ANGLE R

YOO YOUNG KOO AND JAE KUN LIM

Abstract. We generalize the characterizations of the direct sum decomposition of a Hilbert space in terms of the angle R, and, using this result, present another proof of the equality concerning the norms of the projections P and $I-P$. As an application to abstract frame theory, we show that the ranges of the analysis operators of oblique dual frame sequences satisfy the the angle condition.

1. Introduction

Throughout this article $\mathcal{M}, \mathcal{N}, \mathcal{U}$ and $\mathcal{V}$ denote closed subspaces of a Hilbert space $\mathcal{H}$ over $\mathbb{R}$ or $\mathbb{C}$. We let $\mathcal{U}+\mathcal{V}$ denote the sum of $\mathcal{U}$ and $\mathcal{V}$, and $\mathcal{U}\dot{+}\mathcal{V}$ the direct sum of $\mathcal{U}$ and $\mathcal{V}$, i.e., the sum of $\mathcal{U}$ and $\mathcal{V}$ with trivial intersection. If $\mathcal{H}=\mathcal{M}\dot{+}\mathcal{N}$, then $P_{\mathcal{M},\mathcal{N}}$ denote the projection (bounded idempotent) with $\operatorname{ran} P_{\mathcal{M},\mathcal{N}}=\mathcal{M}$ and $\ker P_{\mathcal{M},\mathcal{N}}=\mathcal{N}$ [7, Section II.3]. We also let $P_{\mathcal{M}}:=P_{\mathcal{M},\mathcal{M}^{\perp}}$ be the orthogonal projection onto $\mathcal{M}$. We first recall two definitions of the angles between $\mathcal{U}$ and $\mathcal{V}$. We define ([18, Eqs. (28) and (38)])

$$S(\mathcal{U},\mathcal{V}):=\begin{cases}\sup\left\{\frac{\|P_{\mathcal{V}}u\|}{\|u\|}:u\in\mathcal{U}\setminus\{0\}\right\} & \text{if } \mathcal{U}\neq\{0\},\\ 0 & \text{if } \mathcal{U}=\{0\},\end{cases}$$

$$R(\mathcal{U},\mathcal{V}):=\begin{cases}\inf\left\{\frac{\|P_{\mathcal{V}}u\|}{\|u\|}:u\in\mathcal{U}\setminus\{0\}\right\} & \text{if } \mathcal{U}\neq\{0\},\\ 1 & \text{if } \mathcal{U}=\{0\}.\end{cases}$$

The following is the definition of *Dixmier angle* ([9], [8, Definition 2]).

$$c_0(\mathcal{U},\mathcal{V}):=\sup\left\{|\langle u,v\rangle|:u\in\mathcal{U},\|u\|\leq 1,v\in\mathcal{V},\|v\|\leq 1\right\},$$

where neither $\mathcal{U}$ nor $\mathcal{V}$ is assumed to be trivial. It is well-known and easy to see that $c_0=S$. In particular, $S(\mathcal{U},\mathcal{V})=S(\mathcal{V},\mathcal{U})$. On the other hand, if $\mathcal{U}$ is a non-trivial proper closed subspace of $\mathcal{V}$, then $R(\mathcal{U},\mathcal{V})=1$ and $R(\mathcal{V},\mathcal{U})=0$. See Section 2 for further facts on the asymmetry of R.

There are vast literature and an interesting survey article ([8]) on S, whereas many known facts on R are scattered throughout research papers [18, 17, 12, 3]. S can be used to characterize when the sum of two closed subspaces is closed [8], whereas R can be used to construct two biorthogonal wavelet frames or Riesz bases of $L^2(\mathbb{R}^d)$ [1, 17, 12].

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The purpose of this article is to generalize [3, Proposition 3.3] and [17, Theorem 2.3], which connect some properties of the angle R with those of projections (idempotents [7, Section II.3]) in $\mathcal{B}(\mathcal{H})$ (Theorem 2.1). As an application of this generalization, we give another proof of Theorem 3.1, which is the main focus of [16]. We also apply our results to abstract frame theory (Proposition 3.2). Further remarks on R are also included.

2. Main result

In this section we prove the following theorem which generalizes [3, Proposition 3.3] which, in turn, is a generalization of [17, Theorem 2.3].

Theorem 2.1. *Let $\mathcal{U}$ and $\mathcal{V}$ be closed subspaces of $\mathcal{H}$ such that at least one of which is non-trivial. Then the the following statements are equivalent:*

(1) $0 < R(\mathcal{U},\mathcal{V})$ *and* $0 < R(\mathcal{V},\mathcal{U})$;
(2) $0 < R(\mathcal{U},\mathcal{V}) = R(\mathcal{V},\mathcal{U})$;
(3) $A := P_{\mathcal{V}} : \mathcal{U} \to \mathcal{V}$ *is invertible;*
(4) $B := P_{\mathcal{U}} : \mathcal{V} \to \mathcal{U}$ *is invertible;*
(5) $\mathcal{H} = \mathcal{U} \dotplus \mathcal{V}^{\perp}$;
(6) $\mathcal{H} = \mathcal{V} \dotplus \mathcal{U}^{\perp}$;
(7) $0 < R(\mathcal{U}^{\perp},\mathcal{V}^{\perp})$ *and* $0 < R(\mathcal{V}^{\perp},\mathcal{U}^{\perp})$;
(8) $0 < R(\mathcal{U}^{\perp},\mathcal{V}^{\perp}) = R(\mathcal{V}^{\perp},\mathcal{U}^{\perp})$;
(9) $C := P_{\mathcal{V}^{\perp}} : \mathcal{U}^{\perp} \to \mathcal{V}^{\perp}$ *is invertible;*
(10) $D := P_{\mathcal{U}^{\perp}} : \mathcal{V}^{\perp} \to \mathcal{U}^{\perp}$ *is invertible;*
(11) $0 < R(\mathcal{U},\mathcal{V}) = R(\mathcal{V},\mathcal{U}) = R(\mathcal{U}^{\perp},\mathcal{V}^{\perp}) = R(\mathcal{V}^{\perp},\mathcal{U}^{\perp})$;
(12) *There is a projection P_1 whose range is $\mathcal{U}$ such that $P_1 : \mathcal{V} \to \mathcal{U}$ is invertible;*
(13) *There is a projection P_2 whose range is $\mathcal{V}$ such that $P_2 : \mathcal{U} \to \mathcal{V}$ is invertible;*
(14) *There is a projection P_3 whose range is $\mathcal{U}^{\perp}$ such that $P_3 : \mathcal{V}^{\perp} \to \mathcal{U}^{\perp}$ is invertible;*
(15) *There is a projection P_4 whose range is $\mathcal{V}^{\perp}$ such that $P_4 : \mathcal{U}^{\perp} \to \mathcal{V}^{\perp}$ is invertible.*

Moreover, if any one of the above conditions holds, then the following statement holds. The bounded operators $P_{\mathcal{U},\mathcal{V}^{\perp}} : \mathcal{V} \to \mathcal{U}, P_{\mathcal{V},\mathcal{U}^{\perp}} : \mathcal{U} \to \mathcal{V}, P_{\mathcal{V}^{\perp},\mathcal{U}} : \mathcal{U}^{\perp} \to \mathcal{V}^{\perp}, P_{\mathcal{U}^{\perp},\mathcal{V}} : \mathcal{V}^{\perp} \to \mathcal{U}^{\perp}$ are invertible (hence the spaces are isomorphic), and

$$R(\mathcal{U},\mathcal{V}) = \left\|A^{-1}\right\|^{-1} = \left\|B^{-1}\right\|^{-1} = \left\|C^{-1}\right\|^{-1} = \left\|D^{-1}\right\|^{-1} \tag{2.1}$$

$$= \left\|P_{\mathcal{U},\mathcal{V}^{\perp}}\right\|^{-1} = \left\|P_{\mathcal{V},\mathcal{U}^{\perp}}\right\|^{-1} = \left\|P_{\mathcal{V}^{\perp},\mathcal{U}}\right\|^{-1} = \left\|P_{\mathcal{U}^{\perp},\mathcal{V}}\right\|^{-1}. \tag{2.2}$$

We note that the equivalences of Items (1) to (6) are established in [3, Proposition 3.3] and [17, Theorem 2.3]. Moreover, the first two equalities in (2.1) are proved in [3, Proposition 3.3]. We recall the following equations [18, p. 2922].

$$R(\mathcal{U},\mathcal{V}) = \sqrt{1 - S\left(\mathcal{U},\mathcal{V}^{\perp}\right)^2} \tag{2.3}$$

$$= R\left(\mathcal{V}^{\perp},\mathcal{U}^{\perp}\right). \tag{2.4}$$

(2.4) implies that Items (1) to (6) are equivalent to Items (7) to (11) and the equalities in (2.1) hold. It is reported in [8, p. 119] that Ljance [15] showed that, if $\mathcal{H} = \mathcal{M} \dotplus \mathcal{N}$, then $S(\mathcal{M}, \mathcal{N})^2 = 1 - \|P_{\mathcal{M},\mathcal{N}}\|^{-2}$. By using (2.3) and (2.4), the first equality in (2.2) follows. We derive the first equality in (2.2) by using the fact that $P_{\mathcal{U},\mathcal{V}^\perp} : \mathcal{V} \to \mathcal{U}$ is invertible in Lemma 2.4.

Lemma 2.2. *If $\mathcal{H} = \mathcal{U} \dotplus \mathcal{V}^\perp$, then $P_{\mathcal{U},\mathcal{V}^\perp} : \mathcal{V} \to \mathcal{U}$ and $P_{\mathcal{V},\mathcal{U}^\perp} : \mathcal{U} \to \mathcal{V}$ are invertible.*

Proof. Let $u \in \mathcal{U}$ be arbitrary. Define $v := P_{\mathcal{V}} u \in \mathcal{V}$. Then

$$P_{\mathcal{U},\mathcal{V}^\perp} v = P_{\mathcal{U},\mathcal{V}^\perp} P_{\mathcal{V}} u = P_{\mathcal{U},\mathcal{V}^\perp} \left(P_{\mathcal{V}} u + P_{\mathcal{V}^\perp} u\right) = P_{\mathcal{U},\mathcal{V}^\perp} u = u.$$

Hence $P_{\mathcal{U},\mathcal{V}^\perp} : \mathcal{V} \to \mathcal{U}$ is onto. Suppose that $v \in \mathcal{V}$ and $P_{\mathcal{U},\mathcal{V}^\perp} v = 0$. Then $v \in \ker P_{\mathcal{U},\mathcal{V}^\perp} = \mathcal{V}^\perp$. Hence $v \in \mathcal{V} \cap \mathcal{V}^\perp = \{0\}$. Therefore $P_{\mathcal{U},\mathcal{V}^\perp}|_{\mathcal{V}}$ is one-to-one. By the open mapping theorem, $P_{\mathcal{U},\mathcal{V}^\perp} : \mathcal{V} \to \mathcal{U}$ is invertible. Similarly, $P_{\mathcal{V},\mathcal{U}^\perp} : \mathcal{U} \to \mathcal{V}$ is invertible. □

The converse of Lemma 2.2 holds.

Lemma 2.3. *If there is a projection P such that* $\operatorname{ran} P = \mathcal{U}$ *and $P : \mathcal{V} \to \mathcal{U}$ is invertible, then $\mathcal{H} = \mathcal{U} \dotplus \mathcal{V}^\perp$.*

Proof. Since P is a projection, $P = P_{\mathcal{U},\mathcal{W}}$, where $\mathcal{W} := \ker P$. Of course, $\mathcal{H} = \mathcal{U} \dotplus \mathcal{W}$. Since Items (3) and (5) are equivalent, it suffices to show that $P_{\mathcal{V}} : \mathcal{U} \to \mathcal{V}$ is invertible. By our assumption

$$P_{\mathcal{U},\mathcal{W}} : \mathcal{V} \to \mathcal{U} \quad \text{is invertible.} \tag{2.5}$$

Since $(P_{\mathcal{U},\mathcal{W}})^* = P_{\mathcal{W}^\perp,\mathcal{U}^\perp}$, the adjoint of $P_{\mathcal{U},\mathcal{W}}$ as an element of $\mathcal{B}(\mathcal{U}, \mathcal{V})$ is $P_{\mathcal{W}^\perp,\mathcal{U}^\perp} : \mathcal{U} \to \mathcal{V}$, which is also invertible. Hence $\mathcal{V} \subset \operatorname{ran} P_{\mathcal{W}^\perp,\mathcal{U}^\perp} = \mathcal{W}^\perp$. In particular, $\mathcal{W} \subset \mathcal{V}^\perp$. Let $v \in \mathcal{V}$ be arbitrary. Then,

$$\begin{aligned} v &= P_{\mathcal{U},\mathcal{W}} v + (I - P_{\mathcal{U},\mathcal{W}})\, v = P_{\mathcal{U},\mathcal{W}} v + P_{\mathcal{W},\mathcal{U}} v, \quad \text{i.e.,} \\ v &= P_{\mathcal{V}} v = P_{\mathcal{V}} P_{\mathcal{U},\mathcal{W}} v + P_{\mathcal{V}} P_{\mathcal{W},\mathcal{U}} v = P_{\mathcal{V}} P_{\mathcal{U},\mathcal{W}} v \end{aligned}$$

since $P_{\mathcal{W},\mathcal{U}} v \in \mathcal{W} \subset \mathcal{V}^\perp$. That is,

$$v = P_{\mathcal{V}} P_{\mathcal{U},\mathcal{W}} v \quad \forall\, v \in \mathcal{V}. \tag{2.6}$$

This shows that $P_{\mathcal{V}} : \mathcal{U} \to \mathcal{V}$ is onto since $P_{\mathcal{U},\mathcal{W}} v \in \mathcal{U}$. Now, suppose that $u \in \mathcal{U}$ and $P_{\mathcal{V}} u = 0$, i.e., $u \in \mathcal{U} \cap \mathcal{V}^\perp$. By (2.5), there exists $v \in \mathcal{V}$ such that $P_{\mathcal{U},\mathcal{W}} v = u$. Then, by (2.6), $0 = P_{\mathcal{V}} u = P_{\mathcal{V}} P_{\mathcal{U},\mathcal{W}} v = v$. Hence $u = P_{\mathcal{U},\mathcal{W}} v = P_{\mathcal{U},\mathcal{W}} 0 = 0$. This shows that $P_{\mathcal{V}} : \mathcal{U} \to \mathcal{V}$ is one-to-one. Hence it is invertible by the open mapping theorem. □

Lemma 2.4. *If $\mathcal{H} = \mathcal{U} \dotplus \mathcal{V}^\perp$ and $\mathcal{U}$ is not trivial, then*

$$R(\mathcal{U}, \mathcal{V}) = \left\|P_{\mathcal{U},\mathcal{V}^\perp}\right\|^{-1}. \tag{2.7}$$

Proof. By Lemma 2.2 $P_{\mathcal{U},\mathcal{V}^\perp} : \mathcal{V} \to \mathcal{U}$ is invertible. Let $v \in \mathcal{V}$. Since $I - P_{\mathcal{U},\mathcal{V}^\perp} = P_{\mathcal{V}^\perp,\mathcal{U}}$, $v = Iv = P_{\mathcal{U},\mathcal{V}^\perp} v + P_{\mathcal{V}^\perp,\mathcal{U}} v$. In particular,

$$v = P_{\mathcal{V}} v = P_{\mathcal{V}} P_{\mathcal{U},\mathcal{V}^\perp} v + P_{\mathcal{V}} P_{\mathcal{V}^\perp,\mathcal{U}} v = P_{\mathcal{V}} P_{\mathcal{U},\mathcal{V}^\perp} v.$$

Since $P_{\mathcal{U},\mathcal{V}^\perp} : \mathcal{V} \to \mathcal{U}$ is invertible, we have

$$\begin{aligned} R(\mathcal{U},\mathcal{V}) &= \inf_{u \in \mathcal{U} \setminus \{0\}} \frac{\|P_{\mathcal{V}} u\|}{\|u\|} = \inf_{v \in \mathcal{V} \setminus \{0\}} \frac{\left\|P_{\mathcal{V}} P_{\mathcal{U},\mathcal{V}^\perp} v\right\|}{\left\|P_{\mathcal{U},\mathcal{V}^\perp} v\right\|} = \inf_{v \in \mathcal{V} \setminus \{0\}} \frac{\|v\|}{\left\|P_{\mathcal{U},\mathcal{V}^\perp} v\right\|} \\ &= \left(\sup_{v \in \mathcal{V} \setminus \{0\}} \frac{\left\|P_{\mathcal{U},\mathcal{V}^\perp} v\right\|}{\|v\|} \right)^{-1} = \left\|P_{\mathcal{U},\mathcal{V}^\perp}\right\|^{-1}. \end{aligned}$$ □

Proof of Theorem 2.1. Lemmas 2.2 and 2.3 show that Items (5) and (12) are equivalent. The remaining equivalences follow easily. Now, suppose that any of the Items holds. Then the first equality in (2.2) holds by Lemma 2.4 (which was already observed by Ljance [15]). The second equality in (2.2) holds by taking the adjoint of $P_{\mathcal{U},\mathcal{V}^\perp}$. The remaining equalities in (2.2) hold by Theorem 3.1 since its proof uses only Lemma 2.4. □

We see that $R(\mathcal{U},\mathcal{V}) = R(\mathcal{V},\mathcal{U})$ if both of them are positive. On the other hand, we also see that $R(\mathcal{V},\mathcal{U})$ can be 0 while $R(\mathcal{U},\mathcal{V}) = 1$ if $\mathcal{U}$ is a non-trivial proper closed subspace of $\mathcal{V}$. It is shown in [3, Lemma 3.2] that if $0 = R(\mathcal{V},\mathcal{U}) < R(\mathcal{U},\mathcal{V})$, then $\mathcal{V} \ominus \mathcal{U}$ is not trivial. Finally, [3, Lemma 3.1] shows that if $\mathcal{U}$ is not trivial, then

$$R(\mathcal{U},\mathcal{V}) = \begin{cases} 0, & \text{if } P_{\mathcal{V}}|_{\mathcal{U}} \text{ is not bounded below,} \\ \left\|(P_{\mathcal{V}}|_{\mathcal{U}})^\dagger\right\|^{-1}, & \text{if } P_{\mathcal{V}}|_{\mathcal{U}} \text{ is bounded below,} \end{cases}$$

where $T^\dagger$ denotes the Moore-Penrose generalized inverse of a bounded operator T with closed range.

3. Applications

As applications of Theorem 2.1, we first present yet another proof of the main result in the survey article [16], which is used in the proof of the third equality in (2.2). Then we show that the ranges of the analysis operators of two oblique dual frame sequences satisfy the decomposition in Theorem 2.1. This recovers a result in [13] that the excesses of the oblique dual frame sequences are the same.

Theorem 3.1 (Theorem 2.1 [16]). *Let P be a projection in $\mathcal{B}(\mathcal{H})$ such that neither* $\operatorname{ran} P$ *nor* $\ker P$ *is $\mathcal{H}$. Then $\|P\| = \|I - P\|$.*

Proof. Since P is a projection, $\mathcal{H} = \operatorname{ran} P \dotplus \ker P$. Moreover, $I - P$ is also a projection such that $\operatorname{ran}(I - P) = \ker P$ and $\ker(I - P) = \operatorname{ran} P$. Let $\mathcal{M} := \operatorname{ran} P$ and $\mathcal{N} := \ker P$. Then $P = P_{\mathcal{M},\mathcal{N}}$ and $I - P = P_{\mathcal{N},\mathcal{M}}$. By our assumptions, neither $\mathcal{M}$ nor $\mathcal{N}$ is trivial and

$\mathcal{H} = \mathcal{M} \dotplus (\mathcal{N}^\perp)^\perp$. Hence, by (2.7), $\|P\| = \|P_{\mathcal{M},\mathcal{N}}\| = R(\mathcal{M},\mathcal{N}^\perp)^{-1}$. On the other hand, by (2.4), $R(\mathcal{M},\mathcal{N}^\perp) = R(\mathcal{N},\mathcal{M}^\perp)$. By (2.7),

$$\|I - P\| = \|P_{\mathcal{N},\mathcal{M}}\| = \left\|P_{\mathcal{N},(\mathcal{M}^\perp)^\perp}\right\| = R(\mathcal{N},\mathcal{M}^\perp)^{-1} = R(\mathcal{M},\mathcal{N}^\perp)^{-1} = \|P\|. \qquad \square$$

We now give an application of Theorem 2.1 to abstract frame theory. We refer to [4] for the basic facts on frames and frame sequences. For a sequence $X := \{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$, define $\mathcal{H}_X := \overline{\operatorname{span}} X$. X is said to be a *Bessel sequence* if there exists a positive constant β_X such that, for each $h \in \mathcal{H}$, $\sum |\langle h, x_n\rangle|^2 \le \beta_X \|h\|^2$. For a Bessel sequence X, define its *synthesis operator* $T_X : \ell^2 \to \mathcal{H}$ by $T_X a := \sum a(n) x_n$. It is known that T_X is a well-defined bounded operator. Its adjoint T_X^* is called the *analysis operator* of X and $T_X^* h = (\langle h, x_n\rangle)_n$ for $h \in \mathcal{H}$. The *frame operator* of X is defined to be $S_X := T_X T_X^*$. X is said to be a *frame sequence* if there exist positive constants α_X and β_X such that, for any $h \in \mathcal{H}_X$, $\alpha_X \|h\|^2 \le \sum |\langle h, x_n\rangle|^2 \le \beta_X \|h\|^2$. A frame sequence is a *frame* for $\mathcal{H}$ if $\mathcal{H}_X = \mathcal{H}$. It is known that a Bessel sequence is a frame sequence if and only if T_X has closed range. In this case, $\operatorname{ran} T_X = \mathcal{H}_X$ and $S_X : \mathcal{H}_X \to \mathcal{H}_X$ is invertible. Suppose that X is a frame sequence. Then $T_X^\dagger$ and $(T_X^*)^\dagger$ are bounded since T_X and T_X^* have closed range. We recall that [6]

$$(T_X^*)^\dagger = (T_X^\dagger)^* \quad \text{and} \quad (T_X T_X^*)^\dagger = (T_X^*)^\dagger T_X^\dagger. \tag{3.1}$$

For a frame sequence X, a Bessel sequence $Y := \{y_n\}_{n\in\mathbb{N}}$ is said to be a *dual* of X if $T_X T_Y^*|_{\mathcal{H}_X} = I|_{\mathcal{H}_X}$. Two Bessel sequences are said to be *oblique duals* of each other if they are both frame sequences and they are duals of each other [10]. In this case, the following equations hold [10]:

$$T_X T_Y^* = P_{\mathcal{H}_X,\mathcal{H}_Y^\perp} \quad \text{and} \quad T_Y T_X^* = P_{\mathcal{H}_Y,\mathcal{H}_X^\perp}. \tag{3.2}$$

In particular, $\mathcal{H} = \mathcal{H}_X \dotplus \mathcal{H}_Y^\perp$. The following proposition, which is a generalization of [5, Proposition 7.2], shows that if X and Y are oblique duals of each other, then $\ell^2 = \operatorname{ran} T_X^* \dotplus (\operatorname{ran} T_Y^*)^\perp$. Hence, by Theorem 2.1, $\ker T_X = (\operatorname{ran} T_X^*)^\perp$ and $\ker T_Y = (\operatorname{ran} T_Y^*)^\perp$ are isomorphic. For a frame sequence $\dim\ker T_X$ is called the *excess* of X and it is equal to the following quantity [11, 2]: $\sup\{\operatorname{card} X' : X' \subset X, \mathcal{H}_{X\setminus X'} = \mathcal{H}_X\}$. In a sense, the excess of X measures the redundancy of X. The following proposition implies that if X and Y are oblique dual frames of each other, then $\ell^2 = \operatorname{ran} T_X^* \dotplus (\operatorname{ran} T_Y^*)^\perp$. Hence $\dim\ker T_X = \dim(\operatorname{ran} T_X^*)^\perp = \dim(\operatorname{ran} T_Y^*)^\perp = \dim\ker T_Y$ by Theorem 2.1. Therefore X and Y have the same excesses. This result on the excess of oblique duals is also proved in [13] using different method.

Proposition 3.2. *Let X and Y be oblique dual frame sequences and $P := P_{\operatorname{ran} T_X^*}$ and $Q := P_{\operatorname{ran} T_Y^*}$. The the following hold:*

(1) $P = T_X^* S_X^\dagger T_X$;
(2) $P T_Y^* = T_X^\dagger P_{\mathcal{H}_X,\mathcal{H}_Y^\perp}$.

Moreover, $\operatorname{ran} T_X^*$ *and* $\operatorname{ran} T_Y^*$ *satisfy the angle condition in Theorem 2.1. In particular,* $\ell^2 = \operatorname{ran} T_X^* \dotplus (\operatorname{ran} T_Y^*)^\perp$.

Proof. Since X and Y are oblique dual frame sequences, (3.2) hold. Since $S_X : \mathcal{H}_X \to \mathcal{H}_X$ is invertible, $\ker S_X^\dagger = \mathcal{H}_X^\perp$ and $\operatorname{ran} S_X^\dagger = \mathcal{H}_X$.

(1): We have, by (3.1) and the properties of Moore-Penrose generalized inverses,

$$\begin{aligned} T_X^* S_X^\dagger T_X &= T_X^*(T_X T_X^*)^\dagger T_X = \left(T_X^*(T_X^*)^\dagger\right)\left(T_X^\dagger T_X\right) \\ &= P_{\operatorname{ran} T_X^*} P_{\operatorname{ran} T_X^*} = P_{\operatorname{ran} T_X^*} = P. \end{aligned}$$

(2): (1), (3.2) and the properties of Moore-Penrose generalized inverses imply that

$$\begin{aligned} PT_Y^* &= \left(T_X^* S_X^\dagger T_X\right) T_Y^* = T_X^*(T_X T_X^*)^\dagger \,(T_X T_Y^*) = \left(T_X^*(T_X^*)^\dagger\right) T_X^\dagger P_{\mathcal{H}_X, \mathcal{H}_Y^\perp} \\ &= P_{\operatorname{ran} T_X^*} T_X^\dagger P_{\mathcal{H}_X, \mathcal{H}_Y^\perp} = T_X^\dagger P_{\mathcal{H}_X, \mathcal{H}_Y^\perp}. \end{aligned}$$

By Theorem 2.1, to show that $\operatorname{ran} T_X^*$ and $\operatorname{ran} T_Y^*$ satisfy the angle condition, it suffices to show that $P : \operatorname{ran} T_Y^* \to \operatorname{ran} T_X^*$ is invertible. It is elementary to see that $T_X^\dagger : \mathcal{H}_X \to \operatorname{ran} T_X^*$ is invertible (see, for example, [14]). Hence, by (2),

$$\operatorname{ran} T_X^* = T_X^\dagger\,(\mathcal{H}_X) \subset \operatorname{ran} T_X^\dagger P_{\mathcal{H}_X, \mathcal{H}_Y^\perp} = \operatorname{ran} PT_Y^* \subset P(\operatorname{ran} T_Y^*) \subset \operatorname{ran} P = \operatorname{ran} T_X^*.$$

In particular, $P(\operatorname{ran} T_Y^*) = \operatorname{ran} T_X^*$. Now, suppose that $a \in \operatorname{ran} T_Y^*$ and $Pa = 0$. Then, there exists $h \in \mathcal{H}$ such that $a = T_Y^* h$ and $PT_Y^* h = 0$. We may assume that $h \in (\ker T_Y^*)^\perp = \operatorname{ran} T_Y = \mathcal{H}_Y$. By (2), $0 = PT_Y^* h = T_X^\dagger P_{\mathcal{H}_X, \mathcal{H}_Y^\perp} h$. Since $\ker T_X^\dagger = \ker T_X^*$,

$$\operatorname{ran} T_X \ni P_{\mathcal{H}_X, \mathcal{H}_Y^\perp} h \in \ker T_X^\dagger = \ker T_X^* = (\operatorname{ran} T_X)^\perp,$$

i.e., $P_{\mathcal{H}_X, \mathcal{H}_Y^\perp} h = 0$. Therefore $\mathcal{H}_Y \ni h \in \ker P_{\mathcal{H}_X, \mathcal{H}_Y^\perp} = \mathcal{H}_Y^\perp$. Hence $h = 0$. Therefore, $a = T_Y^* h = 0$. This shows that $P : \operatorname{ran} T_Y^* \to \operatorname{ran} T_X^*$ is invertible by the open mapping theorem. □

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(Y. Y. Koo) University College, Yonsei University, Seoul 120-749, Republic of Korea
E-mail address: yykoo@yonsei.ac.kr

(J. K. Lim) Department of Applied Mathematics, Hankyong National University, Anseongsi, Gyeonggido, 456-749, Republic of Korea
E-mail address: jaekun@hknu.ac.kr