$ exponent is p^β. Hence $\langle l\rangle^b \neq \langle l\rangle$, $\langle l, l^b\rangle = \langle l^b\rangle \times \langle l\rangle \leq \langle b, b^g\rangle$, we have $r^2 \mid |\langle b, b^g\rangle : H|$, contrary to the condition that $|\langle b, b^g\rangle : H|$ is square-free.

Assume that $G \cong G_5$. Then $l = c_1^{vu^p-1}c_2^{vu^{p-1}-1}$, $l^b = c_1^{vu^p(vu^p-1)}c_2^{vu^{p-1}(vu^{p-1}-1)}$, the order of l is r. Suppose that $l^b \in \langle l\rangle$. Then $l^b = l^m$, where $0 \leq m \leq r-1$, thus

$$c_1^{vu^p(vu^p-1)}c_2^{vu^{p-1}(vu^{p-1}-1)} = c_1^{(vu^p-1)m}c_2^{(vu^{p-1}-1)m},$$

$$c_1^{(vu^p-1)(vu^p-m)} = c_2^{(vu^{p-1}-1)(m-vu^{p-1})} = 1.$$

Hence $r \mid (vu^p - 1)(vu^p - m)$ and $r \mid (vu^{p-1} - 1)(m - vu^{p-1})$. Since v, $u(mod\ r)'s$ exponent are p^β and p respectively, we have $v^{p^\beta} \equiv 1(\ mod\ r)$ and $u^p \equiv 1(\ mod\ r)$. If $r \mid vu^p - 1$, then $r \mid vu^p - 1 - (u^p - 1)$, that is, $r \mid u^p(v-1)$, so $r \mid v - 1$, contrary to the condition that $v(mod\ r)'s$ exponent is p^β. So $r \mid vu^p - m$. If $r \mid vu^{p-1} - 1$, then $r \mid u^p - 1 - (vu^{p-1} - 1)$, that is, $r \mid u - v$, so $r \mid v^p - 1$, thus $\beta = \alpha = 1$, we have the complement of Sylow r-subgroup in G is abelian, a contradiction. Hence $r \mid m - vu^{p-1}$, thus $r \mid vu^p - vu^{p-1}$, so $r \mid u - 1$, again a contradiction. Hence $l^b \notin \langle l \rangle$, $\langle l, l^b \rangle = \langle l^b \rangle \times \langle l \rangle \leq \langle b, b^g \rangle$, we have $r^2 \mid |\langle b, b^g \rangle : H|$, contrary to the condition that $|\langle b, b^g \rangle : H|$ is square-free.

Assume that $G \cong G_6$. Then $l = c_1^{v-1} c_2^{v^u - 1}$, $l^b = c_1^{v(v-1)} c_2^{v^u(v^u-1)}$ and the order of l is r. Suppose that $l^b \notin \langle l \rangle$. Then $l^b = l^m$, where $0 \leq m \leq r - 1$, that is, $c_1^{v(v-1)} c_2^{v^u(v^u-1)} = c_1^{(v-1)m} c_2^{(v^u-1)m}$, we have $c_1^{(v-1)(v-m)} = c_2^{(v^u-1)(m-v^u)} = 1$. Hence $r \mid (v-1)(v-m)$ and $r \mid (v^u - 1)(m - v^u)$. Since $v(mod\ r)'s$ exponent is q, we have $v^q \equiv 1(\ mod\ r)$, $r \nmid v$ and $r \mid v - m$. If $r \mid v^u - 1$, then $q \mid u$, contrary to the condition that $u(mod\ q)'s$ exponent is p. So $r \mid v^u - m$, thus $r \mid v^u - v$ and so $r \mid (v^{u-1} - 1)$. As before, $q \mid u - 1$, again a contradiction. Hence $l^b \notin \langle l \rangle$, $\langle l, l^b \rangle = \langle l^b \rangle \times \langle l \rangle \leq \langle b, b^g \rangle$. Therefore, $r^2 \mid |\langle b, b^g \rangle : H|$, contrary to the condition that $|\langle b, b^g \rangle : H|$ is square-free.

These contradictions complete the proof of this theorem. □

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