Fundamental regular semigroups with quasi-ideal regular $*$-transversals*

Shou-feng Wang†

Department of Mathematics, Yunnan Normal University,

Kunming, Yunnan, 650500, P. R. China.

Abstract

Let S be a semigroup and "$*$" a unary operation on S which satisfies the following identities

$$xx^*x = x, x^*xx^* = x^*, x^{***} = x^*, (xy^*)^* = y^{**}x^*, (x^*y)^* = y^*x^{**}.$$

Then $S^* = \{x^* | x \in S\}$ is called a *regular $*$-transversal* of S in the literatures. Following Munn and Hall's idea, in this paper we construct fundamental regular semigroups with quasi-ideal regular $*$-transversals by which fundamental representations of regular semigroups with quasi-ideal regular $*$-transversals are obtained.

Keywords: Regular $*$-transversals, Fundamental semigroups, Fundamental representations

AMS Subject Classification: 20M10, 20M17.

1 Introduction

Let S be a semigroup. We denote the set of all idempotents of S by $E(S)$ and the set of all inverses of $x \in S$ by $V(x)$. Recall that

$$V(x) = \{a \in S | xax = x, axa = a\}$$

for all $x \in S$.

A semigroup S is called *regular* if $V(x) \neq \emptyset$ for any $x \in S$, and a regular semigroup S is called *inverse* if $E(S)$ is a commutative subsemigroup of S, or equivalently, the cardinal of $V(x)$ is equal to 1 for all $x \in S$.

*This paper is supported jointly by a Nature Science Foundation of Yunnan Province (2012FB139) and a Nature Science Foundation of China (11226049).

†E-mail: wsf1004@163.com

Recall from Petrich-Reilly [14] that *a unary semigroup* is a (2,1)-algebra $(S, \cdot, ^*)$ where $(S, \cdot)$ is a semigroup and the mapping $a \mapsto a^*$ is a unary operation on S. For brevity, we denote $(S, \cdot, ^*)$ by $(S,\ ^*)$. It is well known that a regular semigroup S is inverse if and only if there exists a unary operation "$*$" on S satisfying the following identities:

$$xx^*x = x, (x^*)^* = x,\ (xy)^* = y^*x^*, xx^*yy^* = yy^*xx^*. \tag{1.1}$$

Thus, inverse semigroups can be regarded as a class of unary semigroups.

Inspired the above identity (1.1), *regular $*$-semigroups* were introduced in Nordahl-Scheiblich [13]. Recall that a unary semigroup $(S,\ ^*)$ is called *a regular $*$-semigroup* if the following identities are satisfied:

$$xx^*x = x, (x^*)^* = x, (xy)^* = y^*x^*. \tag{1.2}$$

Obviously, the class of regular $*$-semigroups forms a class of unary semigroups and contains the class of inverse semigroups as a subclass. Regular $*$-semigroups and their generalizations are investigated in many papers (see [6, 7, 8, 13, 21, 22]).

On the other hand, Blyth-McFadden [1] introduced the concept of inverse transversals for regular semigroups. A subsemigroup S° of a semigroup S is called an *inverse transversal* of S if $V(x) \cap S^\circ$ contains one element exactly for all $x \in S$. Clearly, in this case, S° is an inverse subsemigroup of S. From the remarks following Theorem 2 in Tang [16] and Theorem 4.8 in Tang [17], we can deduce easily that a regular semigroup S contains an inverse transversal if and only if there exists a unary operation "$*$" on S satisfying the following identities:

$$\begin{gathered} xx^*x = x, x^*xx^* = x^*, x^{***} = x^*, (x^*y)^* = y^*x^{**}, \\ (xy^*)^* = y^{**}x^*, x^*x^{**}y^*y^{**} = y^*y^{**}x^*x^{**}. \end{gathered} \tag{1.3}$$

In this case, $S^\circ = \{x^* | x \in S\}$ is an inverse transversal of S. Therefore, the class of regular semigroups with inverse transversals is a class of unary semigroups which also contains the class of inverse semigroups as a subclass. Inverse transversals of regular semigroups are studied extensively (for example, see [1, 2, 3, 15, 16, 17]).

Now, let $(S,\ ^*)$ be a unary semigroup and the unary operation "$*$" satisfy the following identities

$$xx^*x = x, x^*xx^* = x^*, x^{***} = x^*, (xy^*)^* = y^{**}x^*, (x^*y)^* = y^*x^{**}. \tag{1.4}$$

Then $S^* = \{x^* | x \in S\}$ is called a *regular $*$-transversal* of S from Li [10]. Clearly, $(S^*,\ ^*)$ is a regular $*$-semigroup in this case. Moreover, combining the facts (1.2) and (1.3), we can see that regular semigroups having regular $*$-transversals are generalizations of regular $*$-semigroups and regular semigroups with inverse transversals. In fact, there exists a regular semigroup with quasi-ideal regular $*$-transversals which is neither a regular $*$-semigroup nor a regular semigroup with inverse transversals (see Section 2 in [10]).

Regular $*$-transversals also have received serious attention in the literatures, see e.g. [9, 10, 11, 18]. Recently, the author considered algebraic structures of regular semigroups with quasi-ideal regular $*$-transversals in [19] and gave a classification of regular $*$-transversals in [20], respectively.

A semigroup S is *fundamental* if the maximum idempotent-separating congruence μ_S on S is the identity congruence. Structure theorems for certain important subclasses of the class of fundamental regular semigroups are already known. The one initiating the work in this direction is due to Munn [12]. He proved that given a semilattice E, the *Munn semigroup* T_E of all isomorphisms of principal ideals of E is "maximal" in the class of all fundamental inverse semigroups whose semilattice of idempotents is E, that is, every semigroup belonging to this class is isomorphic to a full inverse subsemigroup of T_E. In [5] Hall introduced a fundamental regular semigroup H_C for any regular semigroup C generated by the set of idempotents $E(C)$ which is called a *Hall semigroup*. Following these directions, fundamental regular $*$-semigroups are studied in the texts [6], [7] and [22], and fundamental regular semigroups with inverse transversals are investigated in Song-Zhu [15].

Inspired by the above works, in this paper we shall initiate the investigations of regular semigroups with regular $*$-transversals by the above so-called *fundamental* approach. After giving some necessary preliminaries in Section 2, we construct fundamental regular semigroups with a quasi-ideal regular $*$-transversal in Section 3. Finally, fundamental representations of regular semigroups with quasi-ideal regular $*$-transversals are obtained in section 4.

2 Preliminaries

This section will collect some useful results related to regular $*$-semigroups and regular $*$-transversals which will used in the next sections. Let $(S, {}^*)$ be a regular $*$-semigroup. Then we write $(S, {}^*) \in \mathbf{r}$, $F_{(S,\ ^*)} = \{e \in E(S) | e^* = e\}$ and call $F_{(S,\ ^*)}$ the set of idempotent projections of $(S, {}^*)$. It is easy to see that

$$F_{(S,\ ^*)} = \{xx^* | x \in S\} = \{x^*x | x \in S\}.$$

On regular $*$-semigroups, we have the following basic results.

Lemma 2.1 ([13],[21]). *Let $(S, {}^*) \in \mathbf{r}$. Then*

(1) $(\forall e, f \in F_{(S,\ ^*)})\ ef \in F_{(S,\ ^*)} \Longrightarrow ef = fe \in F_{(S,\ ^*)}$;

(2) $(\forall x \in S)\ x \in E(S) \Longleftrightarrow x^* \in E(S)$;

(3) $(F_{(S,\ ^*)})^2 \subseteq E(S)$ *and* $xF_{(S,\ ^*)}x^*, x^*F_{(S,\ ^*)}x \subseteq F_{(S,\ ^*)}$ *for all* $x \in S$.

Now, let $(S, {}^*)$ be a unary semigroup and S^* be a regular $*$-transversal of S. Then we write $(S, {}^*) \in \mathbf{rt}$. Thus, $(S^*, {}^*) \in \mathbf{r}$ if $(S, {}^*) \in \mathbf{rt}$. In this case, we denote $F_{(S^*, {}^*)}$ by F_{S^*} for simplicity. For idempotent-separating congruences on S with $(S, {}^*) \in \mathbf{rt}$, we have the following results by Proposition 4.8 and Exercise 15 of Chapter 2 in [4].

Lemma 2.2 ([4])**.** *Let $(S, {}^*) \in \mathbf{rt}$, $\langle E(S)\rangle$ be the subsemigroup of S generated by $E(S)$ and ρ a congruence on S. Then*

(1) ρ is idempotent-separating if and only if $\rho \subseteq \mathcal{H}$.

(2) $x \in \langle E(S)\rangle$ if and only if $x^* \in \langle E(S)\rangle$.

A *quasi-ideal* of a semigroup S is a subsemigroup T of S which satisfies that $TST \subseteq T$. If $(S, {}^*) \in \mathbf{rt}$ and S^* is a quasi-ideal of S, then we denote $(S, {}^*) \in \mathbf{qit}$. In this case, we denote $I_S = \{aa^* | a \in S\}, \Lambda_S = \{a^*a | a \in S\}$.

For $(S, {}^*) \in \mathbf{qit}$, we have the following important result.

Lemma 2.3 (Theorem 1 in [11])**.** *Let $(S, {}^*) \in \mathbf{qit}$. Then $(xy)^* = y^*(x^*xyy^*)^*x^*$ for all $x, y \in S$.*

The following several results consider the sets I_S and Λ_S.

Lemma 2.4 (Lemmas 4.1 and 4.2 in [10])**.** *Let $(S, {}^*) \in \mathbf{qit}$.*

(1) $I_S = \{e \in E(S) | e\mathcal{L}e^*\}$, $\Lambda_S = \{f \in E(S) | f\mathcal{R}f^*\}$ and $F_{S^*} = I_S \cap \Lambda_S$.

(2) $g^{**} = g^* \in F_{S^*}$ for all $g \in I_S \cup \Lambda_S$.

(3) $fg \in S^*$ and so $fg = (fg)^{**}$for all $f \in \Lambda_S$ and $g \in I_S$.

Corollary 2.5. *Let $(S, {}^*) \in \mathbf{qit}$. Then $(xy)^{**} = x^{**}x^*xyy^*y^{**}$ for all $x, y \in S$.*

Proof. This follows from Lemma 2.3 and Lemma 2.4 (3). □

By Lemma 2.3 and Lemma 2.4, we can easily obtain the following results.

Lemma 2.6. *Let $(S, {}^*) \in \mathbf{qit}$ and $e \in I_S, f \in \Lambda_S$.*

(1) $(fe)^* = e^*(fe)^* = (fe)^*f^* = e^*(fe)^*f^*$.

(2) $e(fe)^*f \in E(S)$ and $(e(fe)^*f)^* = fe$.

(3) $fe(fe)^*f \in \Lambda_S, e(fe)^*fe \in I_S$.

Proof. (1) This follows from the identity

$$(fe)^* = e^*(f^*fee^*)^*f^* = e^*(fe)^*f^*$$

obtained by Lemma 2.3 and Lemma 2.4.

(2) Obviously, $e(fe)^*f \in E(S)$. By Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}
(e(fe)^*f)^* &= f^*((e(fe)^*)^*e(fe)^*ff^*)^*(e(fe)^*)^* \\
&= f^*((fe)^{**}e^*e(fe)^*ff^*)^*(fe)^{**}e^* \\
&= f^*((fe)e^*e(fe)^*ff^*)^*fee^* \\
&= f^*(fe(fe)^*f^*)^*fe \\
&= f^*(f^*(fe)^{**}(fe)^*)fe \\
&= f^*fe(fe)^*fe = f^*fe = fe.
\end{aligned}$$

(3) This follows from item (2). □

Lemma 2.7. *Let $(S, {}^*) \in$ **qit** and $a, b \in S^*, x \in S$. Then $axb = ax^{**}b$. In particular, we have $efg = ef^*g^*$ (resp. $efg = e^*f^*g$) and $(efg)^* = g^*f^*e^*$ for $e, f, g \in I_S$ (resp. $e, f, g \in \Lambda_S$).*

Proof. Since $(S, {}^*) \in$ **qit**, we have $axb \in S^*$. Observe that $(S^*, {}^*) \in$**r**, it follows that $a = a^{**}, b = b^{**}$ and $axb = (axb)^{**}$. This implies that

$$axb = (axb)^{**} = ((a^{**}xb^{**})^*)^* = (b^{***}(a^{**}x)^*)^*$$

$$= (b^{***}x^*a^{***})^* = (b^*x^*a^*)^* = a^{**}x^{**}b^{**} = ax^{**}b.$$

Now, let $e, f, g \in I_S$. Then by Lemma 2.4, we have

$$efg = ee^*ff^*gg^* = ee^*f^*f^*g^*g^* = ef^*g^*$$

whence $(efg)^* = (ef^*g^*)^* = (f^*g^*)^*e^* = g^*f^*e^*$. By symmetry, we can obtain the corresponding result for Λ_S. □

Lemma 2.8 (Lemma 2.6 in [20])**.** *Let $(S, {}^*) \in$ **qit**. Then I_S is $\mathcal{R}$-unipotent and Λ_S is $\mathcal{L}$-unipotent, respectively.*

As direct consequences of Lemma 2.2 (1) and Lemma 2.8, we have the corollary below.

Corollary 2.9. *Let $(S, {}^*) \in$ **qit**, $a, b \in S$ and ρ be an idempotent-separating congruence on S.*

(1) $a\mathcal{L}b$ (reps. $a\mathcal{R}b$) if and only if $a^*a = b^*b$ (resp. $aa^* = bb^*$).

(2) If $a\rho b$, then $a^*\rho b^*$.

Proof. (1) If $a\mathcal{L}b$, then $a^*a\mathcal{L}a\mathcal{L}b\mathcal{L}b^*b$. Since $a^*a, b^*b \in \Lambda_S$ and Λ_S is $\mathcal{L}$-unipotent by Lemma 2.8, we have $a^*a = b^*b$. The converse is clear.

(2) If $a\rho b$, then $a\mathcal{H}b$ by Lemma 2.2 (1). In view of item (1), we have $aa^* = bb^*$ and $a^*a = b^*b$. This implies that

$$b^* = b^*bb^* = a^*ab^*\rho a^*bb^* = a^*aa^* = a^*,$$

as required. □

Now, let $(S, {}^*) \in \mathbf{qit}$. For $e \in I_S$ and $f \in \Lambda_S$, denote

$$\langle e \rangle = eI_Se = \{eie | i \in I_S\}, \langle f \rangle = f\Lambda_S f = \{f\lambda f | \lambda \in \Lambda_S\}.$$

In the end of this section, we present some results on the sets $\langle e \rangle$ and $\langle f \rangle$.

Lemma 2.10. *Let $(S, {}^*) \in \mathbf{qit}$, $a \in S$, $e \in I_S$, $f \in \Lambda_S$ and $p \in F_{S^*}$.*

(1) $\langle e \rangle = eF_{S^*}e^* = \{x \in I_S | exe = x\}$.

(2) $\langle f \rangle = f^*F_{S^*}f = \{x \in \Lambda_S | fxf = x\}$.

(3) $xyx \in \langle e \rangle$ for all $x, y \in \langle e \rangle$.

(4) $xyx \in \langle f \rangle$ for all $x, y \in \langle f \rangle$.

(5) $a^*xa \in \langle a^*a \rangle$ for all $x \in \langle aa^* \rangle$.

(6) $aya^* \in \langle aa^* \rangle$ for all $y \in \langle a^*a \rangle$.

(7) $\langle p \rangle \subseteq F_{S^*}$.

Proof. (1) If $x = eie \in \langle e \rangle$ for some $i \in I_S$, then by Lemma 2.7, we have $x = eie = ei^*e^* \in eF_{S^*}e^*$. Now, let $x = ese^*$ for some $s \in F_{S^*}$. Then $s = s^*$ and so $x = esse^* = es(es)^* \in I_S$. Moreover, $exe = eese^*e = ese^* = x$. This shows that $eF_{S^*}e^* \subseteq \{x \in I_S | exe = x\}$. Obviously, $\{x \in I_S | exe = x\} \subseteq \langle e \rangle$. Thus the reslut holds.

(2) This is the dual of item (2).

(3) Since $x, y \in \langle e \rangle \subseteq I_S$, we have $xyx = xy^*x^* \in xF_{S^*}x^*$ by Lemma 2.7. This implies that $xyx \in \langle x \rangle$ by item (1) in this lemma. Thus, $xyx \in I_S$ and $exyxe = xyx$ whence $xyx \in \langle e \rangle$ by item (1) in this lemma again.

(4) This is the dual of item (3).

(5) Let $x \in \langle aa^* \rangle$. Then $x \in I_S$ and so $x = xx^*, x^* = x^{**}$ by Lemma 2.4. This implies that

$$a^*xa = a^*xx^*a = a^*x^{**}x^*a = (x^*a)^*x^*a \in \Lambda_S.$$

Since $a^*a(a^*xa)a^*a = a^*xa$, we have $a^*xa \in \langle a^*a \rangle$ by item (2).

(6) This is the dual of (5).

(7) Since $p \in F_{S^*}$, we have $\langle p \rangle = pF_{S^*}p^* \subseteq F_{S^*}$ by item (1) and Lemma 2.1 (3).

$\square$

3 Fundamental regular semigroups with a quasi-ideal regular $*$-transversal

In this section, we shall construct fundamental regular semigroups with quasi-ideal regular $*$-transversals by some kind of partial transformations. Recall that a *semi-band* is a semigroup which is generated by its idempotents. Throughout this section, we let C be a regular semi-band, $(C, {}^*) \in$**qit** and use I and Λ to denote I_C and Λ_C, respectively. In view of Lemma 2.10, we have $xyx \in \langle e \rangle$ for all $x, y \in \langle e \rangle$ and $e \in I \cup \Lambda$.

Now, let $e, f \in I \cup \Lambda$. A bijection α from $\langle e \rangle$ onto $\langle f \rangle$ is called a *pre-isomorphism* if

$$(\forall x, y \in \langle e \rangle) \ \ (xyx)\alpha = (x\alpha)(y\alpha)(x\alpha). \tag{3.1}$$

Clearly, $e\alpha = f$. Moreover, we say that $\langle e \rangle$ is *pre-isomorphic* to $\langle f \rangle$ if there exists a pre-isomorphism from $\langle e \rangle$ onto $\langle f \rangle$. In this case, we write $\langle e \rangle \simeq \langle f \rangle$ and denote the set of all pre-isomorphisms from $\langle e \rangle$ onto $\langle f \rangle$ by $T_{e,f}$.

The following result shows that pre-isomorphisms exist indeed. As usual, we use ι_M to denote the identity transformation on the non-empty set M.

Proposition 3.1. *Let $a \in C$ and*

$$\pi_a : \langle aa^* \rangle \to \langle a^*a \rangle, x \mapsto a^*xa.$$

Then $\pi_a \in T_{aa^*,a^*a}$. Moreover, the inverse mapping of π_a is

$$\pi_a^{-1} : \langle a^*a \rangle \to \langle aa^* \rangle, y \mapsto aya^*$$

and $\pi_a^{-1} \in T_{a^*a,aa^*}$. In particular, we have $\pi_p = \iota_{\langle p \rangle} = \pi_p^{-1}$ for any $p \in F_{C^*}$.

Proof. The results can be checked by straight calculations by using Lemma 2.10. $\square$

On pre-isomorphisms in general, we have the following results.

Lemma 3.2. *Let $e \in I, f \in \Lambda$, $x \in \langle e \rangle$, $y \in \langle f \rangle$ and $\alpha \in T_{e,f}$.*

(1) $\alpha^{-1} \in T_{f,e}$.

(2) $\langle x \rangle\alpha = \langle x\alpha \rangle$ and $\langle y \rangle\alpha^{-1} = \langle y\alpha^{-1} \rangle$.

(3) $(x\alpha)^* = (x\alpha)f^*, x\alpha = (x\alpha)^*f$.

(4) $(y\alpha^{-1})^* = e^*(y\alpha^{-1}), y\alpha^{-1} = e(y\alpha^{-1})^*$.

Proof. (1) Obviously, α^{-1} is a bijection. Now, let $x', y' \in \langle f\rangle$. Then $x' = x\alpha$ and $y' = y\alpha$ for some $x, y \in \langle e\rangle$. Since $\alpha \in T_{e,f}$, we have

$$(xyx)\alpha = (x\alpha)(y\alpha)(x\alpha)$$

whence

$$(x'\alpha^{-1})(y'\alpha^{-1})(x'\alpha^{-1}) = (x'y'x')\alpha^{-1}.$$

This implies that $\alpha^{-1} \in T_{f,e}$.

(2) Obviously, $\langle x\rangle \subseteq \langle e\rangle$ and $x\alpha \in \langle f\rangle$. Let $u \in \langle x\rangle$. Then $u\alpha \in \langle f\rangle$. Observe that

$$(x\alpha)(u\alpha)(x\alpha) = (xux)\alpha = u\alpha,$$

it follows that $u\alpha \in \langle x\alpha\rangle$ by Lemma 2.10. This shows that $\langle x\rangle\alpha \subseteq \langle x\alpha\rangle$. Conversely, let $u' \in \langle x\alpha\rangle$. Then $u' \in \langle f\rangle$ and so $u' = u\alpha$ for some u in $\langle e\rangle$ and

$$u\alpha = u' = (x\alpha)u'(x\alpha) = (x\alpha)(u\alpha)(x\alpha) = (xux)\alpha.$$

Since α is bijective, we have $xux = u$ and so $u \in \langle x\rangle$ by Lemma 2.10. This implies that $\langle x\alpha\rangle \subseteq \langle x\rangle\alpha$. By similar method, we can prove the other identity.

(3) Since $x\alpha \in \langle f\rangle \subseteq \Lambda$, we have $x\alpha = f(x\alpha)f = f^*(x\alpha)^*f$ and so $(x\alpha)^* = f^*(x\alpha)^{**}f^*$ by Lemma 2.7. Observe that $f^* \in S^*$ and $x\alpha = f(x\alpha)f$, it follows that

$$(x\alpha)^* = f^*(x\alpha)^{**}f^* = f^*(x\alpha)f^* = f^*f(x\alpha)ff^* = f(x\alpha)ff^* = (x\alpha)f^*$$

by Lemma 2.7 and Lemma 2.4. This also implies that

$$(x\alpha)^*f = (x\alpha)f^*f = (x\alpha)f = x\alpha.$$

(4) This is the dual of (3). □

Denote $\mathcal{U} = \{(e, f) \in I \times \Lambda | \langle e\rangle \simeq \langle f\rangle\}$ and define a multiplication "$\circ$" on the set

$$T_C = \bigcup_{(e,f)\in\mathcal{U}} T_{e,f}$$

as follows: For $\alpha \in T_{e,f}$ and $\beta \in T_{g,h}$,

$$\alpha \circ \beta = \alpha\pi^{-1}_{g(fg)^*f}\beta,$$

where the composition is that in the symmetric inverse semigroup on the set $I \cup \Lambda$.

Lemma 3.3. *If $\alpha, \beta \in T_C$ and $\alpha \in T_{e,f}$, $\beta \in T_{g,h}$, then $\alpha \circ \beta \in T_{j,k}$, where $j = (fg(fg)^*f)\alpha^{-1}$, $k = (g(fg)^*fg)\beta$. As a consequence, the above " $\circ$ " is well-defined.*

Proof. By Lemma 2.6 (2), $(g(fg)^*f)^* = fg$. This implies that $\mathrm{ran}(\pi^{-1}_{g(fg)^*f}) = \langle g(fg)^*fg\rangle$ and so

$$\mathrm{dom}(\pi^{-1}_{g(fg)^*f}\beta) = (\langle g(fg)^*fg\rangle \cap \langle g\rangle)\pi_{g(fg)^*f} = \langle g(fg)^*fg\rangle\pi_{g(fg)^*f} = \langle fg(fg)^*f\rangle$$

by Lemma 3.2 (2). This implies that

$$\mathrm{dom}(\alpha \circ \beta) = (\mathrm{dom}(\pi^{-1}_{g(fg)^*f}\beta) \cap \langle f\rangle)\alpha^{-1} = \langle fg(fg)^*f\rangle\alpha^{-1} = \langle j\rangle,$$

$$\mathrm{ran}(\alpha \circ \beta) = (\mathrm{dom}(\pi^{-1}_{g(fg)^*f}\beta) \cap \langle f\rangle)\pi^{-1}_{g(fg)^*f}\beta = \langle g(fg)^*fg\rangle\beta = \langle k\rangle$$

by Lemma 3.2 (2) again. Since $\alpha, \beta, \pi^{-1}_{g(fg)^*f}$ all satisfy the condition (3.1) by Proposition 3.1, it follows that $\alpha \circ \beta$ also satisfies this condition. This implies that $\alpha \circ \beta \in T_{j,k}$. □

Lemma 3.4. *The multiplication "$\circ$" is associative. Therefore, T_C is a semigroup with respect to " $\circ$ ".*

Proof. Now, let $\alpha \in T_{e,f}$, $\beta \in T_{g,h}$, $\gamma \in T_{s,t}$ and

$$\alpha \circ \beta \in T_{j,k}, (\alpha \circ \beta) \circ \gamma \in T_{m,n}, \beta \circ \gamma \in T_{p,q}, \alpha \circ (\beta \circ \gamma) \in T_{a,b},$$

where

$$j = (fg(fg)^*f)\alpha^{-1}, k = (g(fg)^*fg)\beta, p = (hs(hs)^*h)\beta^{-1}, q = (s(hs)^*hs)\gamma,$$

$$m = (ks(ks)^*k)(\alpha\circ\beta)^{-1}, n = (s(ks)^*ks)\gamma, a = (fp(fp)^*f)\alpha^{-1}, b = (p(fp)^*fp)(\beta\circ\gamma).$$

On one hand, by Lemma 3.2 (3) and the fact that $k \in \langle h\rangle$, we have $k = k^*h = kh$. Moreover, we can obtain that $(hs)^* = (hs)^*h^*$ and $(hs)^*h^{**}k^* = (hs)^*hk^*$ by Lemma 2.6 (1) and Lemma 2.7, respectively. Thus,

$$\begin{aligned}
(ks)(ks)^*k &= (khs)(k^*hs)^*k \quad (\text{since } k = k^*h = kh)\\
&= k(hs)(hs)^*k^{**}k \quad (\text{ since } (k^*hs)^* = (hs)^*k^{**})\\
&= k(hs)(hs)^*h^{**}k^*k \quad (\text{ since } (hs)^* = (hs)^*h^*, h^* = h^{**}, k^{**} = k^*)\\
&= k(hs)(hs)^*hk^*k \quad (\text{ since } (hs)^*h^{**}k^* = (hs)^*hk^*)\\
&= k((hs)(hs)^*h)k \quad (\text{ since } k^*k = k)\\
&= k(p\beta)k. \quad (\text{ since } p = ((hs)(hs)^*h)\beta^{-1})
\end{aligned}$$

Since $k = (g(fg)^*fg)\beta$ and β is a pre-isomorphism, we have

$$ks(ks)^*k = k(p\beta)k = (g(fg)^*fg)\beta \cdot p\beta \cdot (g(fg)^*fg)\beta = (g(fg)^*fgpg(fg)^*fg)\beta,$$

whence

$$m = (ks(ks)^*k)(\alpha\circ\beta)^{-1} = (ks(ks)^*k)\beta^{-1}\pi_{g(fg)^*f}\alpha^{-1} = (fgpg(fg)^*f)\alpha^{-1}.$$

On the other hand, by Lemma 3.2 (4) and the fact that $p \in \langle g\rangle$, we have $gp^* = p = gp$. Moreover, we can obtain that $(fg)^* = g^*(fg)^*$ and $p^*g^*(fg)^* = p^*g^{**}(fg)^* = p^*g(fg)^*$ by Lemma 2.6 (1) and Lemma 2.7, respectively. Thus,

$$\begin{aligned}
(fp)(fp)^*f &= (fgp)(fgp^*)^*f \quad (\text{since } gp = p = gp^*)\\
&= fgpp^{**}(fg)^*f \quad (\text{ since } (fgp^*)^* = p^{**}(fg)^*)\\
&= fgpp^*g^*(fg)^*f \quad (\text{ since } (fg)^* = g^*(fg)^*, p^* = p^{**})\\
&= fgpp^*g(fg)^*f \quad (\text{ since } p^*g^*(fg)^* = p^*g(fg)^*)\\
&= fgpg(fg)^*f. \quad (\text{ since } pp^* = p)
\end{aligned}$$

This implies that $a = (fp(fp)^*f)\alpha^{-1} = m$. Dually, we can show that $n = b$.

Let $x \in \langle m\rangle$ and denote $y = x\alpha$. On one hand, since $k = kh = k^*h$, we have $ks = k^*hs$ and

$$(ks)^*k = (k^*hs)^*k = (hs)^*k^{**}k = (hs)^*k^*k = (hs)^*k.$$

Observe that

$$k\cdot(g(fg)^*fyfg)\beta\cdot k = (g(fg)^*fg)\beta\cdot(g(fg)^*fyfg)\beta\cdot(g(fg)^*fg)\beta = (g(fg)^*fyfg)\beta,$$

it follows that

$$\begin{aligned}
x[(\alpha\circ\beta)\circ\gamma] &= (s(ks)^*k\cdot(g(fg)^*fyfg)\beta\cdot ks)\gamma\\
&= (s(hs)^*k\cdot(g(fg)^*fyfg)\beta\cdot khs)\gamma\\
&= (s(hs)^*[k\cdot(g(fg)^*fyfg)\beta\cdot k]hs)\gamma\\
&= (s(hs)^*\cdot(g(fg)^*fyfg)\beta\cdot hs)\gamma.
\end{aligned}$$

On the other hand, since $p = gp^* = gp$, we have

$$p(fp)^*f = p(fgp^*)^*f = pp^{**}(fg)^*f = pp^*g^*(fg)^*f = pp^*g(fg)^*f = pg(fg)^*f.$$

Observe that $p\beta = hs(hs)^*h$ and β is a pre-isomorphism, it follows that

$$\begin{aligned}
x[\alpha\circ(\beta\circ\gamma)] &= (s(hs)^*h\cdot(p(fp)^*fyfp)\beta\cdot hs)\gamma\\
&= (s(hs)^*h\cdot(pg(fg)^*fyfgp)\beta\cdot hs)\gamma\\
&= (s(hs)^*h\cdot p\beta\cdot(g(fg)^*fyfg)\beta\cdot p\beta\cdot hs)\gamma\\
&= ((s(hs)^*h\cdot p\beta)\cdot(g(fg)^*fyfg)\beta\cdot(p\beta\cdot hs))\gamma\\
&= (s(hs)^*h\cdot(g(fg)^*fyfg)\beta\cdot hs)\gamma\\
&= (s(hs)^*(h\cdot(g(fg)^*fyfg)\beta)\cdot hs)\gamma\\
&= (s(hs)^*\cdot(g(fg)^*fyfg)\beta\cdot hs)\gamma\\
&= x[(\alpha\circ\beta)\circ\gamma].
\end{aligned}$$

Thus, $(\alpha\circ\beta)\circ\gamma = \alpha\circ(\beta\circ\gamma)$. This implies that the operation "$\circ$" is associative and so $(T_C, \circ)$ is a semigroup. □

Theorem 3.5. *Under the above notations,* $(T_C, {}^*) \in$**qit** *with respect to the unary operation "$*$" on T_C defined by the rule:*

$$(\forall \alpha \in T_{e,f}) \quad \alpha^* = \pi_f \alpha^{-1} \pi_e.$$

In this case, $T_C^* = \{\alpha \in T_C | \alpha \in T_{p,q}, p, q \in F_{C^*}\}$.

Proof. By Lemma 3.3 and Lemma 3.4, $(T_C, \circ)$ is a semigroup. For any $\alpha \in T_{e,f}$, it is routine to check that α^* is a bijection from $\langle f^* \rangle$ onto $\langle e^* \rangle$ and satisfies the condition (3.1) and so $\alpha^* \in T_{f^*,e^*}$. Thus, the unary operation "$*$" is well-defined.

Now, let $\alpha \in T_{e,f}, \beta \in T_{g,h} \in T_C$. Then we have the following facts:

(1) Since $\alpha^* \in T_{f^*,e^*}$, we have

$$\alpha \circ \alpha^* = \alpha \pi^{-1}_{f^*(ff^*)^* f} \alpha^* = \alpha \pi_f^{-1} \pi_f \alpha^{-1} \pi_e = \pi_e. \tag{3.2}$$

By similar method, we can show that $\pi_e \circ \alpha = \alpha$ and $\alpha^* \circ \pi_e = \alpha^*$. This implies that $\alpha \circ \alpha^* \circ \alpha = \alpha$ and $\alpha^* \circ \alpha \circ \alpha^* = \alpha^*$.

(2) Since $\alpha^* = \pi_f \alpha^{-1} \pi_e \in T_{f^*,e^*}$ and $\pi_{e^*} = \iota_{\langle e^* \rangle}, \pi_{f^*} = \iota_{\langle f^* \rangle}$ by Lemma 2.4 and Proposition 3.1, it follows that

$$\alpha^{**} = \pi_{e^*}(\alpha^*)^{-1} \pi_{f^*} = \pi_{e^*} \pi_e^{-1} \alpha \pi_f^{-1} \pi_{f^*} = \pi_e^{-1} \alpha \pi_f^{-1} \in T_{e^*,f^*}. \tag{3.3}$$

This implies that

$$\alpha^{***} = \pi_{f^*} \pi_f \alpha^{-1} \pi_e \pi_{e^*} = \pi_f \alpha^{-1} \pi_e = \alpha^*.$$

(3) Since $\alpha^* \in T_{f^*,e^*}, \beta^* \in T_{h^*,g^*}$ and

$$fh^*(fh^*)^* f = fh^* f^* f = fh^* f, h^*(fh^*)^*(fh^*) = h^* h^* f^* fh^* = h^* fh^* = h^* f^* h^*$$

by Lemma 2.7, we have $\alpha \circ \beta^* \in T_{(fh^*f)\alpha^{-1},(h^*f^*h^*)\beta^*}$ by Lemma 3.3. Since $(h^*f^*h^*)\beta^* \in \langle g^* \rangle$ and $g^* \in F_{C^*}$, it follows that $(h^*f^*h^*)\beta^* \in F_{C^*}$ by Lemma 2.10, and so $((h^*f^*h^*)\beta^*)^* = (h^*f^*h^*)\beta^*$. Moreover, by Lemma 3.2 and Lemma 2.7, we have

$$((fh^*f)\alpha^{-1})^* = e^*[(fh^*f)\alpha^{-1}] = e^*(f^*fh^*f)\alpha^{-1} = e^*(f^*f^{**}h^*f)\alpha^{-1} = e^*(f^*h^*f)\alpha^{-1}. \tag{3.4}$$

This yields that

$$(\alpha \circ \beta^*)^* \in T_{(h^*f^*h^*)\beta^*, e^*(f^*h^*f)\alpha^{-1}}.$$

On the other hand, since $\beta^{**} \in T_{g^*,h^*}$ and

$$(\beta^{**})^{-1} = (\pi_g^{-1} \beta \pi_h^{-1})^{-1} = \pi_h \beta^{-1} \pi_g = \beta^*,$$

it follows that

$$(h^*f^*(h^*f^*)^*h^*)(\beta^{**})^{-1} = (h^*f^*h^*)(\beta^{**})^{-1} = (h^*f^*h^*)\beta^*.$$

Moreover,

$$(f^*(h^*f^*)^*h^*f^*)\alpha^* = (f^*h^*f^*)\alpha^* = (f^*h^*f^*)\pi_f\alpha^{-1}\pi_e$$

$$= e^* \cdot (f^*f^*h^*f^*f)\alpha^{-1} \cdot e = e^* \cdot (f^*h^*f)\alpha^{-1}.$$

This implies that $\beta^{**} \circ \alpha^* \in T_{(h^*f^*h^*)\beta^*, e^*(f^*h^*f)\alpha^{-1}}$.

Now, let $x \in \langle (h^*f^*h^*)\beta^* \rangle$. Since

$$\alpha \circ \beta^* = \alpha\pi^{-1}_{h^*(fh^*)^*f}\beta^* = \alpha\pi^{-1}_{h^*f}(\pi_h\beta^{-1}\pi_g) \in T_{(fh^*f)\alpha^{-1},(h^*f^*h^*)\beta^*},$$

we have

$$(\alpha \circ \beta^*)^* = \pi_{(h^*f^*h^*)\beta^*}(\alpha \circ \beta^*)^{-1}\pi_{(fh^*f)\alpha^{-1}} = \pi_{(h^*f^*h^*)\beta^*}\pi_g^{-1}\beta\pi_h^{-1}\pi_{h^*f}\alpha^{-1}\pi_{(fh^*f)\alpha^{-1}}.$$

Since $(h^*f^*h^*)\beta^* \in F_{C^*}$, we have $\pi_{(h^*f^*h^*)\beta^*} = \iota_{\langle (h^*f^*h^*)\beta^* \rangle}$. By the identity (3.4), it follows that $((fh^*f)\alpha^{-1})^* = e^*(f^*h^*f)\alpha^{-1}$. Since

$$f^*h^*ff^*h^*h = f^*h^*f^*h^*h = f^*h, \ \ h^*h^*ffh^*f = h^*fh^*f = h^*f^*h^*f^*f = h^*f^*f = h^*f$$

by Lemma 2.1, Lemma 2.4 and Lemma 2.7, we have

$$\begin{aligned} x(\alpha \circ \beta^*)^* &= ((fh^*f)\alpha^{-1})^* \cdot (f^*h^*h((gxg^*)\beta)h^*h^*f)\alpha^{-1} \cdot (fh^*f)\alpha^{-1} \\ &= e^*(f^*h^*f)\alpha^{-1} \cdot (f^*h^*h((gxg^*)\beta)h^*h^*f)\alpha^{-1} \cdot (fh^*f)\alpha^{-1} \\ &= e^* \cdot (f^*h \cdot (gxg^*)\beta \cdot h^*f)\alpha^{-1}. \end{aligned}$$

Observe that

$$\beta^{**} \circ \alpha^* = \beta^{**}\pi^{-1}_{f^*(h^*f^*)^*h^*}\alpha^* = \beta^{**}\pi^{-1}_{f^*h^*}\alpha^* = \pi_g^{-1}\beta\pi_h^{-1}\pi^{-1}_{f^*h^*}\pi_f\alpha^{-1}\pi_e,$$

it follows that

$$x(\beta^{**} \circ \alpha^*) = e^* \cdot (f^*f^*h^*h((gxg^*)\beta)h^*h^*f^*f)\alpha^{-1} \cdot e = e^* \cdot (f^*h \cdot (gxg^*)\beta \cdot h^*f)\alpha^{-1}.$$

This implies that $x(\alpha \circ \beta^*)^* = x(\beta^{**} \circ \alpha^*)$. Thus, $(\alpha \circ \beta^*)^* = \beta^{**} \circ \alpha^*$. Similarly, we can see that $(\alpha^* \circ \beta)^* = \beta^* \circ \alpha^{**}$.

(4) From items (1), (2) and (3), we obtain that $(T_C, \ ^*) \in \mathbf{rt}$. Now, we first assert that $T_C^* = \{\alpha | \alpha \in T_{p,q}, p, q \in F_{C^*}\}$. Obviously, $T_C^* \subseteq \{\alpha | \alpha \in T_{p,q}, p, q \in F_{C^*}\}$. Conversely, if $\alpha \in T_{p,q}, p, q \in F_{C^*}$, then $\pi_p^{-1} = \iota_{\langle p \rangle}, \pi_q^{-1} = \iota_{\langle q \rangle}$ and so $\alpha^{**} = \pi_p^{-1}\alpha\pi_q^{-1} = \alpha \in T_C^*$. This implies that $\{\alpha | \alpha \in T_{p,q}, p, q \in F_{C^*}\} \subseteq T_C^*$. Now, let $\alpha \in T_{p,q}, \gamma \in T_{s,t}$ for some $p, q, s, t \in F_{C^*}$ and $\beta \in T_{g,h}$ for $g \in I$ and $h \in \Lambda$. Then $\langle p \rangle, \langle q \rangle, \langle s \rangle, \langle t \rangle \subseteq F_{C^*}$ by Lemma 2.10 (7). By the proof of Lemma 3.4, $\alpha \circ \beta \circ \gamma \in T_{a,b}$ for some $a \in \mathrm{ran}(\alpha^{-1})$ and $b \in \mathrm{ran}(\gamma)$. Since $\mathrm{ran}(\alpha^{-1}) = \langle p \rangle \subseteq F_{C^*}$ and $\mathrm{ran}(\gamma) = \langle t \rangle \subseteq F_{C^*}$, we have $\alpha \circ \beta \circ \gamma \in T_C^*$. This shows that T_C^* is a quasi-ideal of T_C. Thus, $(T_C, \ ^*) \in \mathbf{qit}$. □

The following examples illustrate the above Theorem 3.5.

Example 3.6. Let $C = E$ be a semilattice. Then $(C, {}^*) \in \mathbf{qit}$ with respect to the operation "$*$" defined by $u^* = u$ for all u in C. Obviously,

$$I = \Lambda = E = F_C = C = C^* = F_{C^*}$$

and $\langle e \rangle = eE$ is a subsemilattice of E for all $e \in C$. Now, let $e, f, g, h, \in E$ and $\alpha \in T_{e,f}, \beta \in T_{g,h}$. Since $gf \in F_{C^*}$, we have $\pi^{-1}_{g(fg)^*f} = \pi^{-1}_{gf} = \iota_{\langle gf \rangle}$ by Proposition 3.1, this implies $\alpha \circ \beta = \alpha\iota_{\langle gf \rangle}\beta = \alpha\beta$. Observe the condition (3.1), it follows that the pre-isomorphisms between $\langle e \rangle$ and $\langle f \rangle$ are exactly isomorphisms between them for all $e, f \in E$. Thus, T_C is exactly the well-known Munn semigroup determined by the semilattice E.

Example 3.7. Let C be a rectangular band. Then $(C, {}^*) \in \mathbf{qit}$ with respect to the operation "$*$" defined by $u^* = e^\circ$ for all $u \in C$ where e° is a fixed element in C and thus $C^* = \{e^\circ\}$. By Lemma 2.4, $I = \{e \in C | e\mathcal{L}e^\circ\}, \Lambda = \{f \in C | f\mathcal{R}e^\circ\}$ in the case. Moreover, for any $e \in I$ and $f \in \Lambda$, we have $\langle e \rangle = \{e\}$ and $\langle f \rangle = \{f\}$. Denote

$$\sigma_{e,f} : \langle e \rangle \to \langle f \rangle, e \mapsto f.$$

Then $T_C = \{\sigma_{e,f} | e \in I, f \in \Lambda\}$ and

$$\sigma_{e,f} \circ \sigma_{g,h} \in T_{(fg(fg)^*f)\alpha^{-1},(g(fg)^*fg)\beta} = T_{f\alpha^{-1},g\beta} = T_{e,h}$$

for any $\sigma_{e,f}, \sigma_{g,h} \in T_C$ by Lemma 3.3 and the fact that C is a rectangular band. This implies that $\sigma_{e,f} \circ \sigma_{g,h} = \sigma_{e,h}$. Thus, T_C is isomorphic to the rectangular band $I \times \Lambda$ and $T^*_C = \{\iota_{\{e^\circ\}}\}$. In fact, T_C is isomorphic to C.

Example 3.8. Let $C = \{e, f, p, q\}$ be a rectangular band with $p\mathcal{R}e\mathcal{L}q\mathcal{R}f\mathcal{L}p$. Then $(S, {}^*) \in \mathbf{qit}$ with the respect to the unary operation $p^* = p, q^* = q, e^* = f, f^* = e$. It is routine to check that $C = C^*$ and $I = \Lambda = F_{C^*} = \{p, q\}$. Moreover, we have $\langle p \rangle = \{p\}$ and $\langle q \rangle = \{q\}$. Thus, $T_C = T^*_C = \{\iota_{\{p\}}, \iota_{\{q\}}, \sigma_{p,q}, \sigma_{q,p}\}$. In fact, T_C is also isomorphic to C.

By the proof of Theorem 3.5, we have some information on the semigroup T_C.

Corollary 3.9. *Let $\alpha \in T_{e,f}$, $\beta \in T_{g,h}$.*

(1) $\alpha^* \in T_{f^*,e^*}$, $\alpha^{**} \in T_{e^*,f^*}$.

(2) $\alpha \circ \alpha^* = \pi_e, \alpha^* \circ \alpha = \pi_f$ and so $I_{T_C} = \{\pi_e | e \in I\}$ and $\Lambda_{T_C} = \{\pi_f | f \in \Lambda\}$.

(3) $\alpha \circ \alpha^* = \iota_{\langle e \rangle}, \alpha^* \circ \alpha = \iota_{\langle f \rangle}$ if $e, f \in F_{C^*}$ and so $F_{T^*_C} = \{\iota_{\langle c \rangle} | c \in F_{C^*}\}$.

(4) $\alpha\mathcal{R}^{T_C}\beta$ (resp. $\alpha\mathcal{L}^{T_C}\beta$) if and only if $e = g$ (resp. $f = h$).

Proof. Items (1-3) can be obtained by the proof of Theorem 3.5 and Proposition 3.1 directly. Item (4) follows from item (2) and Corollary 2.9 (1). □

The following theorem shows that the above T_C is fundamental.

Theorem 3.10. *Let U be a subsemigroup of $(T_C, {}^*)$ and $(U, {}^*) \in \mathbf{qit}$ such that $F_{U^*} = F_{T_C^*}$. Then U is fundamental. In particular, T_C itself is fundamental.*

Proof. By Corollary 3.9 (3), $F_{T_C^*} = \{\iota_{\langle c\rangle} | c \in F_{C^*}\} = F_{U^*}$. Now, let $\alpha, \beta \in U$ and $\alpha \in T_{e,f}, \beta \in T_{g,h}$ with $\alpha\mu_U\beta$. Then $\alpha\mathcal{H}^U\beta$ by Lemma 2.2 (1). This implies that $e = g$ and $f = h$ by Corollary 3.9 (4). Thus, $\alpha^{**}, \beta^{**} \in T_{e^*,f^*}$ by Corollary 3.9 (1).

On the other hand, it follows that $\alpha^*\mu_U\beta^*$ and $\alpha^{**}\mu_U\beta^{**}$ by Corollary 2.9 (2). Let $c \in F_{C^*}$. Then $\alpha^* \circ \iota_{\langle c\rangle} \circ \alpha^{**}\mu_U\beta^* \circ \iota_{\langle c\rangle} \circ \beta^{**}$. Since μ_U is idempotent-separating and

$$\alpha^* \circ \iota_{\langle c\rangle} \circ \alpha^{**}, \beta^* \circ \iota_{\langle c\rangle} \circ \beta^{**} \in E(U^*)$$

by Lemma 2.1, we obtain that

$$\alpha^* \circ \iota_{\langle c\rangle} \circ \alpha^{**} = \beta^* \circ \iota_{\langle c\rangle} \circ \beta^{**}.$$

Observe that

$$\langle (e^*ce^*)\alpha^{**}\rangle = \mathrm{ran}(\alpha^* \circ \iota_{\langle c\rangle} \circ \alpha^{**}) = \mathrm{ran}(\beta^* \circ \iota_{\langle c\rangle} \circ \beta^{**}) = \langle (e^*ce^*)\beta^{**}\rangle$$

by Lemma 3.4, it follows that $(e^*ce^*)\alpha^{**} = (e^*ce^*)\beta^{**}$. Since c is arbitrary, this implies that $\alpha^{**} = \beta^{**}$ and so $\alpha^* = \beta^*$. Since $\alpha\mathcal{H}^U\beta$ and $\alpha^* = \beta^* \in V(\alpha) \cap V(\beta)$, we have $\alpha = \beta$. We have shown that μ_U is the identity relation on U and so U is fundamental. □

4 Fundamental representations of regular semigroups with quasi-ideal regular $*$-transversals

Throughout this section, we let $(S, {}^*) \in\mathbf{qit}$ and C be the semi-band generated by $E(S)$. The aim of this section is to give a fundamental representation of S. We first give the following

Lemma 4.1. *Let $(S, {}^*) \in\mathbf{qit}$. Then $(C, {}^*) \in \mathbf{qit}$. In this case, $C^* = C \cap S^*$, $I_S = I_C$, $\Lambda_S = \Lambda_C$ and $F_{S^*} = F_{C^*}$.*

Proof. This can be proved easily by using Lemma 2.2 (2) and Lemma 2.4. □

For simplicity, we let $I = I_C$ and $\Lambda = \Lambda_C$. By Theorem 3.5, we have a fundamental regular semigroup T_C with a quasi-ideal regular $*$-transversal T_C^*. Let $a \in S$. Define

$$\rho_a : \langle aa^*\rangle \to \langle a^*a\rangle, x \mapsto a^*xa.$$

Then

$$\rho_a^{-1} : \langle a^*a\rangle \to \langle aa^*\rangle, y \mapsto aya^*.$$

Observe that $\rho_a = \pi_a$ for every $a \in C$ where π_a is defined as in Proposition 3.1. Moreover, we can prove the following

Lemma 4.2. *Let $a, b \in S$. Then $\rho_a \in T_{aa^*,a^*a}$ and $\rho_a^{-1} \in T_{a^*a,aa^*}$. Moreover, $\rho_a \circ \rho_b = \rho_{ab}$.*

Proof. It is routine to check that $\rho_a \in T_{aa^*,a^*a}$ and $\rho_a^{-1} \in T_{a^*a,aa^*}$ by Lemma 2.10. Obviously, $\rho_{ab} \in T_{ab(ab)^*,(ab)^*ab}$. By Lemma 3.3, $\rho_a \circ \rho_b \in T_{j,k}$ where

$$j = (a^*abb^*(a^*abb^*)^*a^*a)\rho_a^{-1} = a(a^*abb^*(a^*abb^*)^*a^*a)a^* = ab(b^*(a^*abb^*)^*a^*) = ab(ab)^*$$

and

$$k = (bb^*(a^*abb^*)^*a^*abb^*)\rho_b = b^*(bb^*(a^*abb^*)^*a^*abb^*)b = b^*(a^*abb^*)^*a^*ab = (ab)^*ab$$

by Lemma 2.3. This implies that ρ_{ab} and $\rho_a \circ \rho_b$ have the same domains and ranges. Now, let $x \in \mathrm{dom}\rho_{ab}$. Then we have $x\rho_{ab} = (ab)^*xab$ and

$$\begin{aligned} x[\rho_a \circ \rho_b] &= x\rho_a\pi^{-1}_{bb^*(a^*abb^*)^*a^*a}\rho_b \\ &= b^*(bb^*(a^*abb^*)^*a^*a)a^*xa(bb^*(a^*abb^*)^*a^*a)^*b \\ &= (ab)^*xa(bb^*(a^*abb^*)^*a^*a)^*b \\ &= (ab)^*xa(a^*abb^*)b \\ &= (ab)^*xab \end{aligned}$$

by Lemma 2.3 and Lemma 2.6 (2). Thus, $\rho_a \circ \rho_b = \rho_{ab}$. □

Let $(U, {}^*)$ and $(V, {}^\circ)$ be two unary semigroups. A mapping (resp. a bijective mapping) ψ from U to V is called *a unary homomorphism* (resp. *a unary isomorphism*) if $(ab)\psi = (a\psi)(b\psi)$ and $(a^*\psi) = (a\psi)^\circ$ for all a, b in U. We say that $(U, {}^*)$ is *unary isomorphic* to $(V, {}^\circ)$ if there exists a unary isomorphism from U onto V. The following result provides a fundamental representation of S.

Theorem 4.3. *Define $\rho : S \to T_C, a \mapsto \rho_a$. Then ρ is a unary homomorphism from $(S, {}^*)$ to $(T_C, {}^*)$ such that $\ker\rho$ is the maximum idempotent-separating congruence of S. Moreover, ρ satisfies the following conditions:*

(1) $\rho|_{E(S)}$ is a bijection from $E(S)$ onto $E(T_C)$.

(2) $S^*\rho \subseteq T_C^*$ and $\rho|_{F_{S^*}}$ is a bijection from F_{S^*} onto $F_{T_C^*}$.

(3) $\rho|_C$ is a unary homomorphism from $(C, {}^*)$ onto $(\langle E(T_C)\rangle, {}^*)$.

(4) $\rho|_I$ (resp. $\rho|_\Lambda$) is a bijection from I onto I_{T_C} (resp. Λ_{T_C}).

Proof. By Lemma 4.2, $\rho_a \in T_{aa^*,a^*a} \subseteq T_C$. This shows that ρ is well-defined. According to Lemma 4.2 again, we have $(ab)\rho = \rho_{ab} = \rho_a \circ \rho_b$ and so ρ is a homomorphism. Moreover, by Corollary 3.9 (1), we have

$$(\rho_a)^* \in T_{(a^*a)^*,(aa^*)^*} = T_{a^*a^{**},a^{**}a^*} \ni \rho_{a^*}.$$

Observe that $(\rho_a)^* = \pi_{a^*a}\rho_a^{-1}\pi_{aa^*}$, it follows that

$$x(\rho_a)^* = x\pi_{a^*a}\rho_a^{-1}\pi_{aa^*} = (aa^*)^*a(a^*a)^*xa^*aa^*aa^* = a^{**}xa^* = x\rho_{a^*}$$

for all $x \in \mathrm{dom}\rho_{a^*}$. This implies that $(\rho_a)^* = \rho_{a^*}$. Thus, ρ preserves the unary operation "$*$".

On the other hand, we have

$$\ker\rho = \{(a,b) \in S \times S | \rho_a = \rho_b\}.$$

If $\rho_a = \rho_b$, then $\langle aa^*\rangle = \mathrm{dom}\rho_a = \mathrm{dom}\rho_b = \langle bb^*\rangle$ and $\langle a^*a\rangle = \mathrm{ran}\rho_a = \mathrm{ran}\rho_b = \langle b^*b\rangle$, this implies that $aa^* = bb^*$ and $a^*a = b^*b$. This shows that $a\mathcal{H}b$ by Corollary 2.9 (1). Thus $\ker\rho \subseteq \mathcal{H}$ and so $\ker\rho$ is idempotent-separating by Lemma 2.2. Now, let σ be an idempotent-separating congruence on S and $a\sigma b$. Since σ is idempotent-separating, we have $a^*\sigma b^*$ by Corollary 2.9 (2). This implies that $aa^*\sigma bb^*$ and $a^*a\rho b^*b$ whence $aa^* = bb^*$ and $a^*a = b^*b$. Therefore, $\mathrm{dom}\rho_a = \mathrm{dom}\rho_b$ and $\mathrm{ran}\rho_a = \mathrm{ran}\rho_b$. Moreover, for any $x \in \mathrm{dom}\rho_a$, we have $x\rho_a = a^*xa \in E(S)$ and $x\rho_b = b^*xb \in E(S)$. Observe that $a^*xa\sigma b^*xb$, it follows that $a^*xa = b^*xb$ since σ is idempotent-separating. This implies that $\rho_a = \rho_b$ and so $\sigma \subseteq \ker\rho$. Thus, $\ker\rho$ is the maximum idempotent-separating congruence.

(1) Obviously, $E(S)\rho \subseteq E(T_C)$. Since $\ker\rho$ is idempotent-separating, $\rho|_{E(S)}$ is injective. If $\alpha \in T_{e,f}$ and $\alpha \in E(T_C)$, then $\alpha = \alpha \circ \alpha = \alpha\pi_{e(fe)^*f}^{-1}\alpha$. By Lemma 3.3, $\alpha \circ \alpha \in T_{j,k}$ where $j = (fe(fe)^*f)\alpha^{-1}$ and $k = (e(fe)^*fe)\alpha$. This implies that $j = e$ and $k = f$ and so $e = e(fe)^*fe, f = fe(fe)^*f$. Since $(e(fe)^*f)^* = fe$ by Lemma 2.6, it follows that

$$\rho_{e(fe)^*f} \in T_{e(fe)^*fe,fe(fe)^*f} = T_{e,f} \ni \alpha.$$

Now, for any $x \in \langle e\rangle$, we have

$$x\alpha = x\alpha \circ \alpha = x\alpha\pi_{e(fe)^*f}^{-1}\alpha = (e(fe)^*f(x\alpha)(e(fe)^*f)^*)\alpha = (e(fe)^*f(x\alpha)fe)\alpha.$$

Since α is bijective, we have $x = e(fe)^*f(x\alpha)fe$. This implies that

$$x\rho_{e(fe)^*f} = (fe)x(e(fe)^*f) = fe(e(fe)^*f(x\alpha)fe)e(fe)^*f = f(x\alpha)f = x\alpha$$

by the facts $(e(fe)^*f)^* = fe, fe(fe)^*f = f$ and $x\alpha \in \langle f\rangle$. Thus, $\alpha = \rho_{e(fe)^*f}$. Since $e(fe)^*f \in E(S)$ by Lemma 2.6,

$$\alpha = \rho_{e(fe)^*f} = (e(fe)^*f)\rho \in E(S)\rho.$$

This shows that $\rho|_{E(S)}$ is surjective.

(2) If $a \in S^*$, then $aa^*, a^*a \in F_{C^*}$ and $\rho_a \in T_{aa^*,a^*a}$. This implies that $a\rho = \rho_a \in T_C^*$. Moreover, by Proposition 3.1 and Corollary 3.9 (3), the mapping

$$\rho|_{F_{S^*}} : F_{S^*} \to F_{T_C^*}, c \mapsto \rho_c = \pi_c = \iota_{\langle c \rangle}$$

is a bijection.

(3) This follows from item (1).

(4) This follows from Corollary 3.9 (2). □

Corollary 4.4. *Let C be a semi-band and $(C, {}^*) \in$ **qit**. Assume that $(S, {}^*) \in$ **qit** and $(\langle E(S) \rangle, {}^*)$ is unary isomorphic to $(C, {}^*)$. Then S is fundamental if and only if $(S, {}^*)$ is unary isomorphic to a subsemigroup (or a (2,1) subalgebra) $(U, {}^*)$ of T_C with $F_{U^*} = F_{T_C^*}$.*

Proof. If S is fundamental, then $\ker \rho$ in the above Theorem 4.3 is the identity congruence on S and so ρ is injective. Moreover, ρ is a unary isomorphism from $(S, {}^*)$ onto the subsemigroup (or a (2,1) subalgebra) $(U, {}^*) = (S\rho, {}^*)$ of $(T_C, {}^*)$. Obviously, $F_{U^*} = F_{T_C^*}$ by (2) of Theorem 4.3.

Conversely, let $(S, {}^*)$ be unary isomorphic to a subsemigroup (or a (2,1) subalgebra) $(U, {}^*)$ of T_C with $F_{U^*} = F_{T_C^*}$. Since $(S, {}^*) \in$ **qit**, it follows that $(U, {}^*) \in$ **qit**. By Theorem 3.10, $(U, {}^*)$ is fundamental. This implies that S is also fundamental. □

Acknowledgement: This study is initiated in the author's Ph.D. dissertation, and the author wishes to thank his supervisor Professor Yuqi Guo for his encouragement and help. The author also expresses his profound gratitude to the referees for the valuable comments, which improve the presentation of this article. Thanks also go to Professor K. P. Shum for the timely communications.

References

[1] Blyth, T. S.; McFadden, R. B., Regular semigroups with a multiplicative inverse transversal, Proc. Roy. Soc. Edinburgh 92(A), 1982: 253-270.

[2] Blyth, T. S.; Almeida Santos, M. H., On amenable orders and inverse transversals, Comm. Algebra 39(6), 2011: 2189-2209.

[3] Blyth, T. S.; Almeida Santos, M. H., $\mathcal{H}$-cohesive orders associated with inverse transversals, Comm. Algebra 40(8), 2012: 2771-2785.

[4] Howie, J. M., An Introduction to Semigroup Theory, Academic Press, London, 1976.

[5] Hall, T. E., On regular semigroups, J. Algebra 2, 1973: 1-24.

[6] Imaoka, T., On fundamental regular $*$-semigroups, Mem. Fac. Sci. Shimane Univ. 14, 1980: 19-23.

[7] Jones, P. R., A common framework for restriction semigroups and regular $*$-semigroups, Journal of Pure and Applied Algebra 216, 2012: 618-632.

[8] Kong, X. Z.; Shum, K. P., Regular cryptic super r-ample semigroups, Bull. Malays. Math. Sci. Soc. (2) 35, 2012: 859-868.

[9] Li, Y. H., A class of semigroups with regular $*$-transversals, Semigroup Forum 65, 2002: 43-57.

[10] Li, Y. H., On regular semigoups with a quasi-ideal regular $*$-transversal, Advances in Mathematics(China) 32 (6), 2003: 727-738.

[11] Li, Y. H.; Wang, S. F.; Zhang, R. H., Regular semigroups with regular $*$-transversals, J. Southwest China Normal University(Natural Science) 31(5), 2006: 52-56.

[12] Munn, W. D., Uniform semilattices and bisimple inverse semigroups, Quart. J. Math. Oxford Ser. 17, 1966: 151-159.

[13] Nordahl T. E.; Scheiblich H. E., Regular $*$-semigroups, Semigroup Forum 16, 1978: 369-377.

[14] Petrich, M.; Reilly, N. R., Completely regular semigroups, A Wiley-Interscience Publication, 1999.

[15] Song, G. T.; Zhu, F. L., Fundamental regular semigroups with inverse transversals, Semigroup Forum 67(1), 2003: 37-49.

[16] Tang, X. L., Regular semigroups with inverse transversals, Semigroup Forum 55(1), 1997: 24-32.

[17] Tang, X. L., Identities for a class of regular unary semigroups, Comm. Algebra 36, 2008: 2487-2502.

[18] Wang, S. F.; Liu, Y., On $\mathcal{P}$-regular semigroups having regular $*$-transversals, Semigroup Forum, 76(3), 2008: 561–575.

[19] Wang, S. F.; Zhang, D., Regular semigroups with regular $*$-transversals, Acta. Math. Sinica (Chinese Series) 54(4), 2011: 591–600.

[20] Wang, S. F., A classification of regular $*$-transversals, Advance in Mathematics (China) 41(5), 2012: 574–582.

[21] Yamada, M., $\mathcal{P}$-systems in regular semigroups, Semigroup Forum 24, 1982: 173-178.

[22] Yamada, M., On the structure of fundamental regular $*$-semigroups, Studia Sci. Math. Hungar. 16(3-4), 1981: 281-288.