QUALITATIVE UNCERTAINTY PRINCIPLES FOR THE GENERALIZED FOURIER TRANSFORM ASSOCIATED TO A CHEREDNIK TYPE OPERATOR ON THE REAL LINE

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Abstract. In this paper, we prove various mathematical aspects of the qualitative uncertainty principle, including Hardy's, Cowling-Price and its variants, Beurling and its variants, Gelfand-Shilov and Miyachi theorems, for the generalized Fourier transform associated to a Cherednik type operator on the real line.

1. Introduction

We consider the first order singular differential-difference operator on $\mathbb{R}$:

$$\Lambda f(x) = \frac{d}{dx}f(x) + \frac{A'(x)}{A(x)}(\frac{f(x)-f(-x)}{2}) - \rho f(x), \tag{1}$$

where

$$A(x) = |x|^{2k}B(x), \quad k > 0, \tag{2}$$

B being a positive C^∞ even function on $\mathbb{R}$, with $B(0) = 1$, and $\rho > 0$.
We suppose in addition that the function A satisfies the following conditions.

i) For all $x \geq 0, A(x)$ is increasing and $\lim_{x\to\infty} A(x) = \infty$.

ii) For all $x > 0$, $\frac{A'(x)}{A(x)}$ is decreasing and $\lim_{x\to\infty} \frac{A'(x)}{A(x)} = 2\rho$.

iii) There exists a constant $\delta > 0$ such that for all $x \in [x_0, \infty[, x_0 > 0$, we have

$$\frac{A'(x)}{A(x)} = 2\rho + e^{-\delta x}D(x),$$

where D is a C^∞-function, bounded together with its derivatives.

For

$$\begin{cases} A(x) = (\sinh|x|)^{2k}(\cosh x)^{2k'}, k \geq k' \geq 0, k \neq 0 \\ \rho = k + k', \end{cases} \tag{3}$$

we have the differential-difference operator

$$\Lambda_{k,k'} f(x) = \frac{d}{dx}f(x) + (k\coth(x) + k'\tanh(x))\{f(x) - f(-x)\} - \rho f(x), \tag{4}$$

which is refereed to as the Jacobi-Cherednik operator (see [11]).
This operator is more general than the Cherednik operator in the one dimensional case. Indeed for a root system $\mathcal{R}$ in $\mathbb{R}^d$, $\mathcal{R}_+$ a fixed positive subsystem and k a nonnegative multiplicity function defined on $\mathcal{R}$, the Cherednik operators $T_j, j = 1, 2, ..., d$, [5], are defined for f of class C^1 on $\mathbb{R}^d$ by

$$T_j f(x) = \frac{\partial}{\partial x_j}f(x) + \sum_{\alpha\in\mathcal{R}_+} \frac{k_\alpha \alpha_j}{1 - e^{-\langle\alpha,x\rangle}}\{f(x) - f(\sigma_\alpha(x))\} - \rho_j f(x), \tag{5}$$

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where $\langle .,. \rangle$ is the usual scalar produit, σ_α is the orthogonal reflection in the hyperplane orthogonal to $\alpha, \rho_j = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} k_\alpha \alpha_j$, and the function k is invariant by the finite reflection group W generated by the reflections $\sigma_\alpha, \alpha \in \mathcal{R}$.
For $d = 1$, the root systems are $\mathcal{R} = \{-\alpha, \alpha\}$ or $\mathcal{R} = \{-2\alpha, -\alpha, \alpha, 2\alpha\}$ with α the positive root. We take the normalization $\alpha = 2$.

- For $\mathcal{R}_+ = \{\alpha\}$, we have the Cherednik operator

$$T_1 f(x) = \frac{d}{dx} f(x) + \frac{2k_\alpha}{1 - e^{-2x}} \{f(x) - f(-x)\} - \rho f(x),$$

with $\rho = k_\alpha$. This operator can also be written in the form

$$T_1 f(x) = \frac{d}{dx} f(x) + k_\alpha \coth(x) \{f(x) - f(-x)\} - k_\alpha f(x), \tag{6}$$

which is of the form (4) with $k = k_\alpha$, and $k' = 0$

- For $\mathcal{R}_+ = \{\alpha, 2\alpha\}$, we have the Cherednik operator

$$T_1 f(x) = \frac{d}{dx} f(x) + \left(\frac{2k_\alpha}{1 - e^{-2x}} + \frac{4k_{2\alpha}}{1 - e^{-4x}} \right) \{f(x) - f(-x)\} - \rho f(x),$$

with $\rho = k_\alpha + 2k_{2\alpha}$. It is also equal to

$$T_1 f(x) = \frac{d}{dx} f(x) + ((k_\alpha + k_{2\alpha}) \coth(x) + k_{2\alpha} \tanh(x)) \{f(x) - f(-x)\} - \rho f(x). \tag{7}$$

This operator is therefore of the form (4) with $k = k_\alpha + k_{2\alpha}$, and $k' = k_{2\alpha}$.

- Another interesting case is $\mathcal{R} = \{-2\alpha, 2\alpha\}, \mathcal{R}_+ = \{2\alpha\}$, with the Cherednik operator

$$\begin{aligned} T_1 f(x) &= \frac{d}{dx} f(x) + \frac{4k_{2\alpha}}{1 - e^{-4x}} \{f(x) - f(-x)\} - \rho f(x), \\ &= \frac{d}{dx} f(x) + (k_{2\alpha} \coth(x) + k_{2\alpha} \tanh(x)) \{f(x) - f(-x)\} - \rho f(x). \end{aligned} \tag{8}$$

with $\rho = 2k_{2\alpha}$. This operator is also of the form (4) with $k = k' = k_{2\alpha}$.

The operators $T_j, j = 1, 2, ..., d$, have been used by Heckmann and Opdam to develop a theory generalizing the harmonic analysis on symmetric spaces (cf. [14, 22]). For recent important results in this direction we refer to [25].

In [20] the author provides a new harmonic analysis on the real line corresponding to the differential-difference operator Λ. In particular he has introduced the transmutation operators V and ${}^tV$ between the first derivative operator and the operator Λ. The operators V and ${}^tV$ are integral operators given for regular functions on $\mathbb{R}$, by

$$V g(x) = \begin{cases} \displaystyle\int_{-|x|}^{|x|} K(x,y) g(y) dy, & \text{if } x \neq 0, \\ g(0), & \text{if } x = 0, \end{cases}$$

$${}^tV f(y) = \int_{|x| \geq |y|} K(x,y) f(x) A(x) dx, \quad y \in \mathbb{R},$$

where $K(x,y)$ is a continuous function on $]-|x|, |x|[$, with support in $[-|x|, |x|]$, given by the relation (2.12) of [20].

In the case of the Jacobi-Cherednik operator (4), the operators V and ${}^tV$ have been defined and studied in [11].

Recently Trimèche in [28] has proved the positivity of the the transmutation operators V and ${}^tV$.

Classical uncertainty principles give us information about a function and its Fourier transform. If we try to limit the behavior of one we lose control of the other. Uncertainty principles have implications in two main areas: quantum physics and signal analysis. In quantum physics they tell us that a particles speed and position cannot both be measured with infinite precision. In signal analysis they tell us that if we observe a signal only for a finite period of time, we will lose information about the frequencies the signal consists of. The mathematical equivalent is that a function and its Fourier transform cannot both be arbitrarily localized. There is two categories of uncertainty principles: Quantitative uncertainty principles and Qualitative uncertainty principles.

Quantitative uncertainty principles is just another name for some special inequalities. These inequalities give us information about how a function and its Fourier transform relate. They are called uncertainty principles since they are similar to the classical Heisenberg Uncertainty Principle, which has had a big part to play in the development and understanding of quantum physics. For example: Benedicks [2], Slepian and Pollak [26], Slepian[27], and Donoho and Stark [8] paid attention to the supports of functions and gave quantitative uncertainty principles for the Fourier transforms.

Qualitative uncertainty principles are not inequalities, but are theorems that tell us how a function (and its Fourier transform) behave under certain circumstances. For example: Hardy [13], Morgan [19], Cowling and Price [6], Beurling [3], Miyachi [18] theorems enter within the framework of the qualitative uncertainty principles.

The quantitative and qualitative uncertainty principles has been studied by many authors for various Fourier transforms, for examples (cf. [11, 16, 17, 29]) and others.

In this paper, we prove Hardy's theorem, Cowling-Price's theorem, Ray-Sarkar's theorem, Miyachi's theorem, Beurling's theorem and Gelfand-Shilov's theorem for the generalized Fourier transform associated to the Cherednik type operator on the real line. We note that Mourou in [21] has studied only partial version of Hardy's and Cowling-Price's theorems to the Cherednik type operator on the real line. The method used is based on the relation between the generalized Fourier transform and the classical Fourier transform on $\mathbb{R}$, and on the positivity of the transmutation operators relating to the Cherednik type operator on $\mathbb{R}$. This method allows to give the analogue of the uncertainty principles within the framework of the generalized Fourier transform, contrary to the method of the Mourou, which is based on the estimations of the eigenfunction of the operator Λ, and the generalized heat kernel, which give only partial versions of Hardy's and Cowling-Price's theorem.

The remaining part of the paper is organized as follows. In §2, we recall the main results about the Cherednik type operator on the real line. In §3 we prove an L^p version of Hardy's theorem for the generalized Fourier transform. §4 is devoted to generalize Cowling-Price's theorem for the generalized Fourier transform $\mathcal{F}$. §5 is devoted to obtain Beurling's theorem for $\mathcal{F}$ and in §6 we generalize Miyachi's theorem.

2. Preliminaries

This section gives an introduction to the harmonic analysis associated with the Cherednik type operator. Main references are [20, 28].

2.1. **The eigenfunction of the operator Λ.**

Notations. We denote by

$\mathcal{P}_m(\mathbb{R})$	the set of homogeneous polynomials of degree m .
$C(\mathbb{R})$	the space of continuous functions on $\mathbb{R}$.
$C_c(\mathbb{R})$	the space of continuous functions on $\mathbb{R}$ with compact support.
$C^p(\mathbb{R})$	the space of functions of class C^p on $\mathbb{R}$.
$C^p_b(\mathbb{R})$	the space of bounded functions of class C^p.
$\mathcal{E}(\mathbb{R})$	the space of C^∞-functions on $\mathbb{R}$.
$\mathcal{S}(\mathbb{R})$	the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.
$D(\mathbb{R})$	the space of C^∞-functions on $\mathbb{R}$ which are of compact support.
$\mathcal{S}^2(\mathbb{R})$	$:= (\cosh x)^{-\rho} S(\mathbb{R})$, the generalized Schwartz space.
$PW(\mathbb{C})$	the space of entire functions on $\mathbb{C}$, rapidly decreasing and of exponential type.

To present the eigenfunction $\Phi_\lambda, \lambda \in \mathbb{C}$, of Λ satisfying the condition $\Phi_\lambda(0) = 1$, we consider first the eigenfunction $\varphi_\lambda, \lambda \in \mathbb{C}$, of the second order singular differential operator Δ on $]0, \infty[$:

$$\Delta = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)}\frac{d}{dx}.$$

The function φ_λ, $\lambda \in \mathbb{C}$, is the unique analytic solution of the differential equation

$$\begin{cases} \Delta u(x) = -(\lambda^2 + \rho^2)u(x), \\ u(0) = 1, u'(0) = 0. \end{cases}$$

We denote also by φ_λ the even function on $\mathbb{R}$ equal to φ_λ on $[0, \infty[$.

For every $\lambda \in \mathbb{C}$, let us denote by Φ_λ the unique solution of the equation

$$\begin{cases} \Lambda f(x) = i\lambda f(x), \\ f(0) = 1. \end{cases} \tag{2.1}$$

It is given for all $\lambda \in \mathbb{C}$, by

$$\forall\, x \in \mathbb{R}, \Phi_\lambda(x) = \begin{cases} \varphi_\lambda(x) + \frac{1}{i\lambda - \rho}\frac{d}{dx}\varphi_\lambda(x), & \text{if } \lambda \neq i\rho, \\ 1 + \frac{2\rho}{A(x)}\int_0^x A(t)dt, & \text{if } \lambda = i\rho. \end{cases}$$

For $\lambda \neq -i\rho$, we can write it in the form

$$\forall\, x \in \mathbb{R}, \Phi_\lambda(x) = \varphi_\lambda(x) + sgn(x)\,\frac{i\lambda + \rho}{A(x)}\int_0^{|x|}\varphi_\lambda(z)A(z)dz.$$

It possesses the following properties

i) For every $x \in \mathbb{R}$, the function $\lambda \to \Phi_\lambda(x)$ is entire on $\mathbb{C}$.

ii) There exists a positive constant M such that

$$\forall\, x \in \mathbb{R}, \;\; \forall\, \lambda \in \mathbb{R}, \; |\Phi_\lambda(x)| \leq M(1 + |x|)(1 + \sqrt{\lambda^2 + \rho^2})e^{-\rho|x|}.$$

iii) For all $x \in \mathbb{R}\backslash\{0\}$ and $\lambda \in \mathbb{C}$, the function $\Phi_\lambda(x)$ admits the Laplace type integral representation

$$\Phi_\lambda(x) = \int_{-|x|}^{|x|} K(x, y)e^{i\lambda y}dy, \tag{2.2}$$

where $K(x, .)$ is a continuous function on $] - |x|, |x|[$, with support in $[-|x|, |x|]$, given by the relation (9).

Example 1. *The Laplace type integral representation of the Φ_λ corresponding to the Jacobi-Cherednik operator (4), has been obtained in* [11], *and it is of the form (2.2) with $K(x, y)$ possessing the expressions in the three following cases*

- If $k' = 0, k > 0$, we have

$$K(x, y) = \frac{2^{k-1}\Gamma(k + \frac{1}{2})}{\sqrt{\pi}\Gamma(k)}(\sinh|x|)^{-2k}(\cosh x - \cosh y)^{k-1}sgn(x)(e^x - e^{-y}),$$

for all $x \in \mathbb{R}\backslash\{0\}$ and $-|x| \leq y \leq |x|$, and this function is positive.

- If $k' = k > 0$, we have

$$K(x, y) = \frac{2^k\Gamma(k + \frac{1}{2})}{\sqrt{\pi}\Gamma(k)}(\sinh 2|x|))^{-2k}(\cosh(2x) - \cosh(2y))^{k-1}$$
$$\times sgn(x)(e^{2x} - e^{-2y}),$$

for all $x \in \mathbb{R}\backslash\{0\}$, and $-|x| \leq y \leq |x|$, and this function is positive.

- If $k > k' > 0$, the function $K(x, y)$ is given by the relation (2.54) of [11] *p. 178. Its expression does not show that it is positive.*

Proposition 1. *([20]). Let p be polynomial of degree m. Then there exists a positive constant C such that for all $\lambda \in \mathbb{C}$ and for all $x \in \mathbb{R}$, we have*

$$|p(\frac{\partial}{\partial\lambda})\Phi_\lambda(x)| \leq C(1+|\lambda|))(1+|x|)^{m+2}e^{(|Im\lambda|-\varrho)|x|}. \tag{2.3}$$

2.2. **Generalized Fourier transform.** For a Borel positive measure μ on $\mathbb{R}$, and $1 \leq p \leq \infty$, we write $L^p_\mu(\mathbb{R})$ for the Lebesgue space equipped with the norm $\|\cdot\|_{L^p_\mu(\mathbb{R})}$ defined by

$$\|f\|_{L^p_\mu(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p \, d\mu(x)\right)^{1/p}, \quad \text{if } p < \infty,$$

and $\|f\|_{L^\infty_\mu(\mathbb{R})} =$ ess $\sup_{x\in\mathbb{R}}|f(x)|$. When $\mu(x) = w(x)dx$, with w a nonnegative function on $\mathbb{R}$, we replace the μ in the norms by w.

For $f \in C_c(\mathbb{R})$, the generalized Fourier transform is defined by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)\Phi_\lambda(x)A(x)dx, \quad \text{for all } \lambda \in \mathbb{C}. \tag{2.4}$$

Remark 1. *For $\lambda \in \mathbb{C}$ and $g \in C_c(\mathbb{R})$, we have*

$$\mathcal{F}(g)(\lambda) = \mathcal{F}_\Delta(g_e)(\lambda) + (-\varrho + i\lambda)\mathcal{F}_\Delta(\mathcal{I}g_o)(\lambda), \tag{2.5}$$

where $\mathcal{F}_\Delta$ denotes the stands for the Fourier transform related to the differential operator Δ, g_e (resp. g_o) denotes the even (resp. odd) part of g, and

$$\mathcal{I}g_o(x) = \int_{-\infty}^{x} g_o(t)dt.$$

Theorem 1. *For all $f \in D(\mathbb{R})$,*

$$\mathcal{F}^{-1}f(x) = \int_{\mathbb{R}} f(\lambda)\Phi_{-\lambda}(x)d\sigma(\lambda), \tag{2.6}$$

where

$$d\sigma(\lambda) = (1 - \frac{i\varrho}{\lambda})\frac{d\lambda}{|c(|\lambda|)|^2}, \tag{2.7}$$

with c is a continuous function on $(0,\infty)$ such that

$$c(s)^{-2} \sim \begin{cases} C_1 s^{2k} & as \quad s \to \infty \\ C_2 s^2 & as \quad s \to 0, \end{cases} \tag{2.8}$$

for some $C_1, C_2 \in \mathbb{C}$.

Remark 2. *For $A(x) = (\sinh|x|)^{2k}(\cosh x)^{2k'}$, $k \geq k' > 0$, we have*

$$d\sigma(\lambda) = \frac{d\lambda}{|c(\lambda)|^2}$$

where

$$c(\lambda) := \frac{2^{\rho - i\lambda}\Gamma(k+\frac{1}{2})\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(k-k'+1+i\lambda))}, \quad \lambda \in \mathbb{C}\backslash i\mathbb{N}.$$

Next, we give some properties of this transform.

i) For f in $L^1_A(\mathbb{R})$ we have

$$\forall\, \lambda \in \mathbb{R}, \quad |\mathcal{F}(f)(\lambda)| \leq C(1+|\lambda|)||f||_{L^1_A(\mathbb{R})}, \tag{2.9}$$

ii) For f in $\mathcal{S}^2(\mathbb{R})$ we have

$$\mathcal{F}(\mathcal{L}_A f)(y) = -y^2\mathcal{F}(f)(y), \quad \text{for all } y \in \mathbb{R}, \tag{2.10}$$

where $\mathcal{L}_A$ is the generalized Laplace operator on $\mathbb{R}$ given by

$$\mathcal{L}_A f(x) := \Lambda^2 f(x) \tag{2.11}$$

Proposition 2. *([20]).* i*)* **Plancherel formula for** $\mathcal{F}$. *For all* f, g *in* $\mathcal{S}^2(\mathbb{R})$ *we have*

$$\int_{\mathbb{R}} f(x)g(-x)A(x)\,dx = \int_{\mathbb{R}} \mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi)d\sigma(\xi). \tag{2.12}$$

Theorem 2. *([25]). The generalized Fourier transform* $\mathcal{F}$ *is a topological isomorphism from*

i) $D(\mathbb{R})$ *onto* $PW(\mathbb{C})$.

ii) $\mathcal{S}^2(\mathbb{R})$ *onto* $\mathcal{S}(\mathbb{R})$.

2.3. **Transmutation operators associated with the operators** Λ**.** The generalized intertwining operator is the operator V defined on $\mathcal{E}(\mathbb{R})$ by

$$V f(x) = \begin{cases} \displaystyle\int_{-|x|}^{|x|} K(x,y)f(y)\,dy & \text{if} \quad x \in \mathbb{R}\setminus\{0\}, \\ f(0) & \text{if} \quad x = 0. \end{cases} \tag{2.13}$$

We have

$$\forall \lambda \in \mathbb{C},\ \forall x \in \mathbb{R}, \quad \Phi_\lambda(x) = V(e^{i\lambda\cdot})(x). \tag{2.14}$$

The operator V is a topological automorphism of $\mathcal{E}(\mathbb{R})$ satisfying

$$\forall f \in \mathcal{E}(\mathbb{R}), \quad \Lambda(V f)(x) = V\left(\frac{d}{dy}f\right)(x). \tag{2.15}$$

The operator ${}^tV$ is defined on $D(\mathbb{R})$ by

$$\forall y \in \mathbb{R}, \quad {}^tV(f)(y) = \int_{|x|\geq|y|} K(x,y)f(x)A(x)\,dx. \tag{2.16}$$

The operator ${}^tV$ is a topological automorphism of $D(\mathbb{R})$ satisfying

$$\forall f \in D(\mathbb{R}),\ \forall y \in \mathbb{R}, \quad \frac{d}{dy}{}^tV(f)(y) = {}^tV(\Lambda + 2\rho S)(f)(y), \tag{2.17}$$

where S is the operator defined by

$$\forall\, x \in \mathbb{R}, S(f)(x) = f(-x), \quad f \in D(\mathbb{R}).$$

The operators V and ${}^tV$ possess the following properties:

For all $f \in D(\mathbb{R})$ and $g \in \mathcal{E}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} {}^tV(f)(y)g(y)dy = \int_{\mathbb{R}} f(x)V g(x)A(x)dx. \tag{2.18}$$

Proposition 3. *([20]). For all* $f \in D(\mathbb{R})$ *we have*

$$\mathcal{F}(f) = \mathcal{F}_c \circ {}^tV(f), \tag{2.19}$$

where $\mathcal{F}_c$ *is the classical Fourier transform defined on* $D(\mathbb{R})$ *by*

$$\forall\, \lambda \in \mathbb{C}, \quad \mathcal{F}_c(f)(\lambda) = \int_{\mathbb{R}} f(x)e^{-i\lambda x}dx.$$

Proposition 4. *Let* $f \in L^1_A(\mathbb{R})$*. For almost all* y*, the function*

$$y \longmapsto {}^tV(f)(y) = \int_{|x|\geq|y|} K(x,y)f(x)A(x)\,dx, \tag{2.20}$$

is defined almost everywhere on $\mathbb{R}$ *and belongs to* $L^1(\mathbb{R})$*. Moreover, for all bounded continuous function* g *on* $\mathbb{R}$*, we have the following formula :*

$$\int_{\mathbb{R}} {}^tV(f)(y)g(y)dy = \int_{\mathbb{R}} f(x)V g(x)A(x)dx. \tag{2.21}$$

Proof. The functions $(x,y) \longmapsto K(x,y)f(x)A(x)$ and $(x,y) \longmapsto K(x,y)f(x)g(y)A(x)$ are Lebesgue integrable on $\mathbb{R}^2$. Then by using Fubini's theorem, we get the result. □

Proposition 5. *([28]). The generalized intertwining operator* V *and its dual* ${}^tV$ *are positive.*

2.4. **The generalized heat kernel.**

Definition 1. *Let $t > 0$. The heat kernel E_t associated with the operator Λ is defined by*

$$\forall x \in \mathbb{R}, \quad E_t(x) = \mathcal{F}^{-1}(e^{-t\lambda^2})(x). \tag{2.22}$$

Remark 3. *As the function $\lambda \mapsto e^{-t\lambda^2}$ is an even function on $\mathbb{R}$, then from the relation (2.5), we deduce that*

$$\forall x \in \mathbb{R}, \quad E_t(x) = \frac{1}{2}\mathcal{F}_\Delta^{-1}(e^{-t\lambda^2})(x). \tag{2.23}$$

We introduce also the generalized heat functions $N_n(t,.)$, $n \in \mathbb{N}$ are defined on $\mathbb{R}$ by

$$N_n(t,x) = (-i)^n \int_{\mathbb{R}} \lambda^n e^{-t\lambda^2}\Phi_\lambda(x)d\sigma(\lambda). \tag{2.24}$$

These functions satisfies the following properties.
i) For all $t > 0$, $N_n(t,.)$ is an C^∞-function on $\mathbb{R}$.
ii) For all $t > 0$, $N_0(t,.) = E_t > 0$.
iii) For all $t > 0$, $\|E_t\|_{L^1_A(\mathbb{R})} = 1$.
iv) For all $t > 0$, $\forall \lambda \in \mathbb{R}$, $\mathcal{F}\Big(N_n(t,.)\Big)(\lambda) = (-i)^n\lambda^n e^{-t\lambda^2}$.
v) For all $t > 0$ and $\forall x \in \mathbb{R}$, $\quad \mathcal{L}_A N_n(t,x) = \frac{\partial}{\partial t}N_n(t,x)$.

Proposition 6. *Let $t > 0$. We have*

$$\forall y \in \mathbb{R}, \quad {}^tV(E_t)(y) = \frac{1}{2\sqrt{\pi t}}e^{-\frac{y^2}{4t}}. \tag{2.25}$$

Proof. From the relations (2.22) and (2.19), we have

$$\forall y \in \mathbb{R}, \quad {}^tV(E_t)(y) = \mathcal{F}_c^{-1}(e^{-t\lambda^2})(y) = \frac{1}{2\sqrt{\pi t}}e^{-\frac{y^2}{4t}}.$$

□

Proposition 7. *Let $p \in [1,\infty)$. There exists a positive constant $C(p,t)$ such that*

$$\forall x \in \mathbb{R}, \quad (E_t(x))^p \le C(p,t)E_{\frac{t}{p}}(x). \tag{2.26}$$

Proof. From [10], p. 251, there exists $C_1(t) > 0$ and $C_2(t) > 0$ such that

$$\forall x \in \mathbb{R}, \quad C_1(t)\frac{e^{-\frac{x^2}{4t}}}{\sqrt{B(x)}} \le E_t(x) \le C_2(t)\frac{e^{-\frac{x^2}{4t}}}{\sqrt{B(x)}}. \tag{2.27}$$

Using the hypothesis on the function A, there exist $C > 0$ such that for all $x \in \mathbb{R}$, $B(x) \ge C$. Thus, according (2.27), we obtain (2.26). □

3. An L^p version of Hardy's theorem

We denote by $L^p_\nu(\mathbb{R}), 1 \le p \le \infty$, the space of measurable functions on $\mathbb{R}$, satisfying

$$\begin{aligned}
\|f\|_{L^p_\nu(\mathbb{R})} &= \left(\int_{\mathbb{R}} |f(x)|^p d|\sigma|(x)\right)^{1/p} < \infty, \quad 1 \le p < \infty,\\
\|f\|_{L^\infty_\nu(\mathbb{R})} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| < \infty, \quad p = \infty.
\end{aligned}$$

Proposition 8. *Let $p \in [1,\infty]$ and f a measurable function on $\mathbb{R}$ such that $\Big(E_{\frac{1}{4a}}\Big)^{-1} f$ belongs to $L^p_A(\mathbb{R})$ for some $a > 0$. Then*

$$e^{ay^2}\Big({}^tV(f)\Big) \in L^p(\mathbb{R}).$$

Proof. We consider two cases.
1^{st} case : If $p \in [1, \infty[$, from (2.16), we have

$$\|e^{ay^2}\left({}^tV\left(f\right)\right)\|^p_{L^p(\mathbb{R})} \leq \int_{\mathbb{R}} e^{apy^2}\left(\int_{|x|\geq|y|} K(x,y)\left[\left(E_{\frac{1}{4a}}\right)^{-1}(x)|f(x)|\right]E_{\frac{1}{4a}}(x)A\left(x\right)dx\right)^p dy.$$

By applying Hölder's inequality to the middle integral, we obtain

$$\|e^{ay^2}\left({}^tV\left(f\right)\right)\|^p_{L^p(\mathbb{R})} \leq \int_{\mathbb{R}} e^{apy^2}\,{}^tV\left(|\left(E_{\frac{1}{4a}}\right)^{-1}f|^p\right)(y)\left[{}^tV\left[\left(E_{\frac{1}{4a}}\right)^{p'}\right](y)\right]^{\frac{p}{p'}} dy,$$

where p' is the conjugate exponent of p. By the relations (2.26), (2.25), and (2.21), we deduce that

$$\|e^{ay^2}\left({}^tV\left(f\right)\right)\|_{L^p(\mathbb{R})} \leq M\|\left(E_{\frac{1}{4a}}\right)^{-1}f\|_{L^p_A(\mathbb{R})} < \infty,$$

where $M = \left(C(p', \frac{1}{4a})\sqrt{\frac{p'a}{\pi}}\right)^{\frac{1}{p'}}$.

2^{nd} case : If $p = \infty$, using (2.16), we obtain for almost all y in $\mathbb{R}$:

$$\begin{aligned}|{}^tV\left(f\right)(y)| &\leq \int_{|x|\geq|y|} K(x,y)\left(\left(E_{\frac{1}{4a}}\right)^{-1}(x)|f(x)|\right)E_{\frac{1}{4a}}(x)A\left(x\right)dx\\ &\leq \|\left(E_{\frac{1}{4a}}\right)^{-1}f\|_{L^\infty_A(\mathbb{R})}{}^tV(E_{\frac{1}{4a}})(y).\end{aligned}$$

By the relation (2.25), we deduce that

$$\|e^{ay^2}\,{}^tV\left(f\right)(y)\|_{L^\infty_A(\mathbb{R})} \leq M_0\|\left(E_{\frac{1}{4a}}\right)^{-1}f\|_{L^\infty_A(\mathbb{R})} < \infty,$$

where $M_0 = \sqrt{\frac{a}{\pi}}$. This completes the proof. ☐

Proposition 9. *Let $p \in [1, \infty]$ and f a measurable function on $\mathbb{R}$ such that $\left(E_{\frac{1}{4a}}\right)^{-1}f$ belongs to $L^p_A(\mathbb{R})$ for some $a > 0$. Then the function $\mathcal{F}\left(f\right)$ given for all $\lambda \in \mathbb{C}$ by*

$$\mathcal{F}\left(f\right)(\lambda) = \int_{\mathbb{R}} f(x)\Phi_\lambda(x)A\left(x\right)dx,$$

is well defined, entire on $\mathbb{C}$, and there exists a positive constant C such that

$$\forall \xi,\ \eta \in \mathbb{R},\quad |\mathcal{F}\left(f\right)(\xi + i\eta)| \leq Ce^{\frac{\eta^2}{4a}}. \tag{3.28}$$

Proof. The first assertion follows from Hölder's inequality, the relation (2.3), and the derivation theorem under the integral sign. As the function f belongs to $L^1_A(\mathbb{R})$, we deduce from the relation (2.19) that for all $\xi,\ \eta \in \mathbb{R}$, we have

$$|\mathcal{F}\left(f\right)(\xi + i\eta)| \leq e^{\frac{\eta^2}{4a}}\int_{\mathbb{R}} e^{ay^2}|{}^tV\left(f\right)(y)|e^{-a\left(y-\frac{\eta}{2a}\right)^2}dy.$$

Using Proposition 8 and Hölder's inequality, we obtain (3.28) with

$$C = \left(\frac{\pi}{ap'}\right)^{\frac{1}{2p'}}\|e^{ay^2}\left({}^tV\left(f\right)\right)\|_{L^p(\mathbb{R})},$$

where p' is the conjugate exponent of p. ☐

Theorem 3. *Let f be a measurable function on $\mathbb{R}$ such that*

$$\left(E_{\frac{1}{4a}}\right)^{-1}f \in L^p_A(\mathbb{R}) \text{ and } e^{b\lambda^2}\mathcal{F}\left(f\right) \in L^q_\nu(\mathbb{R}), \tag{3.29}$$

for some constants $a, b > 0$, $1 \leq p, q \leq \infty$, and at least one of p and q is finite. Then

- *If $ab \geq \frac{1}{4}$, we have $f = 0$, almost everywhere.*

- *If $ab < \frac{1}{4}$, for all $t \in]b, \frac{1}{4a}[$, the functions $f = E_t$, satisfy the relations (3.29).*

For prove this theorem we need the following lemmas.

Lemma 1. *Let h be an entire function on $\mathbb{C}$ such that*

$$\forall z \in \mathbb{C}, \quad |h(z)| \le Ce^{a(Rez)^2} \tag{3.30}$$

and

$$\forall x \in \mathbb{R}, \quad |h(x)| \le C, \tag{3.31}$$

for some $a, C > 0$. Then h is constant on $\mathbb{C}$.

Lemma 2. *Let $q \in [1, \infty[$ and h an entire function on $\mathbb{C}$ such that*

$$\forall z \in \mathbb{C}, \quad |h(z)| \le Me^{a(Rez)^2} \tag{3.32}$$

and

$$\|h_{|\mathbb{R}}\|_{L^q_\nu(\mathbb{R})} < \infty, \tag{3.33}$$

for some $a, M > 0$. Then $h \equiv 0$.

Proof. of Theorem 3. We will divide the proof in several steps.
1^{st} step : If $ab > \frac{1}{4}$. We consider the function h defined on $\mathbb{C}$ by

$$h(\lambda) = e^{\frac{\lambda^2}{4a}} \mathcal{F}(f)(\lambda).$$

From Proposition 9, there exists a positive constant C such that for all $\xi, \ \eta \in \mathbb{R}$, we have $|h(\xi + i\eta)| \le Ce^{\frac{\xi^2}{4a}}$.

i) If $q < \infty$, we have

$$\|h_{|\mathbb{R}}\|^q_{L^q_\nu(\mathbb{R})} = \int_{\mathbb{R}} |e^{b\lambda^2} \mathcal{F}(f)(\lambda)|^q e^{q(\frac{1}{4a} - b)\lambda^2} \, d\nu(\lambda).$$

The inequality $ab > \frac{1}{4}$ implies

$$\|h_{|\mathbb{R}}\|_{L^q_\nu(\mathbb{R})} \le \|e^{b\lambda^2} \mathcal{F}(f)\|_{L^q_\nu(\mathbb{R})} < \infty.$$

We deduce from Lemma 2 that for all $\lambda \in \mathbb{C}$, $h(\lambda) = 0$.
It follows that for all $\lambda \in \mathbb{R}, \quad \mathcal{F}(f)(\lambda) = 0$ and then from the injectivity of the transform $\mathcal{F}$, we have

$$f = 0, \ a.e., \ \text{on } \mathbb{R}.$$

ii) If $q = \infty$, we have

$$\|h_{|\mathbb{R}}\|_{L^\infty_\nu(\mathbb{R})} = \|e^{b\lambda^2} \mathcal{F}(f) \ e^{(\frac{1}{4a} - b)\lambda^2}\|_{L^\infty_\nu(\mathbb{R})} \le \|e^{b\lambda^2} \mathcal{F}(f)\|_{L^\infty_\nu(\mathbb{R})} < \infty.$$

From Lemma 1, there exists a constant K such that for all $\lambda \in \mathbb{C}$, $h(\lambda) = K$.
It follows that for all $\lambda \in \mathbb{R}, \quad \mathcal{F}(f)(\lambda) = Ke^{-\frac{\lambda^2}{4a}}$. The assumption on $\mathcal{F}(f)$ is expressed as

$$|\mathcal{F}(f)(\lambda)| \le Me^{-b\lambda^2}, \ a.e. \ \lambda \in \mathbb{R},$$

for some constant $M > 0$.
The continuity of $\mathcal{F}(f)$ on $\mathbb{R}$ shows that for all $\lambda \in \mathbb{R}, \quad |\mathcal{F}(f)(\lambda)| \le Me^{-b\lambda^2}$. Then for all $\lambda \in \mathbb{R}, \quad |K| \le Me^{(\frac{1}{4a} - b)\lambda^2}$. It follows from the inequality $ab > \frac{1}{4}$, that $K = 0$. Therefore

$$f = 0, \ a.e., \ \text{on } \mathbb{R}.$$

2^{nd} step : If $ab = \frac{1}{4}$, we have

i) If $q < \infty$. With the same proof as for the point *i)* of the first step, we deduce that

$$f = 0, \ a.e., \ \text{on } \mathbb{R}.$$

ii) If $q=\infty$. Proposition 8 and the relation (2.19) imply that the function ${}^tV(f)$ satisfies

$$e^{ay^2}\Big({}^tV(f)\Big) \in L^p(\mathbb{R}) \textit{ and } e^{b\lambda^2}\mathcal{F}_c\Big({}^tV(f)\Big) \in L^\infty(\mathbb{R}).$$

Then using [9], p. 66, we see that ${}^tV(f)=0$, *a.e.*, on $\mathbb{R}$. From (2.19), it follows that $\mathcal{F}(f)=0$ on $\mathbb{R}$ and then

$$f=0,\ a.e.,\ \text{on } \mathbb{R}.$$

3^{rd} step : If $ab<\frac{1}{4}$. Let $t\in]b,\frac{1}{4a}[$ and $f=E_t$. From the relation (2.27), we get

$$\forall x\in\mathbb{R},\quad K_1e^{-(\frac{1}{4t}-a)x^2}\leq\Big(E_{\frac{1}{4a}}\Big)^{-1}(x)f(x)\leq K_2e^{-(\frac{1}{4t}-a)x^2},$$

for some constants $K_1,\ K_2>0$. As $t<\frac{1}{4a}$, we deduce that $\Big(E_{\frac{1}{4a}}\Big)^{-1}f\in L^p_A(\mathbb{R})$. Using the relation (2.22), we get

$$\forall\lambda\in\mathbb{R},\quad e^{b\lambda^2}\mathcal{F}(f)(\lambda)=e^{-(t-b)\lambda^2}.$$

The condition $t>b$ and the relations (2.8), imply that $e^{b\lambda^2}\mathcal{F}(f)\in L^q_\nu(\mathbb{R})$. This completes the proof of the theorem. □

We determine, in this section, the functions f satisfying the relations (3.29) in the special case $p=q=\infty$. The result obtained for the Jacobi-Cherednik transform is an analogue of the classical Hardy's theorem.

Theorem 4. *Let f be a measurable function on $\mathbb{R}$ such that*

$$|f(x)|\leq ME_{\frac{1}{4a}}(x),\ a.e.\ x\in\mathbb{R}\ \text{and}\ |\mathcal{F}(f)(\lambda)|\leq Me^{-b\lambda^2},\ \text{for all } \lambda\in\mathbb{R}, \tag{3.34}$$

for some constants $a,b,M>0$. Then

- *If $ab>\frac{1}{4}$, we have $f=0$, almost everywhere.*
- *If $ab=\frac{1}{4}$, the function f is of the form $f=C_0E_{\frac{1}{4a}}$, for some real constant C_0.*
- *If $ab<\frac{1}{4}$, there are infinitely many nonzero functions f satisfying the conditions (3.34).*

Proof. 1^{st} step : If $ab>\frac{1}{4}$, the point *ii)* of the first step of the proof of Theorem 3 gives the result.
2^{nd} step : If $ab=\frac{1}{4}$, we deduce from the relations (2.25) and (2.19) that the function ${}^tV(f)$ satisfies

$$|{}^tV(f)(y)|\leq M_0e^{-ay^2},\ a.e.\ y\in\mathbb{R}\ \text{and}\ |\mathcal{F}_c\Big({}^tV(f)\Big)(\lambda)|\leq M_0e^{-b\lambda^2},\ \text{for all } \lambda\in\mathbb{R},$$

for some constant $M_0>0$. Using Hardy's theorem for the usual Fourier transform (see [13], p. 137), we obtain

$${}^tV(f)(y)=M_1e^{-ay^2},\ a.e.\ y\in\mathbb{R},$$

where M_1 is a real constant. From the relation (2.19), it follows that $\mathcal{F}(f)(\lambda)=M_2e^{-\frac{\lambda^2}{4a}}$, for all $\lambda\in\mathbb{R}$, where M_2 is a real constant. We deduce from the relation (2.22), that

$$f=C_0E_{\frac{1}{4a}},$$

for some real constant C_0.
3^{rd} step : If $ab<\frac{1}{4}$, the functions $f=E_t,\ t\in]b,\frac{1}{4a}[$, satisfy the conditions (3.34). This completes the proof of the theorem. □

4. Generalized Cowling-Price theorem for the generalized Fourier transform

Theorem 5. *Let f be a measurable function on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} \frac{\left(E_{\frac{1}{4a}}(x)\right)^{-p}|f(x)|^p}{(1+|x|)^n} A(x)dx < \infty \tag{4.35}$$

and

$$\int_{\mathbb{R}} \frac{e^{bq\xi^2}|\mathcal{F}(f)(\xi)|^q}{(1+|\xi|)^m} d\xi < \infty, \tag{4.36}$$

for some constants $a, b, n > 0$, $m > 1$ and $1 \leq p \leq 2$, $1 \leq q < \infty$. Then

i) *If $ab > \frac{1}{4}$, we have $f = 0$ almost everywhere.*

ii) *If $ab = \frac{1}{4}$, then f is of the form $f = \sum_{j=0}^{d} C_j N_j(b,.)$ where $d \leq \min(\frac{n}{p} + \frac{\gamma}{p'}, \frac{m-1}{q})$, where p' is the conjugate of p. Especially, if*

$$n \leq 1 + 2\rho + p \min(\frac{n}{p} + \frac{\gamma}{p'}, \frac{m-1}{q}),$$

then $f = 0$ almost everywhere. Furthermore, if $n > 2\rho + 1$ and $m \in (1, q+1]$, then f is a constant multiple of E_b.

iii) *If $ab < \frac{1}{4}$, for all $\delta \in (b, \frac{1}{4a})$, the functions of the form $f = \sum_{j=0}^{d} C_j N_j(\delta,.)$, $d \in \mathbb{N}$, satisfy (4.35) and (4.36).*

Proof. We shall show that $\mathcal{F}(f)(z)$ exists and is an entire function in $z \in \mathbb{C}$ and

$$|\mathcal{F}(f)(z)| \leq Ce^{\frac{1}{4a}|\mathrm{Im} z|^2}(1 + |Im\, z|)^s, \quad \text{for all}\, z \in \mathbb{C}, \quad \text{for some} \quad s > 0. \tag{4.37}$$

The first assertion follows from the hypothesis on the function f and Hölder's inequality using (4.35) and the derivation theorem under the integral sign. We want to prove (4.37). Actually, it follows from (2.4) and (2.3) that for all $z = \xi + i\eta \in \mathbb{C}$,

$$\begin{aligned}
|\mathcal{F}(f)(\xi + i\eta)| &\leq \int_{\mathbb{R}} |f(x)||\Phi_{(\xi+i\eta)}(x)|A(x)dx \\
&\leq \int_{\mathbb{R}} \frac{\left(E_{\frac{1}{4a}}(x)\right)^{-1}|f(x)|}{(1+|x|)^{\frac{n}{p}}}(1+|x|)^{\frac{n}{p}} E_{\frac{1}{4a}}(x)e^{(|\eta|-\varrho)|x|}A(x)dx \\
&\leq e^{\frac{|\eta|^2}{4a}} \int_{\mathbb{R}} \frac{\left(E_{\frac{1}{4a}}(x)\right)^{-1}|f(x)|}{(1+|x|)^{\frac{n}{p}}}(1+|x|)^{\frac{n}{p}} e^{-a(|x|-\frac{|\eta|}{2a})^2} e^{-\varrho|x|}A(x)dx.
\end{aligned}$$

Then by using the Hölder inequality, (4.35) and the following.

Remark 4. *There exist $\gamma > 0$ such that for x large we have*

$$A(x) \leq A(1)|x|^\gamma e^{2\varrho|x|}. \tag{4.38}$$

we can obtain

$$\begin{aligned}
|\mathcal{F}(f)(\xi + i\eta)| &\leq e^{\frac{|\eta|^2}{4a}}\Big(\int_{\mathbb{R}}(1+|x|)^{\frac{np'}{p}} e^{-ap'(|x|-\frac{|\eta|}{2a})^2} e^{-\varrho p'|x|}A(x)dx\Big)^{\frac{1}{p'}} \\
&\leq Ce^{\frac{|\eta|^2}{4a}}\Big(\int_0^\infty (1+t)^{\frac{np'}{p}+\gamma} e^{-ap'(t-\frac{|\eta|}{2a})^2} e^{-\varrho(p'-2)t}dt\Big)^{\frac{1}{p'}} \\
&\leq Ce^{\frac{1}{4a}|\mathrm{Im} z|^2}(1+|Im\, z|)^{\frac{n}{p}+\frac{\gamma}{p'}}.
\end{aligned}$$

Thus (4.37) is proved.

If $ab = \frac{1}{4}$, then

$$|\mathcal{F}(f)(\xi + i\eta)| \leq Ce^{b|\mathrm{Im}z|^2}(1 + |Im\, z|)^{\frac{n}{p}+\frac{\gamma}{p'}}.$$

Therefore, if we let $g(z) = e^{bz^2}\mathcal{F}(f)(z)$, then

$$|g(z)| \leq Ce^{b(\mathrm{Re}z)^2}(1 + |\mathrm{Im}z|)^{\frac{n}{p}+\frac{\gamma}{p'}}.$$

Hence it follows from (4.36) that

$$\int_{\mathbb{R}} \frac{|g(\xi)|^q}{(1 + |\xi|)^m} d\xi < \infty.$$

Here we use the following lemma.

Lemma 3. ([24]). *Let h be an entire function on $\mathbb{C}$ such that*

$$|h(z)| \leq Ce^{a|\mathrm{Re}z|^2}(1 + |\mathrm{Im}z|)^m$$

for some $m > 0$, $a > 0$ and

$$\int_{\mathbb{R}} \frac{|h(x)|^q}{(1 + |x|)^s} |Q(x)| dx < \infty$$

for some $q \geq 1$, $s > 1$ and $Q \in \mathcal{P}_M(\mathbb{R})$. Then h is a polynomial with $\deg h \leq \min\{m, \frac{s-M-1}{q}\}$ *and, if $s \leq q + M + 1$, then h is a constant.*

Hence by this lemma g is a polynomial, we say P_b, with $\deg P_b := d \leq \min\{\frac{n}{p} + \frac{\gamma}{p'}, \frac{m-1}{q}\}$. Then

$$\mathcal{F}(f)(x) = P_b(x)e^{-bx^2}$$

and thus,

$$f(x) = \sum_{j=0}^{d} C_j N_j(b, .) \quad \text{for all } x \in \mathbb{R}.$$

Therefore, nonzero f satisfies (4.35) provided that

$$n > 2\varrho + 1 + p \min\Big\{\frac{n}{p} + \frac{\gamma}{p'}, \frac{m-1}{q}\Big\}.$$

Furthermore, if $m \leq q + 1$, then g is a constant by the Lemma 3 and thus

$$\mathcal{F}(f)(x) = Ce^{-bx^2} \quad \text{and} \quad f(x) = C_b E_b(x).$$

When $n > 1$ and $m > 1$, these functions satisfy (4.36) and (4.35) respectively. This proves ii).

If $ab > \frac{1}{4}$, then we can choose positive constants, a_1, b_1 such that $a > a_1 = \frac{1}{4b_1} > \frac{1}{4b}$. Then f and $\mathcal{F}(f)$ also satisfy (4.35) and (4.36) with a and b replaced by a_1 and b_1 respectively. Therefore, it follows that $\mathcal{F}(f)(x) = P_{b_1}(x)e^{-b_1x^2}$. But then $\mathcal{F}(f)$ cannot satisfy (4.36) unless $P_{b_1} \equiv 0$, which implies $f \equiv 0$. This proves i).

If $ab < \frac{1}{4}$, then for all $\delta \in (b, \frac{1}{4a})$, the functions of the form $f(x) = \sum_{j=0}^{d} C_j N_j(\delta, .)$, where $d \in \mathbb{N}$, satisfy (4.35) and (4.36). This proves iii). □

5. Beurling's theorem for the generalized Fourier transform

Beurling's theorem and Bonami, Demange, and Jaming's extension are generalized for the generalized Fourier transform as follows.

Theorem 6. *Let $N \in \mathbb{N}$, $\delta > 0$ and $f \in L^2_A(\mathbb{R})$ satisfy*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)||\mathcal{F}(f)(y)||R(y)|^\delta}{(1+|x|+|y|)^N} e^{|x|\,|y|} A(x)dxdy < \infty, \tag{5.39}$$

where R is a polynomial of degree m. If $N \geq m\delta + 3$, then

$$f(x) = \sum_{s<\frac{N-m\delta-1}{2}} a_s N_s(r,x) \text{ a.e.}, \tag{5.40}$$

where $r > 0$, $a_s \in \mathbb{C}$. Otherwise, $f(x) = 0$ almost everywhere.

Proof. We start the following lemma.

Lemma 4. *We suppose that $f \in L^2_A(\mathbb{R})$ satisfies (5.39). Then $f \in L^1_A(\mathbb{R})$.*

Proof. We may suppose that f is not negligible. (5.39) and the Fubini theorem imply that for almost every $(t, y) \in \mathbb{R}$,

$$\frac{|\mathcal{F}(f)(y)||R(y)|^\delta}{(1+|y|)^N} \int_{\mathbb{R}} \frac{|f(x)|}{(1+|x|)^N} e^{|x|\,|y|} A(x)dx < \infty.$$

Since f and thus, $\mathcal{F}(f)$ are not negligible, there exist $y_0 \in \mathbb{R}$, $y_0 \neq 0$, such that $\mathcal{F}(f)(y_0)R(y_0) \neq 0$. Therefore,

$$\int_{\mathbb{R}} \frac{|f(x)|}{(1+|x|)^N} e^{|x|\,|y_0|} A(x)dx < \infty.$$

Since $\dfrac{e^{|x|\,|y_0|}}{(1+|x|)^N} \geq 1$ for large x, it follows that $\displaystyle\int_{\mathbb{R}} |f(x)|A(x)dx < \infty$. □

This lemma and Proposition 4 imply that ${}^tV(f)$ is well-defined almost everywhere on $\mathbb{R}$. By the same techniques used in [16], we can deduce that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{|x|\,|y|}|{}^tV(f)(x)||\mathcal{F}_c({}^tV)(f)(y)||R(y)|^\delta}{(1+|x|+|y|)^N} A(x)dxdy < \infty.$$

According to Theorem 2.3 in [23], we conclude that for all $x \in \mathbb{R}$,

$${}^tV(f)(x) = P(x)e^{-\frac{x^2}{4s}},$$

where $s > 0$ and P a polynomial of degree strictly lower than $\frac{N-m\delta-1}{2}$. Then by (2.19),

$$\mathcal{F}(f)(y) = \mathcal{F}_c \circ {}^tV(f)(y) = \mathcal{F}_c\Big(P(x)e^{-\frac{x^2}{4s}}\Big)(y) = Q(y)e^{-sy^2},$$

where Q is a polynomial of degree $\deg P$. Then by using properties of the generalized heat kernels functions we can find constants a_s such that

$$\mathcal{F}(f)(y) = \mathcal{F}\Big(\sum_{s<\frac{N-m\delta-1}{2}} a_l^s N_s(r,.)\Big)(y).$$

By the injectivity of $\mathcal{F}$ the desired result follows. □

As an application of Theorem 6, we want to prove the following Gelfand-Shilov type theorem for the generalized Fourier transform.

Corollary 1. *Let $N, m \in \mathbb{N}$, $\delta > 0$, $a, b > 0$ with $ab \geq \frac{1}{4}$, and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^2_A(\mathbb{R})$ satisfy*

$$\int_{\mathbb{R}} \frac{|f(x)|e^{\frac{(2a)^p}{p}|x|^p}}{(1+|x|)^N} A(x)dx < \infty \tag{5.41}$$

and

$$\int_{\mathbb{R}} \frac{|\mathcal{F}(f)(y)| e^{\frac{(2b)^q}{q}|y|^q} |R(y)|^\delta}{(1+|y|)^N} dy < \infty \tag{5.42}$$

for some $R \in \mathcal{P}_m$.

i) If $ab > \frac{1}{4}$ *or* $(p,q) \neq (2,2)$, *then* $f(x) = 0$ *almost everywhere.*

ii) If $ab = \frac{1}{4}$ *and* $(p,q) = (2,2)$, *then* f *is of the form* (5.40) *whenever* $N \geq \frac{m\delta+3}{2}$ *and* $r = 2b^2$. *Otherwise,* $f(x) = 0$ *almost everywhere.*

Proof. Since

$$4ab|x||y| \leq \frac{(2a)^p}{p}|x|^p + \frac{(2b)^q}{q}|y|^q,$$

it follows from (5.41) and (5.42) that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)||\mathcal{F}(f)(y)||R(y)|^\delta}{(1+|x|+|y|)^{2N}} e^{4ab|x|\,|y|} A(x) dx dy < \infty.$$

Then (5.39) is satisfied, because $4ab \geq 1$. Therefore, according to the proof of Theorem 6, we can deduce that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{4ab|x|\,|y|} |{}^tV(f)(x)||\mathcal{F}_c({}^tV)(f)(y)||R(y)|^\delta}{(1+|x|+|y|)^{2N}} A(x) dx dy < \infty,$$

and ${}^tV(f)$ and f are of the forms

$${}^tV(f)(x) = P(x) e^{-\frac{x^2}{4s}} \quad \text{and} \quad \mathcal{F}(f)(y) = Q(y) e^{-sy^2},$$

where $s > 0$ and P, Q are polynomials of the same degree strictly lower than $\frac{2N-m\delta-1}{2}$. Therefore, substituting these from, we can deduce that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{-(\sqrt{s}|y| - \frac{1}{2\sqrt{s}}|x|)^2} e^{(4ab-1)|x|\,|y|} |P(x)||Q(x)||R(y)|^\delta}{(1+|x|+|y|)^{2N}} A(x) dx dy < \infty.$$

When $4ab > 1$, this integral is not finite unless $f = 0$ almost everywhere. Moreover, it follows from (5.41) and (5.42) that

$$\int_{\mathbb{R}} \frac{|P(x)| e^{-\frac{1}{4s}x^2} e^{\frac{(2a)^p}{p}|x|^p}}{(1+|x|)^N} A(x) dx < \infty$$

and

$$\int_{\mathbb{R}} \frac{|Q(y)| e^{-sy^2} e^{\frac{(2b)^q}{q}|y|^q} |R(y)|^\delta}{(1+|y|)^N} dy < \infty.$$

Hence, one of these integrals is not finite unless $(p,q) = (2,2)$. When $4ab = 1$ and $(p,q) = (2,2)$, the finiteness of above integrals implies that $r = 2b^2$ and the rest follows from Theorem 6. ☐

6. Miyachi's theorem for the generalized Fourier transform

Theorem 7. *Let* f *be a measurable function on* $\mathbb{R}$ *such that*

$$\left(E_{\frac{1}{4a}}\right)^{-1} f \in L^p_A(\mathbb{R}) + L^q_A(\mathbb{R}) \tag{6.43}$$

and

$$\int_{\mathbb{R}} \log^+ \frac{e^{b\xi^2} |\mathcal{F}(f)(\xi)|}{\lambda} d\xi < \infty, \tag{6.44}$$

for some constants $a, b, \lambda > 0$, $1 \leq p, q \leq \infty$. *Then*

i) If $ab > \frac{1}{4}$, *we have* $f = 0$ *almost everywhere.*

ii) If $ab = \frac{1}{4}$, *we have* $f = CE_b$ *with* $|C| \leq \lambda$.

iii) If $ab < \frac{1}{4}$, for all $\delta \in (b, \frac{1}{4a})$, the functions of the form $f = \sum_{j=0}^{d} C_j N_j(\delta, .)$, $d \in \mathbb{N}$, satisfy (6.43) and (6.44).

To prove this result we need the following lemmas.

Lemma 5. *([16]). Let h be an entire on $\mathbb{C}$ function such that*

$$|h(z)| \leq Ae^{B|Rez|^2} \text{ and } \int_{\mathbb{R}} \log^+ |h(y)|dy < \infty, \tag{6.45}$$

for some positive constants A, B. Then h is a constant on $\mathbb{C}$.

Lemma 6. *Let r be in $[1, \infty]$. We consider a function g in $L^r_A(\mathbb{R})$. Then there exists a positive constant C such that:*

$$||e^{ax^2}\, {}^tV(E_{\frac{1}{4a}} g)||_{L^r(\mathbb{R})} \leq C||g||_{L^r_A(\mathbb{R})}),$$

where $||\cdot||_{L^r(\mathbb{R})}$ is the norm of the usual Lebesgue space $L^r(\mathbb{R})$ and $a > 0$.

Proof. The proof is immediately from Proposition 8. □

Lemma 7. *Let p, q in $[1, \infty]$ and f a measurable function on $\mathbb{R}$ such that*

$$\left(E_{\frac{1}{4a}}\right)^{-1} f \in L^p_A(\mathbb{R}) + L^q_A(\mathbb{R}), \tag{6.46}$$

for some $a > 0$. Then the function defined on $\mathbb{C}$ by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)\Phi_\lambda(x)A(x)dx, \tag{6.47}$$

is well defined and entire on $\mathbb{C}$. Moreover there exists a positive constant C such that for all ξ, η in $\mathbb{R}$ we have

$$|\mathcal{F}(f)(\xi + i\eta)| \leq Ce^{\frac{\eta^2}{4a}}. \tag{6.48}$$

Proof. The first assertion follows from the hypothesis on the function f and Hölder's inequality using (6.46) and the derivation theorem under the integral sign. We want to prove (6.48). The condition (6.46) implies that the function f belongs to $L^1_A(\mathbb{R})$. Hence we deduce from (2.19) that for all ξ, η in $\mathbb{R}$ we have

$$|\mathcal{F}(f)(\xi + i\eta)| = |\int_{\mathbb{R}} {}^tV(f)(y)e^{-iy(\xi+i\eta)}dy|.$$

$$\leq \int_{\mathbb{R}} \left|{}^tV(f)(y)\right|e^{y\eta}dy.$$

The integral of the second member can also be estimate in the form

$$\int_{\mathbb{R}} e^{ay^2}|{}^tV(f)(y)|e^{-a\left(y-\frac{\eta}{2a}\right)^2} dy.$$

Indeed from (6.46) there exists u in $L^p_A(\mathbb{R})$ and v in $L^q_A(\mathbb{R})$ such that

$$f = E_{\frac{1}{4a}}(u + v).$$

Thus using the Lemma 6 and Hölder's inequality we obtain

$$\int_{\mathbb{R}} e^{ay^2}|{}^tV(f)(y)|e^{-a\left(y-\frac{\eta}{2a}\right)^2} dy \leq C(||u||_{L^p_A(\mathbb{R})} + ||v||_{L^q_A(\mathbb{R})}) < \infty.$$

Therefore, the desired result follows. □

Proof. of Theorem 7.
We will divide the proof in several cases.
1 st case $ab > \frac{1}{4}$.
Consider the function h defined on $\mathbb{C}$ by

$$h(z) = e^{\frac{z^2}{4a}}\mathcal{F}(f)(z). \tag{6.49}$$

This function is entire on $\mathbb{C}$ and using (6.48) we obtain:

$$|h(\xi + i\eta)| \le Ce^{\frac{\xi^2}{4a}}, \tag{6.50}$$

for all $\xi, \eta \in \mathbb{R}$. On the other hand we have

$$\begin{aligned}\int_{\mathbb{R}_+} \log^+ |h(y)|dy &= \int_{\mathbb{R}_+} \log^+ |e^{\frac{y^2}{4a}}\mathcal{F}(f)(y)|dy,\\ &= \int_{\mathbb{R}} \log^+ |\lambda e^{(\frac{1}{4a}-b)y^2}\frac{e^{by^2}\mathcal{F}(f)(y)}{\lambda}|dy\\ &\le \int_{\mathbb{R}} \log^+ |\frac{e^{by^2}\mathcal{F}(f)(y)}{\lambda}|dy + \int_{\mathbb{R}} e^{(\frac{1}{4a}-b)y^2}dy\end{aligned}$$

because $\log^+(cd) \le \log^+(c) + d$ for all $c, d > 0$. Since $ab > \frac{1}{4}$, (6.44) implies that

$$\int_{\mathbb{R}} \log^+ |h(y)|dy < \infty. \tag{6.51}$$

From the relations (6.50) and (6.51), it follows from Lemma 5 that there exists a constant C such that

$$h(\xi + i\eta) = C, \quad \xi, \eta \in \mathbb{R}.$$

Thus

$$\mathcal{F}(f)(y) = Ce^{-\frac{y^2}{4a}}.$$

Using now the condition (6.44) and that $ab > \frac{1}{4}$, we deduce that $C = 0$ and hence from the injectivity of $\mathcal{F}(f)$ we deduce that $f = 0$.
Second case $ab = \frac{1}{4}$.
The same proof as for the the first step give that

$$\mathcal{F}(f)(y) = Ce^{-\frac{y^2}{4a}}.$$

Thus (6.44) holds whenever $|C| \le \lambda$. Hence

$$f = Ce^{-\frac{y^2}{4a}}, \quad \text{with } |C| \le \lambda.$$

Third case $ab < \frac{1}{4}$
If f is a given form, then

$$\mathcal{F}(f)(y) = Q(y)e^{-\frac{y^2}{4a}}$$

for some $Q \in \mathcal{P}$. These functions clearly satisfy the conditions (6.43),(6.44) for all $\delta \in (b, \frac{1}{4a})$. The proof of the Theorem is complete. □

The following is an immediate corollary of Theorem 7.

Corollary 2. *Let f be a measurable function on $\mathbb{R}$ such that*

$$\left(E_{\frac{1}{4a}}\right)^{-1} f \in L^p_A(\mathbb{R}) + L^q_A(\mathbb{R}) \tag{6.52}$$

and

$$\int_{\mathbb{R}} |\mathcal{F}(f)(\xi)|^r e^{br\xi^2} d\xi < \infty, \tag{6.53}$$

for some constants $a, b, r > 0$ and $1 \le p, q \le \infty$. Then
i) If $ab \ge \frac{1}{4}$, we have $f = 0$ almost everywhere.

ii) If $ab < \frac{1}{4}$, then for all $\delta \in (b, \frac{1}{4a})$, all the functions of the form $f = \sum_{j=0}^{d} C_j N_j(\delta, .)$, $d \in \mathbb{N}$, satisfy (6.52) and (6.53).

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