On the permanental polynomials of matrices*

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Abstract

An $m \times n$ $\{0,1\}$-matrix A is said to be totally convertible if there exists a matrix B obtained from A by changing some 1's in A to -1's such that for any submatrix A' of A of order m, the corresponding submatrix B' of B satisfies $\mathrm{per}(xI - A') = \det(xI - B')$. In this paper, motivated by the well-known Pólya's problem, our object is to characterize those totally convertible matrices. Associate a matrix A with a bipartite graph G_A^*. We first prove that a square matrix A is totally convertible if and only if G_A^* is Pfaffian, and then we generalize this result to an $m \times n$ $\{0,1\}$-matrix. Moreover, the characterization of a totally convertible matrix provides an equivalent condition to compute the permanental polynomial of a bipartite graph by the characteristic polynomial of the skew adjacency matrix of its orientation graph. As applications, we give some explicit expressions of the permanental polynomials of two totally convertible matrices by the technique of Pfaffian orientation.

Key Words: Permanent; Permanental polynomial; Pfaffian orientation; Determinant

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1 Introduction

For a square matrix A of order n, the *characteristic polynomial* of A is defined as

$$\det(xI - A) = \sum_{k=0}^{n} c_k x^{n-k},$$

and the *permanental polynomial* of A, by definition, is

$$\mathrm{per}(xI - A) = \sum_{k=0}^{n} b_k x^{n-k}.$$

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Here the *permanent* of a matrix $C = (c_{ij})_{n\times n}$ is given as [15]

$$\mathrm{per}(C) = \sum_{\sigma\in\Lambda_n}\prod_{i=1}^{n} c_{i\sigma(i)}$$

with Λ_n denoting the set of all the permutations of $\{1, 2, \cdots, n\}$. Let $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ and $Q_{r,n} = \{\omega | 1 \leq \omega_1 < \cdots < \omega_r \leq n\}$ the set of increasing sequence. Then [2, 15]

$$b_k = (-1)^k \sum_{\omega\in Q_{k,n}} \mathrm{per}(A[\omega]) \tag{1}$$

and

$$c_k = (-1)^k \sum_{\omega\in Q_{k,n}} \det(A[\omega]), \tag{2}$$

where $A[\omega]$ denotes the $k \times k$ submatrix of A whose (i, j)-entry is a_{ω_i,ω_j}.

As is well known, computing the permanent of a matrix is a #P-complete problem [20]. So far, few work on permanental polynomials has been reported [1, 3, 4, 8, 10, 14, 22]. This may be due to the difficulty to compute permanents and permanental polynomials. However, the determinant can be calculated efficiently using Gaussian elimination and the characteristic polynomial can also be evaluated efficiently [5, 18]. As early as in 1913, Pólya [16] proposed the following problem. If A is a square $\{0, 1\}$-matrix, does there exist a matrix B obtained from A by changing some of the 1's to -1's in such a way that the permanent of A equals the determinant of B? (If the answer is "yes", then A is said to be *convertible*.) Sixty years later, Little answered Pólya's question by characterizing the convertible matrix in terms of Paffian bipartite graph [11]. Robertson et al. [17] and McCuaig [13] independently gave polynomial-time algorithms to determine whether a given bipartite graph has a Pfaffian orientation. Here we consider converting the computation of permanental polynomials into the computation of characteristic polynomials. Concretely, it is described as follows.

For an $m \times n$ $\{0, 1\}$-matrix A $(m \leq n)$, a *signing* of A is a $\{0, 1, -1\}$-matrix obtained from A by replacing some 1's to -1's. Our problem is whether there exists a signing B of A such that for any $\omega \in Q_{m,n}$,

$$\mathrm{per}(xI - A[\omega]) = \det(xI - B[\omega]), \tag{3}$$

where $A[\omega]$ denotes an $m \times m$ submatrix of A whose rows correspond to the rows of A and columns correspond to the columns of A with indexes in ω; If yes, we say that A is *totally convertible*. In this paper, we mainly characterize those totally convertible matrices. Our results are stated in terms of bipartite graphs and Pfaffian orientations.

By a graph G on p vertices we mean a finite simple graph with vertex-set $\{v_1, v_2, \cdots, v_p\}$. A *perfect matching* of G is a set of edges of G covering every vertex exactly once. We use $\phi(G)$ to denote the number of perfect matchings of G. A graph G is *bipartite* if its vertex-set can be partitioned into two sets U and W in such a way that every edge has one endvertex in U and the other one in W, denoted by $G = (U, W; E)$. An *orientation* $\vec{G}$ of a graph G

is an assignment of direction to each edge of G. The *skew adjacency matrix* $A_s(\vec{G})$ of an orientation $\vec{G}$ is the matrix with entry 1 (resp. -1) in row i and column j if an edge is directed from v_i to v_j (resp. from v_j to v_i) in $\vec{G}$, and 0 otherwise. We can see that the skew adjacency matrix of a directed graph is skew symmetric. In particular, the skew adjacency matrix of a bipartite directed graph $\vec{G}$ takes on the form

$$A_s(\vec{G}) = \begin{pmatrix} 0 & D_s \\ -D_s^T & 0 \end{pmatrix},$$

where D_s is the *skew biadjacency matrix* of $\vec{G}$ with rows and columns indexed by vertices in U and W, respectively, such that the ij-entry is 1 if the edge u_iw_j is directed from u_i to w_j, -1 if the edge u_iw_j is directed from w_j to u_i and 0 otherwise.

Let us now assume that $A = (a_{ij})_{n\times n}$ (n is even) is a skew symmetric matrix. Denote by $\mathscr{P}_n$ the set of all partitions of $\{1, 2, \cdots, n-1, n\}$ into pairs. For each partition $P = \{\{i_1, i_2\}, \{i_3, i_4\}, \cdots, \{i_{n-1}, i_n\}\}$, let

$$a_P := \operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ i_1 & i_2 & \cdots & i_{n-1} & i_n \end{pmatrix} a_{i_1i_2}a_{i_3i_4}\cdots a_{i_{n-1}i_n}.$$

The Pfaffian of A is defined as

$$\mathrm{Pf}(A) := \sum_{P\in\mathscr{P}_n} a_P.$$

For an orientation $\vec{G}$ of G, it always holds that

$$|\mathrm{Pf}(A_s(\vec{G}))| \leq \phi(G).$$

If equality holds, we call $\vec{G}$ a *Pfaffian orientation* of G [12]. A graph is said to be *Pfaffian* if it admits a Pfaffian orientation.

For a $\{0,1\}$-matrix A, we define a bipartite graph G_A^* in Section 2, and prove that a square $\{0,1\}$-matrix A is totally convertible if and only if G_A^* is Pfaffian. Further, if G_A^* is Pfaffian, a signing B of A such that $\mathrm{per}(xI-A) = \det(xI-B)$ can be obtained by assigning a Pfaffian orientation of G_A^*; if G_A^* is not Pfaffian, then A is not totally convertible. According to [13, 17], the problem of determining a square matrix is totally convertible or not can be settled by a polynomial-time algorithm. More generally, we consider totally convertible $m\times n$ matrix ($m \leq n$) in Section 3. We prove that an $m \times n$ $\{0,1\}$-matrix A is totally convertible if and only if G_A^* admits a normal orientation such that each left-central ($*$) cycle is oddly oriented. In Section 4 we show that the result on totally convertible matrix provides another characterization of the bipartite graph whose permanental polynomial can be computed by the characteristic polynomial of the skew adjacency matrix of its orientation graph. As applications, in the last section we deduce explicit formulas of the permanental polynomials of two totally convertible matrices.

2 Totally convertible square $\{0, 1\}$-matrices

Little [11] gave an elegant characterization of a convertible matrix in terms of excluded minors (see Proposition 2.2). Following Little's result, in this section we characterize a totally convertible square {0,1}-matrix.

For convenience, we first introduce some notations and definitions.

For a $\{0, 1\}$-matrix $A = (a_{ij})_{m\times n}$ $(m \leq n)$, we associate a bipartite graph $G_A = (U, W; E)$ with $U = \{u_1, u_2, \cdots, u_m\}$ and $W = \{w_1, w_2, \cdots, w_n\}$ whose edges are those pairs $\{u_i, w_j\}$ for which $a_{ij} \neq 0$. We can see that A is the *biadjacency matrix* of G_A, and the constant term b_n of polynomial $\mathrm{per}(xI - A)$ satisfies [12]

$$b_n = (-1)^n \mathrm{per}(A) = (-1)^n \phi(G_A). \tag{4}$$

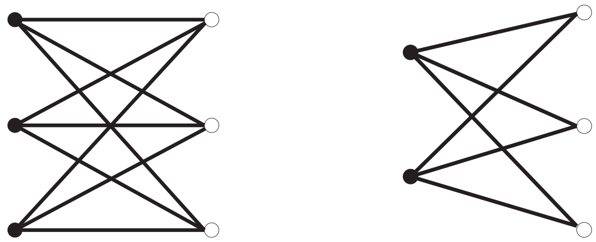
If $u_i \in U$ and $w_j \in W$ are not adjacent in the associated graph G_A and $i \leq j \leq i+n-m$, we call u_iw_j an *auxiliary edge* relative to G_A. The *associated* $(*)$ *bipartite graph* G^*_A is the one obtained from G_A by adding all the auxiliary edges relative to G_A. The auxiliary edges are also said to be the auxiliary edges of G^*_A. For example, see Figure 2(a) (The dashed lines stand for the auxiliary edges). In fact, u_i and w_j, $i \leq j \leq i + n - m$, are always adjacent in G^*_A.

Remark 2.1. *For a square matrix A, G^*_A is obtained from G_A by adding an edge u_iw_i for each i such that G_A does not have u_iw_i.*

Let $B = (b_{ij})_{m\times n}$ be a signing of an $m \times n$ $\{0, 1\}$-matrix A. The oriented bipartite graph $\vec{G}_B$ is obtained from G_A by orienting the edge u_iw_j from u_i to w_j if $b_{ij} = 1$, and from w_j to u_i if $b_{ij} = -1$. The directed bipartite graph $\vec{G}^*_B$ is an orientation of G^*_A such that each auxiliary edge of G^*_A is directed from u_i to w_j, and any other edge is directed from u_i to w_j if $b_{ij} = 1$, and from w_j to u_i if $b_{ij} = -1$. In particular, an orientation of G^*_A is said to be *normal* if for $i = 1, 2, \cdots, m$ and $j = i+r$ $(r \in \{0, 1, \cdots, n-m\})$, each edge u_iw_j is oriented from u_i to w_j.

Proposition 2.2. *[11] For a square $\{0, 1\}$-matrix A, the following three statements are equivalent:*
(1) There exists a signing B of A such that $\mathrm{per}(A) = |\det(B)|$.
(2) The associated bipartite graph G_A is Pfaffian.
(3) G_A contains no even subdivision of $K_{3,3}$ as a central subgraph.
Moreover, in (1) $\vec{G}_B$ is a Pfaffian orientation of G_A.

Figure 1(a) illustrates the graph $K_{3,3}$. We say that a graph G is an *even subdivision* of a graph K if G is obtained from K by replacing some edges of K by internally disjoint paths of odd length. A subgraph H of a graph G is *central* if $G - V(H)$ has a perfect matching. In an oriented graph, a cycle C of even length (*an even cycle*) is *oddly oriented* if it has an odd number of directed edges going in each direction. For two perfect matchings M and M', a cycle in the symmetric difference of M and M' is called an *M-alternating cycle*



(a) (b)

Figure 1. (a) $K_{3,3}$, and (b) $K_{2,3}$.

(or *M'-alternating cycle*). Some equivalent characterizations of a Pfaffian graph in terms of central subgraphs and M-alternating cycles are given as below.

Proposition 2.3. *[12] Let G be a graph with an even number of vertices and $\vec{G}$ an orientation of G. Then the following three properties are equivalent:*
(1) $\vec{G}$ is a Pfaffian orientation of G.
(2) Every central cycle in G is oddly oriented relative to $\vec{G}$.
(3) If G has a perfect matching, then for some perfect matching M, every M-alternating cycle is oddly oriented relative to $\vec{G}$.

Since a central cycle either uses two edges incident with a given vertex or none, in a Pfaffian orientation of a graph, if we reverse the directions of all edges incident with a given vertex, then the resulting orientation remains a Pfaffian orientation.

Lemma 2.4. *For a nonnegative matrix $A = (a_{ij})_{n\times n}$, let $B = (b_{ij})_{n\times n}$ be obtained from A by changing a_{ij} to $-a_{ij}$ for some $i, j \in \{1, 2, \cdots, n\}$ such that* $\operatorname{per}(xI - A) = \det(xI - B)$. *Then $b_{ii} = a_{ii}$ for each $i \in \{1, 2, \cdots, n\}$.*

Proof. Let $\operatorname{per}(xI-A) = \sum_{k=0}^{n} b_k x^{n-k}$ and $\det(xI-B) = \sum_{k=0}^{n} c_k x^{n-k}$. Since $\operatorname{per}(xI-A) = \det(xI - B)$, we have $b_1 = c_1$. By the given condition, we get that $b_{ii} = a_{ii}$ or $-a_{ii}$. By Eqs. (1) and (2), $b_1 = -\sum_{i=1}^{n} a_{ii}$ and $c_1 = -\sum_{i=1}^{n} b_{ii}$. Suppose to the contrary that for some i, $a_{ii} = c$ $(c > 0)$, but $b_{ii} = -c$. Then we get that $b_1 < c_1$, a contradiction. So we obtain that $b_{ii} = a_{ii}$ for all $i \in \{1, 2, \cdots, n\}$. □

Theorem 2.5. *A square $\{0, 1\}$-matrix A is totally convertible if and only if G_A^* is Pfaffian.*

Proof. Let $A = (a_{ij})_{n\times n}$. We fist suppose that A has a signing B such that $\operatorname{per}(xI - A) = \det(xI - B)$. We shall show that $\vec{G}_B^*$ is a Pfaffian orientation of G_A^*.

Case (I): $a_{ii} = 1$ for every $i \in \{1, 2, \cdots, n\}$.

In this case we have that $G_A^* = G_A$ and B is the skew biadjacency matrix of $\vec{G}_B$. Using the fact that $(\operatorname{Pf}(A_s(\vec{G}_B)))^2 = \det(A_s(\vec{G}_B))$ [7, 12], it is easy to check that

$$(\operatorname{Pf}(A_s(\vec{G}_B)))^2 = \det(A_s(\vec{G}_B)) = \det \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} = (\det(B))^2.$$

Since the constant term of $\operatorname{per}(xI - A)$ is equal to the constant term of $\det(xI - B)$, we get that $\operatorname{per}(A) = \det(B)$. By Eq. (4), $\phi(G_A) = \operatorname{per}(A) = \det(B)$. So we obtain that

$\phi^2(G_A) = (\mathrm{Pf}(A_s(\vec{G}_B)))^2$, i.e. $\phi(G_A) = |\mathrm{Pf}(A_s(\vec{G}_B))|$. Therefore, $\vec{G}_B$ is a Pfaffian orientation of G^*_A=G_A.

Case (II): $a_{ii} = 0$ for some $i \in \{1, 2, \cdots, n\}$.

Since $\mathrm{per}(xI-A) = \det(xI-B)$, by Lemma 2.4, $a_{jj} = b_{jj}$ for each $j \in \{1, 2, ..., n\}$. Setting $x = -1$, we get that $\mathrm{per}(A_1) = \det(B_1)$, where $A_1 = I+A$, $B_1 = I+B$. Let $A_1 = (a^1_{ij})_{n\times n}$ and $B_1 = (b^1_{ij})_{n\times n}$. Then the diagonal entries $b^1_{ii} = a^1_{ii} = 1$ or 2, and B_1 is a signing of A_1. Since $\mathrm{per}(A_1) = \sum_{\sigma\in\Lambda_n} a^1_{1\sigma(1)}a^1_{2\sigma(2)}\cdots a^1_{n\sigma(n)}$ and $\det(B_1) = \sum_{\sigma\in\Lambda_n} \mathrm{sgn}(\sigma)b^1_{1\sigma(1)}b^1_{2\sigma(2)}\cdots b^1_{n\sigma(n)}$, we have that for any $\sigma \in \Lambda_n$, $a^1_{1\sigma(1)}a^1_{2\sigma(2)}\cdots a^1_{n\sigma(n)}$=$\mathrm{sgn}(\sigma)b^1_{1\sigma(1)}b^1_{2\sigma(2)}\cdots b^1_{n\sigma(n)}$. Let A'_1 (resp. B'_1) be obtained from A_1 (resp. B_1) by replacing each diagonal entry 2 with 1. Then we obtain a $\{0, 1\}$-matrix A'_1 with a signing B'_1 such that $\mathrm{per}(A'_1) = \det(B'_1)$. Since A'_1 is the biadjacency matrix of G^*_A, we get that $\vec{G}^*_B = \vec{G}_{B'_1}$ is a Pfaffian orientation of G^*_A in the same approach as case (I).

Furthermore, $\vec{G}^*_B$ is also a normal orientation since the all diagonal entries of its skew biadjacency matrix are 1s.

Conversely, suppose that the bipartite graph G^*_A is Pfaffian. By Proposition 2.2, there exists a signing B^*_0 of the biadjacency matrix A^* of G^*_A such that $\mathrm{per}(A^*) = |\det(B^*_0)|$, and the oriented bipartite graph $\vec{G}_{B^*_0}$ is a Pfaffian orientation of G^*_A. For each vertex $u_i \in U$ such that the edge u_iw_i is directed from w_i to u_i, we reverse all the directions of edges incident to u_i. After these operations, all the central cycles are still oddly oriented. By Proposition 2.3, the resulting new orientation, denoted by $\vec{G}$, is a Pfaffian and normal orientation with each edge u_iw_i directed from u_i to w_i.

Let B^* be the skew biadjacency matrix of $\vec{G}$. As $G^*_A = G_{A^*}$ is Pfaffian, for $\omega \in Q_{k,n}$ ($k = 1, 2, \cdots, n$), the subgraph $G_{A^*[\omega]}$ of G^*_A is clearly Pfaffian since $G_{A^*[\omega]}$ is central in G^*_A, i.e. $\mathrm{per}(A^*[\omega]) = |\det(B^*[\omega])|$. Denote by $A^*[\omega] = (a'_{ij})_{k\times k}$ and $B^*[\omega] = (b'_{ij})_{k\times k}$. Then for $i = 1, 2, \cdots, n$, $a'_{ii} = b'_{ii} = 1$. By definitions,

$$\mathrm{per}(A^*[\omega]) = \sum_{\sigma\in\Lambda_k} a'_{1\sigma_1}a'_{2\sigma_2}\cdots a'_{k\sigma_k}, \quad \det(B^*[\omega]) = \sum_{\sigma\in\Lambda_k} \mathrm{sgn}(\sigma)b'_{1\sigma_1}b'_{2\sigma_2}\cdots b'_{k\sigma_k}. \tag{5}$$

Since $B^*[\omega]$ is a signing of $A^*[\omega]$, we get that for any $\sigma \in \Lambda_k$,

$$a'_{1\sigma_1}a'_{2\sigma_2}\cdots a'_{k\sigma_k} = |\mathrm{sgn}(\sigma)b'_{1\sigma_1}b'_{2\sigma_2}\cdots b'_{k\sigma_k}| = 0, 1.$$

Since $\mathrm{per}(A^*[\omega]) = |\det(B^*[\omega])|$ together with Eq. 5, we have that for any $\sigma, \sigma' \in \Lambda_k$,

$$\mathrm{sgn}(\sigma)b'_{1\sigma_1}b'_{2\sigma_2}\cdots b'_{k\sigma_k} = \mathrm{sgn}(\sigma')b'_{1\sigma'_1}b'_{2\sigma'_2}\cdots b'_{k\sigma'_k},$$

whenever they are both non-zeroes. In particular, for the given $\sigma = (1)(2)\cdots(k)$, we have that $a'_{1\sigma_1}a'_{2\sigma_2}\cdots a'_{k\sigma_k} = \mathrm{sgn}(\sigma)b'_{1\sigma_1}b'_{2\sigma_2}\cdots b'_{k\sigma_k}$=1. Hence, for each $\sigma \in \Lambda_k$,

$$a'_{1\sigma_1}a'_{2\sigma_2}\cdots a'_{k\sigma_k} = \mathrm{sgn}(\sigma)b'_{1\sigma_1}b'_{2\sigma_2}\cdots b'_{k\sigma_k}.$$

This shows that $\mathrm{per}(A^*[\omega]) = \det(B^*[\omega])$. For the orientation subgraph of $\vec{G}$ restricted to G_A, let B be its skew biadjacency matrix. Then the above discussions show that $\mathrm{per}(A[\omega]) = \det(B[\omega])$ holds for any $\omega \in Q_{k,n}$.

Since

$$\mathrm{per}(xI - A) = \sum_{k=0}^{n} x^{n-k}(-1)^k \sum_{\omega \in Q_{k,n}} \mathrm{per}(A[\omega])$$

and

$$\det(xI - B) = \sum_{k=0}^{n} x^{n-k}(-1)^k \sum_{\omega \in Q_{k,n}} \det(B[\omega]),$$

$\mathrm{per}(xI - A) = \det(xI - B)$ holds and A is totally convertible. $\square$

Based on this theorem, we immediately deduce that testing totally convertibility of a square matrix A is reduced to testing the Pfaffian property of G_A^*.

By the proof of Theorem 2.5 we have the following immediate corollaries.

Corollary 2.6. *For a totally convertible square matrix A, if there exists a signing B of A such that* $\mathrm{per}(xI - A) = \det(xI - B)$*, then $\vec{G}_B^*$ is a Pfaffian and normal orientation of G_A^*.*

Corollary 2.7. *Let A be a totally convertible square matrix, D the restriction of a Pfaffian and normal orientation of G_A^* on G_A. Then the skew biadjacency matrix B of D is a signing of A such that* $\mathrm{per}(xI - A) = \det(xI - B)$.

3 Totally convertible $m \times n$ $\{0, 1\}$-matrices

In this section we try to characterize those $m \times n$ $\{0, 1\}$-matrices which are totally convertible. If not specified, we suppose $m \le n$.

Lemma 3.1. *An $m \times n$ $\{0, 1\}$-matrix A is totally convertible if and only if there exists a signing B of A such that for any $\omega \in Q_{m,n}$, $\vec{G}_{B[\omega]}^*$ is a Pfaffian and normal orientation of $\vec{G}_{A[\omega]}^*$.*

Proof. The result follows from Corollaries 2.6 and 2.7. $\square$

For a bipartite graph $G = (U, W; E)$ with $|U| \le |W|$, a matching M is *left-perfect* if $|M| = |U|$. A subgraph H of G is *left-central* if $G - V(H)$ has a left-perfect matching. A subgraph H of G_A^* is *left-central* $(*)$ if, for any $\omega \in Q_{m,n}$ such that $G_{A[\omega]}^*$ has H as a subgraph, $G_{A[\omega]}^* - V(H)$ has a perfect matching.

Remark 3.2. *For an $m \times n$ $\{0, 1\}$-matrix A and $\omega \in Q_{m,n}$, $G_{A[\omega]}^*$ is the associated $(*)$ bipartite graph of the $m \times m$ matrix $A[\omega]$ and it is a subgraph of G_A^*. $G_{A[\omega]}^*$ is different from the subgraph $G_A^*[\omega]$ of G_A^* induced by all the vertices in U and the vertices in W with indexes in ω. For example, let*

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

For $\omega = (2, 3, 4)$, the graphs $G_{A_0}^*$, $G_{A_0[\omega]}^*$ and $G_{A_0}^*[\omega]$ are shown in Figure 2.

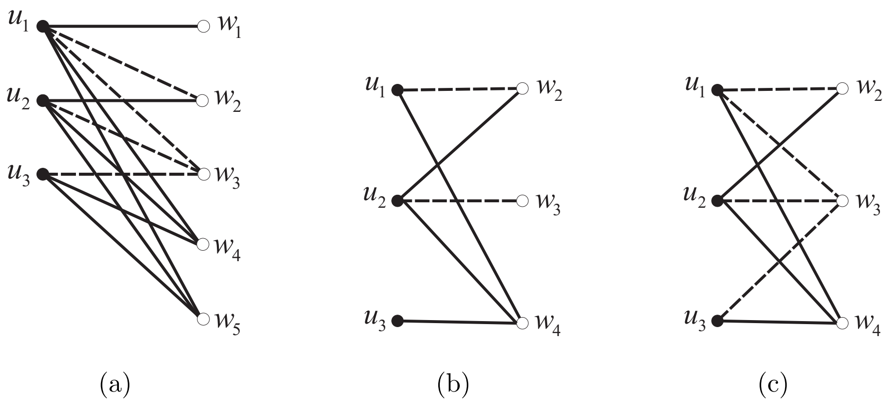


Figure 2. (a) $G^*_{A_0}$, (b) $G^*_{A_0[\omega]}$ and (c) $G^*_{A_0}[\omega]$.

Theorem 3.3. *An $m \times n$ $\{0,1\}$-matrix A is totally convertible if and only if there exists a normal orientation of G^*_A such that each left-central $(*)$ cycle is oddly oriented.*

Proof. If A is totally convertible, let B be a signing of A such that for any $\omega \in Q_{m,n}$, $\mathrm{per}(xI - A[\omega]) = \det(xI - B[\omega])$. By Corollary 2.6 and Lemma 3.1, for any $\omega \in Q_{m,n}$, $\vec{G}^*_{B[\omega]}$ is a Pfaffian and normal orientation of $G^*_{A[\omega]}$. Thus $\vec{G}^*_B$ is normal. Let C be a left-central $(*)$ cycle of G^*_A. Then C is a central cycle of $G^*_{A[\omega]}$ for some $\omega \in Q_{m,n}$. So it is oddly oriented in $\vec{G}^*_{B[\omega]}$ and therefore oddly oriented in $\vec{G}^*_B$.

Let $\vec{G}$ be an orientation of G^*_A such that each left-central $(*)$ cycle is oddly oriented and B the skew biadjacency matrix of the oriented graph obtained from $\vec{G}$ by deleting all the auxiliary edges. Then for any $\omega \in Q_{m,n}$, any central cycle of $\vec{G}^*_{B[\omega]}$ is oddly oriented and $\vec{G}^*_{B[\omega]}$ is a Pfaffian orientation of $G^*_{A[\omega]}$. As $\vec{G}$ is normal, $\vec{G}^*_{B[\omega]}$ is normal for any $\omega \in Q_{m,n}$. By Lemma 3.1, we obtain that A is totally convertible. □

Based on the above results, we have the following consequences.

Corollary 3.4. *Let A be an $m \times n$ $\{0,1\}$-matrix. Then the skew biadjacency matrix B of D is a signing of A satisfying $\mathrm{per}(xI - A[\omega]) = \det(xI - B[\omega])$ for any $\omega \in Q_{m,n}$, where D is an orientation graph obtained by restricting a normal orientation of G^*_A with each left-central $(*)$ cycle being oddly oriented to G_A.*

Corollary 3.5. *If an $m \times n$ $\{0,1\}$-matrix A is totally convertible, then G^*_A contains no even subdivision of $K_{3,3}$ as a left-central $(*)$ subgraph.*

Proof. Suppose to the contrary that G^*_A contains a left-central $(*)$ subgraph H^* which is isomorphic to an even subdivision of $K_{3,3}$. By definition, there exists a $\omega \in Q_{m,n}$ such that H^* is a central subgraph of $G^*_{A[\omega]}$. Then by Proposition 2.2, the graph $G^*_{A[\omega]}$ is not Pfaffian. By Lemma 3.1, A is not totally convertible. This is a contradiction. □

For a bipartite graph $G = (U, W; E)$ with $|U| \leq |W|$, a *totally Pfaffian orientation* of G is an orientation such that each left-central cycle is oddly oriented. If a graph admits a totally Pfaffian orientation, then it is *totally Pfaffian.* In [9] Kakimura gave a characterization of a totally Pfaffian bipartite graph as below.

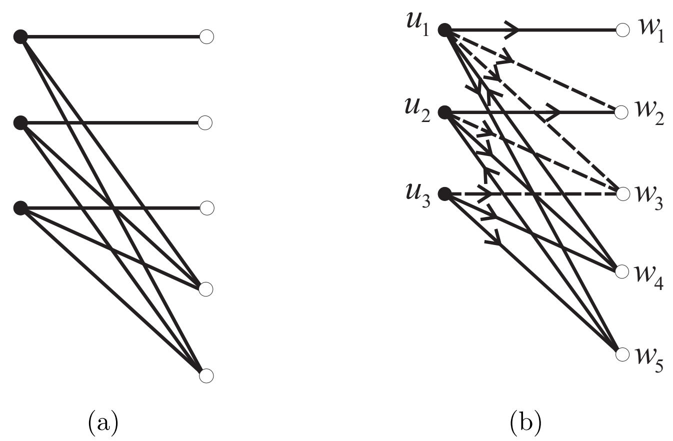


Figure 3. (a) $L_{3,5}$ and (b) an orientation of $L_{3,5}$.

Proposition 3.6. *[9] A bipartite graph is totally Pfaffian if and only if it contains no even subdivision of $K_{3,3}$, $K_{2,3}$ and $L_{3,5}$ as a left-central subgraph.*

See Figure 1(b) and Figure 3(a) for $K_{2,3}$ and $L_{3,5}$, respectively. By definitions, a left-central $(*)$ cycle of G_A^* is a left-central cycle. The following corollary follows immediately.

Corollary 3.7. *Let A be an $m \times n$ $\{0,1\}$-matrix. If G_A^* admits a normal and totally Pfaffian orientation, then A is totally convertible.*

Remark 3.8. *If A is totally convertible, then G_A^* may be not totally Pfaffian. For example, see matrix A_0 in Remark 3.2. Since the graph $G_{A_0}^*$ contains $L_{3,5}$ as a left-central subgraph, $G_{A_0}^*$ is not totally Pfaffian, but it admits an orientation such that each left-central $(*)$ cycle is oddly oriented. As shown in Figure 3(b), the left-central $(*)$ cycles $u_2w_4u_3w_5u_2$, $u_1w_2u_2w_5u_1$, $u_1w_2u_2w_4u_1$, $u_1w_2u_2w_4u_3w_5u_1$ are all oddly oriented. By Corollary 3.4,*

$$B_0 = \begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

is the signed matrix of A_0 such that for any $\omega \in Q_{3,5}$, $\mathrm{per}(xI - A_0[\omega]) = \det(xI - B_0[\omega])$.

4 The permanental polynomial of a bipartite graph

For a graph G on n vertices, the *adjacency matrix* $A(G) = (a_{ij})_{n\times n}$ is the matrix with rows and columns indexed by the vertices of G such that $a_{ij} = 1$ if there is an edge in G joining vertices v_i and v_j, and $a_{ij} = 0$ otherwise. The *permanental polynomial* of G is defined as $\pi(G,x) = \mathrm{per}(xI - A(G))$. Note that the graph G considered here is simple.

In [22] Yan and zhang considered computing the permanental polynomial of a bipartite graph through the characteristic polynomial of the skew adjacency matrix of an oriented graph; in [23] the present authors gave two characterizations (see Theorem 4.1). Now we establish another equivalent characterization of Theorem 4.1 by the result of totally convertible matrix.

Theorem 4.1. *[23] For a bipartite graph G, the following three conditions are equivalent:*
(1) There exists an orientation $\vec{G}$ of G such that $\pi(G,x) = \det(xI - A_s(\vec{G}))$.
(2) There exists an orientation $\vec{G}$ of G such that each cycle is oddly oriented.
(3) G contains no even subdivision of $K_{2,3}$.

Theorem 4.2. *For a bipartite graph G on n vertices, there exists an orientation $\vec{G}$ such that $\pi(G,x) = \det(xI - A_s(\vec{G}))$ if and only if $G^*_{A(G)}$ is Pfaffian.*

Proof. By Theorem 2.5, we only need to show that there exists an orientation $\vec{G}$ such that $\pi(G,x) = \det(xI - A_s(\vec{G}))$ if and only if $A(G)$ is totally convertible. For a graph G, the skew adjacency matrix $A_s(\vec{G})$ of an orientation graph $\vec{G}$ is a signing of the adjacency matrix $A(G)$. Hence if an orientation $\vec{G}$ exists satisfying $\pi(G,x) = \det(xI - A_s(\vec{G}))$, then $A(G)$ is totally convertible. Conversely, if $A(G)$ is totally convertible, then there is a signing $B = (b_{ij})_{n\times n}$ of $A(G)$ such that $\pi(G,x) = \mathrm{per}(xI - A(G)) = \det(xI - A_s(\vec{G}))$, then we show that for any i, j ($i \neq j$), $b_{ij} = -b_{ji}$. Let $\mathrm{per}(xI - A(G)) = \sum_{k=0}^{n} b_k x^{n-k}$ and $\det(xI - A_s(\vec{G})) = \sum_{k=0}^{n} c_k x^{n-k}$. Since $b_2 = c_2$, we get that $\sum_{\omega\in Q_{2,n}} \mathrm{per}(A[\omega]) = \sum_{\omega\in Q_{2,n}} \det(A[\omega])$ by equations (1) and (2), i.e. $\sum_{i,j(i\neq j)} a_{ij}\cdot a_{ji} = \sum_{i,j(i\neq j)} -b_{ij}\cdot b_{ji}$. As $a_{ij} = a_{ji} = 0$ or 1, $b_{ij} = -b_{ji}$ holds. Hence B is skew symmetric and it is the skew adjacency matrix $A_s(\vec{G})$ of some orientation graph $\vec{G}$ of G. □

Based on the above results, we obtain the following corollary.

Corollary 4.3. *A bipartite graph G contains no even subdivision of $K_{2,3}$ if and only if $G^*_{A(G)}$ contains no even subdivision of $K_{3,3}$ as a central subgraph.*

5 Examples

In this section, by establishing Pfaffian orientations, we will compute the permanental polynomials of some totally convertible matrices.

Lemma 5.1. *[19] Define $n\times n$ matrices U and U^{-1} with components $1 \leq k, k' \leq n$:*

$$(U)_{k,k'} = \sqrt{\frac{2}{n+1}}\, i^k \sin(\frac{kk'\pi}{n+1}), \qquad (U^{-1})_{k,k'} = \sqrt{\frac{2}{n+1}}\, (-i)^{k'} \sin(\frac{kk'\pi}{n+1}).$$

Let Q be the $n\times n$ matrix $\begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}$. *Then the matrix $\widetilde{Q} = U^{-1}QU$ has the element $(\widetilde{Q})_{k,k'} = \delta_{k,k'}\cdot 2i\cos\frac{k\pi}{n+1}$ for $1 \leq k, k' \leq n$ and $i^2 = -1$.*

Theorem 5.2. *Let* $A_1 = \begin{pmatrix} 1 & 1 & & & \\ 1 & 1 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \end{pmatrix}$ *be an* $n \times n$ *matrix.*

Then

$$\mathrm{per}(xI - A_1) = \prod_{t=1}^{n} (x - 1 + 2i\cos\frac{t\pi}{n+1}). \tag{6}$$

Proof. We construct the bipartite graph $G^*_{A_1}$ and the orientation graph $\vec{G}^*_{A_1}$ as shown in Figure 4(a). Let M_0 be the perfect matching of G^* containing the edges $(u_1, w_1), (u_2, w_2), \cdots,$ (u_n, w_n). We can see that each M_0-alternating cycle takes the form $(u_i w_i u_{i+1} w_{i+1} u_i)$ $(i \in \{1, 2, \cdots, n-1\})$, and is oddly oriented in $\vec{G}^*_{A_1}$. In addition, each edge (u_i, w_i) is directed from u_i to w_i. So $\vec{G}^*_{A_1}$ is a Pfaffian and normal orientation of $G^*_{A_1}$. Let B_1 be the skew biadjacency matrix of $\vec{G}^*_{A_1} = \vec{G}_{A_1}$. By Corollary 2.7, we have that

$$\mathrm{per}(xI - A_1) = \det(xI - B_1) = \det \begin{pmatrix} x-1 & 1 & & & \\ -1 & x-1 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & x-1 & 1 \\ & & & -1 & x-1 \end{pmatrix}.$$

Conjugate the matrix $(xI - B_1)$ by U_n to obtain $U_n^{-1}(xI - B_1)U_n = \mathrm{diag}(x - 1 + 2i\cos\frac{\pi}{n+1}, x-1+2i\cos\frac{2\pi}{n+1}, \cdots, x-1+2i\cos\frac{n\pi}{n+1})$. So $\mathrm{per}(xI-A_1) = \det(U_n^{-1}(xI-B_1)U_n) = \prod_{t=1}^{n}(x - 1 + 2i\cos\frac{t\pi}{n+1})$. □

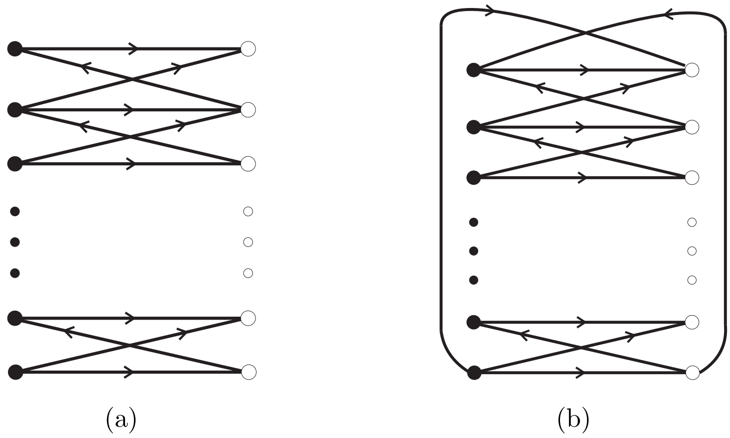


Figure 4. $G^*_{A_1}$ and $G^*_{A_2}$.

Lemma 5.3. *[21] Define* $n \times n$ *matrices* V_n *and* V_n^{-1} *with components* $1 \leq t, j \leq n$*:*

$$(V_n)_{t,j} = \sqrt{\frac{1}{n}} e^{i\frac{(2j-1)t\pi}{n}}, \qquad (V_n^{-1})_{t,j} = \sqrt{\frac{1}{n}} e^{-i\frac{(2t-1)j\pi}{n}}.$$

Let Y_n be the $n \times n$ matrix $\begin{pmatrix} 0 & 1 & & & & 1 \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 1 \\ -1 & & & & -1 & 0 \end{pmatrix}$. *Then the matrix* $\widetilde{Y}_n = V_n^{-1} Y_n V_n$ *has the element* $(\widetilde{Y}_n)_{t,j} = \delta_{t,j} \cdot 2i \sin \frac{(2t-1)\pi}{n}$ *for* $1 \leq t, j \leq n$ *and* $i^2 = -1$.

Theorem 5.4. *Let* $A_2 = \begin{pmatrix} 1 & 1 & & & 1 \\ 1 & 1 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 1 & 1 \\ 1 & & & 1 & 1 \end{pmatrix}$ *be an $n \times n$ matrix (n is even). Then*

$$\mathrm{per}(xI - A_2) = \prod_{t=1}^{n} (x - 1 + 2i \sin \frac{(2t-1)\pi}{n}). \tag{7}$$

Proof. For the graph $G^*_{A_2} = (U, W)$, we give an orientation $\vec{G}^*_{A_2}$ as shown in Figure 4(b). Denote by M_0 the perfect matching $(u_1, w_1), (u_2, w_2), \cdots, (u_n, w_n)$. An M_0-alternating cycle of $G^*_{A_2}$ either takes the form $(u_i w_i u_{i+1} w_{i+1} u_i)$ (mod n) ($i \in \{1, 2, \cdots, n\}$) or contains all the vertices of $G^*_{A_2}$. Since n is even, all the M_0-alternating cycles of $\vec{G}^*_{A_2}$ are oddly oriented. Thus $\vec{G}^*_{A_2}$ is a Pfaffian orientation of $G^*_{A_2}$. As $\vec{G}^*_{A_2}$ is normal and $\vec{G}^*_{A_2} = \vec{G}_{A_2}$, by Theorem 2.5, the skew biadjacency matrix B_2 of $\vec{G}^*_{A_2}$ satisfies that

$$\mathrm{per}(xI - A_2) = \det(xI - B_2) = \det \begin{pmatrix} x-1 & 1 & & & 1 \\ -1 & x-1 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & x-1 & 1 \\ -1 & & & -1 & x-1 \end{pmatrix}.$$

Conjugating $(xI - B_2)$ by V_n, we obtain that $V_n^{-1}(xI - B_2)V_n = \mathrm{diag}(x - 1 + 2i \sin \frac{\pi}{n}, x - 1 + 2i \sin \frac{3\pi}{n}, \cdots, x - 1 + 2i \sin \frac{(2n-1)\pi}{n})$. So $\mathrm{per}(xI - A_2) = \det(V_n^{-1}(xI - B_2)V_n) = \prod_{t=1}^{n}(x - 1 + 2i \sin \frac{(2t-1)\pi}{n})$ holds. $\square$

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