Interval oscillation criteria for second order damped differential equations with mixed nonlinearities *

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Abstract: In this paper, we consider the interval oscillation criteria for second order damped differential equations with mixed nonlinearities

$$\big(r(t)(x'(t))^{\gamma}\big)' + p(t)(x'(t))^{\gamma} + \sum_{i=0}^{n} q_i(t)\,|x(g_i(t))|^{\alpha_i}\operatorname{sgn} x(g_i(t)) = e(t),$$

where γ is a quotient of odd positive integers, $\alpha_0 = \gamma$, $\alpha_i > 0$, $i = 1,\ 2,\ \cdots,\ n$ with $r,\ p,\ e$ and $q_i \in C([t_0,\infty),\mathbb{R})$, $r(t) > 0$, $g_i:\ \mathbb{R}\to\mathbb{R}$ are nondecreasing continuous functions on $\mathbb{R}$ and $\lim_{t\to\infty} g_i(t) = \infty$, $i = 0,\ 1,\ 2,\ \cdots,\ n$. Our results in this paper extend and improve some known results. Some examples are given here to illustrate our main results.
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1 Introduction

In this paper, we are concerned with the interval oscillation criteria for the certain second order damped differential equations containing mixed nonlinearities of the form

$$(r(t)(x'(t))^{\gamma})' + p(t)(x'(t))^{\gamma} + \sum_{i=0}^{n} q_i(t)\,|x(g_i(t))|^{\alpha_i}\operatorname{sgn} x(g_i(t)) = e(t), \quad t \ge t_0, \tag{1.1}$$

where γ is a quotient of odd positive integers, $\alpha_0 = \gamma$, $\alpha_1 > \alpha_2 > \cdots > \alpha_m > \gamma > \alpha_{m+1} > \cdots > \alpha_n > 0$ $(n > m \ge 1)$. We also assume that $r,\ p,\ e$ and $q_i \in C([t_0,\infty),\mathbb{R})$, $r(t) > 0$, $g_i:\ \mathbb{R}\to\mathbb{R}$ are nondecreasing continuous functions on $\mathbb{R}$, $\lim_{t\to\infty} g_i(t) = \infty$, $i = 0,\ 1,\ 2,\ \cdots,\ n$.

By a solution of Eq. (1.1), we mean a function $x \in C^1([t_x,\infty),\mathbb{R})$, $t_x \ge t_0$, which has the property $r(x')^{\gamma} \in C^1([t_x,\infty),\mathbb{R})$ and satisfies Eq. (1.1) on $[t_x,\infty)$. A nontrivial solution of Eq.

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(1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Eq. (1.1) is said to be oscillatory if all its solutions are oscillatory.

In the last few decades, a great deal of effort has been spent in obtaining sufficient conditions for the oscillation or nonoscillation of solutions of the second order and higher order differential equations without forcing terms and it is usually assumed that the potential function q is positive. We refer the reader to the papers [1–18] and the references cited therein.

However, from the Sturm Separation Theorem, we see that oscillation is only an interval property, i.e., if there exists a sequence of subintervals $[a_i, b_i]$ of $[t_o, \infty)$, as $a_i \to \infty$, such that for each i, there exists a solution of equation (1.1) that has at least two zeros in $[a_i, b_i]$, then every solution of equation (1.1) is oscillatory, no matter how equation (1.1) is on the remaining parts of $[t_o, \infty)$ ([19]). Recently, it has been an increasing interest in establishing interval oscillation criteria for second order differential equations with forcing terms, see [19–34].

In 2004, Li [22] considered the problem of interval oscillation of second order quasi-linear differential equation with forced term of the form

$$\left(r(t)|y'(t)|^{\alpha-1}y'(t)\right)' + p(t)|y'(t)|^{\alpha-1}y'(t) + q(t)|y(t)|^{\beta-1}y(t) = e(t), \quad t \geq t_0, \tag{1.2}$$

where $\beta > \alpha > 0$ are constants, $r \in C([t_0, \infty), (0, \infty))$, $p,\ q,\ e \in C([t_0, \infty), \mathbb{R})$. By using two inequalities as well as averaging functions, the author obtained several interval criteria for oscillation, that was, criteria given by the behavior of Eq. (1.2) only on a sequence of subintervals of $[t_0, \infty)$. These oscillation criteria extended some known results.

Li et al. [30] studied the oscillation of second-order functional differential equations with mixed nonlinearities

$$(p(t)x'(t))' + q(t)x(t-\tau) + \sum_{i=1}^{n} q_i(t)\,|x(t-\tau)|^{\alpha_i} \operatorname{sgn} x(t-\tau) = e(t), \quad t \geq t_0,$$

where $\tau \geq 0$. Without assume that the functions $q,\ q_i,\ e$ are nonnegative, the results in this paper extended the results given in [25].

In 2011, Hassan et al. [33] were concerned with the oscillatory behavior of the following forced second order differential equations with mixed nonlinearities

$$(a(t)(x'(t))^\gamma)' + p_0(t)x^\gamma(t) + \sum_{i=1}^{n} p_i(t)\,|x(t)|^{\alpha_i} \operatorname{sgn} x(t) = e(t), \quad t \geq t_0 \tag{1.3}$$

and

$$(a(t)(x'(t))^\gamma)' + p_0(t)x^\gamma(g_0(t)) + \sum_{i=1}^{n} p_i(t)\,|x(g_i(t))|^{\alpha_i} \operatorname{sgn} x(g_i(t)) = e(t), \quad t \geq t_0, \tag{1.4}$$

where γ is a quotient of odd positive integers, $\alpha_i > 0,\ i = 1,\ 2,\ \cdots,\ n$ and $\alpha_i > \gamma$ for $i = 1,\ 2,\ \cdots,\ m$, $\alpha_i < \gamma$ for $i = m+1,\ m+2,\ \cdots,\ n$ with $a,\ e$ and $p_i \in C([t_0, \infty), \mathbb{R})$, $a(t) > 0$, $g_i :\ \mathbb{R} \to \mathbb{R}$ are positive nondecreasing continuous functions on $\mathbb{R}$ and $\lim_{t\to\infty} g_i(t) = \infty$ for $i = 0,\ 1,\ 2,\ \cdots,\ n$. The authors established some sufficient conditions for the oscillation of Eq. (1.3) and Eq. (1.4) that did not assume that e and p_i, $i = 0,\ 1,\ 2,\ \cdots,\ n$ are of definite sign. The results generalized and improved the results in [24], which studied interval oscillation criteria for special case for Eq. (1.3) in case $\gamma = 1$.

In this paper, we intend to use the Riccati transformation technique to obtain some interval oscillation criteria for Eq. (1.1). Our results do not require that the functions p, q_i and e, $i = 0,\ 1,\ 2,\ \cdots,\ n$ are of definite sign and are based on the information only on a sequence of subintervals of $[t_0, \infty)$ rather than the whole half-line. To the best of our knowledge, nothing is known regarding the oscillation criteria for a damped differential equations with mixed nonlinearities and with delayed or advanced arguments. As far as we are aware, these types of equations were not studied earlier, so our results initiate the study. Our results obtained here improve and extend the main results of [22–25, 27, 30, 32, 33].

The paper is organized as follows: In the next section, we present some lemmas which will be used in the following results. In Section 3, using the Riccati transformation technique and inequalities, we establish some new interval criteria for oscillation of Eq. (1.1). In Section 4, we give two examples to illustrate Theorems 3.1 and 3.5, respectively.

2 Some preliminary lemmas

Before stating our main results, we begin with the following lemmas which will play important roles in the proof of the main results.

Lemma 2.1 *([24, Lemma 1]) Let $(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n)$ be an n-tuple satisfying*

$$\alpha_1 > \alpha_2 > \cdots > \alpha_m > \gamma > \alpha_{m+1} > \cdots > \alpha_n > 0.$$

Then there exists an n-tuple $(\eta_1,\ \eta_2,\ \cdots,\ \eta_n)$ with $0 < \eta_i < 1$ satisfying

$$\sum_{i=1}^{n} \alpha_i \eta_i = \gamma, \tag{2.1}$$

and which also satisfies either

$$\sum_{i=1}^{n} \eta_i < 1 \tag{2.2}$$

or

$$\sum_{i=1}^{n} \eta_i = 1. \tag{2.3}$$

Lemma 2.2 *([33, Lemma 2.2]) Let $\alpha,\ \beta,\ u,\ A$ and B be positive real numbers and γ be a quotient of odd positive integers. Then*

$$Au^{\gamma} - Bu^{\gamma-\alpha} \geq -\alpha \left(\left(\frac{\gamma-\alpha}{A} \right)^{\gamma-\alpha} \left(\frac{B}{\gamma} \right)^{\gamma} \right)^{\frac{1}{\alpha}}, \quad 0 < \alpha < \gamma, \tag{2.4}$$

$$Au^{\beta-\gamma} + Bu^{-\gamma} \geq \beta \left(\left(\frac{A}{\gamma} \right)^{\gamma} \left(\frac{B}{\beta-\gamma} \right)^{\beta-\gamma} \right)^{\frac{1}{\beta}}, \quad 0 < \gamma < \beta. \tag{2.5}$$

Lemma 2.3 *Suppose that for any $T \geq t_0$, there exist constants $a_k,\ b_k \in [T,\infty)$ such that $a_k < b_k$, $k = 1,\ 2$, with*

$$q_i(t) \geq 0, \quad \text{for} \quad t \in [G_1(a_1), G_2(b_1)) \cup [G_1(a_2), G_2(b_2)), \quad i = 0,\ 1,\ 2,\ \cdots,\ n$$

and

$$(-1)^k e(t) \geq 0, \quad t \in [G_1(a_k), G_2(b_k)), \quad k = 1,\ 2,$$

where $G_1(t) = \min\{t,\ g_0(t),\ g_1(t),\ \cdots,\ g_n(t)\}$ and $G_2(t) = \max\{t,\ g_0(t),\ g_1(t),\ \cdots,\ g_n(t)\}$. Furthermore, assume that Eq. (1.1) has a nonoscillatory solution x on $[t_0,\infty)$. Then for $t \in [a_k, b_k)$ and $k = 1,\ 2$, we have

$$\frac{x(g_i(t))}{x(t)} \geq \delta_{i,k}(t),$$

where for $i = 0,\ 1,\ 2,\ \cdots,\ n$ and $k = 1,\ 2$, we denote

$$\delta_{i,k}(t) = \begin{cases} \phi_{i,k}(t), & g_i(t) \leq t, \\ \xi_{i,k}(t), & g_i(t) > t, \end{cases} \qquad \zeta(t,a_k) = \exp\left(\int_{G_1(a_k)}^{t} \frac{p(s)}{r(s)} \mathrm{d}s \right),$$

$$\phi_{i,k}(t) = \int_{g_i(a_k)}^{g_i(t)} \frac{\mathrm{d}u}{(r(u)\zeta(u,a_k))^{\frac{1}{\gamma}}} \left(\int_{g_i(a_k)}^{t} \frac{\mathrm{d}u}{(r(u)\zeta(u,a_k))^{\frac{1}{\gamma}}} \right)^{-1}$$

and

$$\xi_{i,k}(t) = \int_{g_i(t)}^{g_i(b_k)} \frac{\mathrm{d}u}{(r(u)\zeta(u,a_k))^{\frac{1}{\gamma}}} \left(\int_{t}^{g_i(b_k)} \frac{\mathrm{d}u}{(r(u)\zeta(u,a_k))^{\frac{1}{\gamma}}} \right)^{-1}.$$

Proof. Let x be an eventually positive solution of Eq. (1.1). Then we can pick $T \in [t_0, \infty)$, such that $x(t) > 0$, $x(g_i(t)) > 0$, $i = 0,\ 1,\ 2,\ \cdots,\ n$, for all $t \geq T$. When $x(t)$ and $x(g_i(t))$, $i = 0,\ 1,\ 2,\ \cdots,\ n$ are eventually negative, the proof follows the same argument using the interval $[G_1(a_2), G_2(b_2))$ instead of $[G_1(a_1), G_2(b_1))$. By assumption, we can choose $b_1 > a_1 > T$, such that $q_i(t) \geq 0$ and $e(t) \leq 0$ on $[G_1(a_1), G_2(b_1))$. From Eq. (1.1), we find that

$$(r(t)(x'(t))^\gamma)' + p(t)(x'(t))^\gamma \leq 0,$$

that is

$$(r(t)(x'(t))^\gamma \zeta(t, a_1))' \leq 0,$$

where $\zeta(t, a_1)$ is defined as in Lemma 2.3. Hence, $r(t)(x'(t))^\gamma \zeta(t, a_1)$ is nonincreasing on $[a_1, G_2(b_1))$.

If $g_i(t) \leq t$, then for $i = 0,\ 1,\ 2,\ \cdots,\ n$ and $t \in [a_1, G_2(b_1))$, we have

$$\begin{aligned} x(t) - x(g_i(t)) &= \int_{g_i(t)}^{t} \frac{(r(u)(x'(u))^\gamma \zeta(u, a_1))^{\frac{1}{\gamma}}}{(r(u)\zeta(u, a_1))^{\frac{1}{\gamma}}} \mathrm{d}u \\ &\leq (r(g_i(t))(x'(g_i(t)))^\gamma \zeta(g_i(t), a_1))^{\frac{1}{\gamma}} \int_{g_i(t)}^{t} \frac{\mathrm{d}u}{(r(u)\zeta(u, a_1))^{\frac{1}{\gamma}}}. \end{aligned}$$

Therefore,

$$\frac{x(t)}{x(g_i(t))} \leq 1 + \frac{(r(g_i(t))(x'(g_i(t)))^\gamma \zeta(g_i(t), a_1))^{\frac{1}{\gamma}}}{x(g_i(t))} \int_{g_i(t)}^{t} \frac{\mathrm{d}u}{(r(u)\zeta(u, a_1))^{\frac{1}{\gamma}}}. \tag{2.6}$$

Also, since $g_i(t)$ are nondecreasing, we see that, for $t \in [a_1, G_2(b_1))$,

$$\begin{aligned} x(g_i(t)) > x(g_i(t)) - x(g_i(a_1)) &= \int_{g_i(a_1)}^{g_i(t)} \frac{(r(u)(x'(u))^\gamma \zeta(u, a_1))^{\frac{1}{\gamma}}}{(r(u)\zeta(u, a_1))^{\frac{1}{\gamma}}} \mathrm{d}u \\ &\geq (r(g_i(t))(x'(g_i(t)))^\gamma \zeta(g_i(t), a_1))^{\frac{1}{\gamma}} \int_{g_i(a_1)}^{g_i(t)} \frac{\mathrm{d}u}{(r(u)\zeta(u, a_1))^{\frac{1}{\gamma}}}, \end{aligned}$$

which implies that

$$\frac{(r(g_i(t))(x'(g_i(t)))^\gamma \zeta(g_i(t), a_1))^{\frac{1}{\gamma}}}{x(g_i(t))} < \left(\int_{g_i(a_1)}^{g_i(t)} \frac{\mathrm{d}u}{(r(u)\zeta(u, a_1))^{\frac{1}{\gamma}}} \right)^{-1}, \quad \text{for} \quad t \in [a_1, G_2(b_1)). \tag{2.7}$$

From (2.6) and (2.7), we get

$$\frac{x(t)}{x(g_i(t))} < \int_{g_i(a_1)}^{t} \frac{\mathrm{d}u}{(r(u)\zeta(u, a_1))^{\frac{1}{\gamma}}} \left(\int_{g_i(a_1)}^{g_i(t)} \frac{\mathrm{d}u}{(r(u)\zeta(u, a_1))^{\frac{1}{\gamma}}} \right)^{-1} = \frac{1}{\phi_{i,1}(t)}.$$

Therefore,

$$x(g_i(t)) > \phi_{i,1}(t) x(t), \quad t \in [a_1, G_2(b_1)). \tag{2.8}$$

On the other hand, if $g_i(t) > t$, then for $i = 0,\ 1,\ 2,\ \cdots,\ n$ and $t \in [a_1, G_2(b_1))$, we obtain

$$\begin{aligned} x(g_i(t)) - x(t) &= \int_{t}^{g_i(t)} \frac{(r(u)(x'(u))^\gamma \zeta(u, a_1))^{\frac{1}{\gamma}}}{(r(u)\zeta(u, a_1))^{\frac{1}{\gamma}}} \mathrm{d}u \\ &\geq (r(g_i(t))(x'(g_i(t)))^\gamma \zeta(g_i(t), a_1))^{\frac{1}{\gamma}} \int_{t}^{g_i(t)} \frac{\mathrm{d}u}{(r(u)\zeta(u, a_1))^{\frac{1}{\gamma}}}. \end{aligned}$$

Therefore,

$$\frac{x(t)}{x(g_i(t))} \leq 1 - \frac{(r(g_i(t))(x'(g_i(t)))^\gamma \zeta(g_i(t), a_1))^{\frac{1}{\gamma}}}{x(g_i(t))} \int_{t}^{g_i(t)} \frac{\mathrm{d}u}{(r(u)\zeta(u, a_1))^{\frac{1}{\gamma}}}. \tag{2.9}$$

Also, since $g_i(t)$ are nondecreasing, we see that, for $t \in [a_1, b_1)$,

$$-x(g_i(t)) < x(g_i(b_1)) - x(g_i(t)) = \int_{g_i(t)}^{g_i(b_1)} \frac{(r(u)(x'(u))^\gamma \zeta(u,a_1))^{\frac{1}{\gamma}}}{(r(u)\zeta(u,a_1))^{\frac{1}{\gamma}}} \mathrm{d}u$$

$$\leq (r(g_i(t))(x'(g_i(t)))^\gamma \zeta(g_i(t),a_1))^{\frac{1}{\gamma}} \int_{g_i(t)}^{g_i(b_1)} \frac{\mathrm{d}u}{(r(u)\zeta(u,a_1))^{\frac{1}{\gamma}}},$$

which implies that

$$-\frac{(r(g_i(t))(x'(g_i(t)))^\gamma \zeta(g_i(t),a_1))^{\frac{1}{\gamma}}}{x(g_i(t))} < \left(\int_{g_i(t)}^{g_i(b_1)} \frac{\mathrm{d}u}{(r(u)\zeta(u,a_1))^{\frac{1}{\gamma}}}\right)^{-1}, \quad t \in [a_1, b_1). \tag{2.10}$$

From (2.9) and (2.10), we get

$$\frac{x(t)}{x(g_i(t))} < \int_t^{g_i(b_1)} \frac{\mathrm{d}u}{(r(u)\zeta(u,a_1))^{\frac{1}{\gamma}}} \left(\int_{g_i(t)}^{g_i(b_1)} \frac{\mathrm{d}u}{(r(u)\zeta(u,a_1))^{\frac{1}{\gamma}}}\right)^{-1} = \frac{1}{\xi_{i,1}(t)}.$$

Therefore,

$$x(g_i(t)) > \xi_{i,1}(t)x(t), \quad t \in [a_1, b_1). \tag{2.11}$$

Combining (2.8) and (2.11), we have

$$x(g_i(t)) \geq \delta_{i,1}(t)x(t), \quad i = 0,\ 1,\ 2,\ \cdots,\ n \ \text{ and } \ t \in [a_1, b_1).$$

This completes the proof.

3 Main results

In this section, we will establish some new criteria for oscillation of Eq. (1.1). In the sequel, we say that a function u belongs to a function class

$$\xi(a,b) = \{u \in C^1[a,b] : \ u(a) = u(b) = 0,\ u(t) \not\equiv 0\}, \quad a,\ b \in [t_0, \infty) \text{ with } a < b,$$

denoted by $u \in \xi(a,b)$.

Theorem 3.1 *Suppose that for any $T \geq t_0$, there exist constants $a_k,\ b_k \in [T, \infty)$ such that $a_k < b_k$, $k = 1,\ 2$, with*

$$q_i(t) \geq 0, \quad \textit{for} \ \ t \in [G_1(a_1), G_2(b_1)) \cup [G_1(a_2), G_2(b_2)), \quad i = 0,\ 1,\ 2,\ \cdots,\ n$$

and

$$(-1)^k e(t) \geq 0, \quad \textit{for} \ \ t \in [G_1(a_k), G_2(b_k)), \quad k = 1,\ 2,$$

where G_1 and G_2 are defined as in Lemma 2.3. Furthermore, assume that there exist functions $\rho \in C^1([t_0,\infty), \mathbb{R}^+)$ and $u \in \xi(a_k, b_k)$, $k = 1,\ 2$, such that

$$\int_{a_k}^{b_k} \left[P_{1,k}(t)u^{\gamma+1}(t) - \frac{\rho(t)r(t)}{(\gamma+1)^{\gamma+1}} P^{\gamma+1}(t)\right] \mathrm{d}t > 0, \quad k = 1,\ 2, \tag{3.1}$$

where

$$P_{1,k}(t) = \rho(t)\left(q_0(t)\delta_{0,k}^{\gamma}(t) + (\eta_0^{-1}|e(t)|)^{\eta_0} \prod_{i=1}^{n} (\eta_i^{-1} q_i(t) \delta_{i,k}^{\alpha_i}(t))^{\eta_i}\right),$$

$$\eta_0 = 1 - \sum_{i=1}^{n} \eta_i, \quad P(t) = (\gamma+1)u'(t) + \left(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\right)u(t),$$

$\eta_i > 0$, $i = 1,\ 2,\ \cdots,\ n$ satisfy (2.1) and (2.2) of Lemma 2.1 and $\delta_{i,k}$, $i = 0,\ 1,\ 2,\ \cdots,\ n$ and $k = 1,\ 2$ are defined as in Lemma 2.3. Then every solution of Eq. (1.1) is oscillatory.

Proof. To arrive at a contradiction, suppose that Eq. (1.1) has a nonoscillatory solution x on $[t_0,\infty)$. Without loss of generality, we assume that there exists a $t_1 \ge t_0$, such that $x(t) > 0$, $x(g_i(t)) > 0$, $i = 0,\ 1,\ 2,\ \cdots,\ n$, for all $t \ge t_1$. By assumption, we can choose $b_1 > a_1 > t_1$, such that $q_i(t) \ge 0$ and $e(t) \le 0$ on the interval $[G_1(a_1), G_2(b_1))$. From Lemma 2.3 and Eq. (1.1), we have, for $t \in [a_1, b_1)$,

$$(r(t)(x'(t))^\gamma)' + p(t)(x'(t))^\gamma + \sum_{i=0}^{n} q_i(t)\delta_{i,1}^{\alpha_i}(t)x^{\alpha_i}(t) \le e(t). \tag{3.2}$$

Define the function ω by

$$\omega(t) = \rho(t)\frac{r(t)(x'(t))^\gamma}{x^\gamma(t)}, \quad t \in [a_1, b_1). \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned}\omega'(t) &= -\rho(t)\frac{p(t)(x'(t))^\gamma}{x^\gamma(t)} - \rho(t)q_0(t)\delta_{0,1}^{\gamma}(t) - \rho(t)\sum_{i=1}^{n} q_i(t)\delta_{i,1}^{\alpha_i}(t)x^{\alpha_i-\gamma}(t)\\ &\quad + \frac{\rho(t)e(t)}{x^\gamma(t)} + \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)\frac{\gamma r(t)(x'(t))^{\gamma+1}}{x^{\gamma+1}(t)}\\ &= -\rho(t)q_0(t)\delta_{0,1}^{\gamma}(t) - \rho(t)\sum_{i=1}^{n} q_i(t)\delta_{i,1}^{\alpha_i}(t)x^{\alpha_i-\gamma}(t)\\ &\quad - \frac{\rho(t)|e(t)|}{x^\gamma(t)} + \left(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\right)\omega(t) - \frac{\gamma\omega^{\frac{\gamma+1}{\gamma}}(t)}{(\rho(t)r(t))^{\frac{1}{\gamma}}}.\end{aligned} \tag{3.4}$$

Corresponding to the exponents α_i, $i = 1,\ 2,\ \cdots,\ n$ in Eq. (1.1), let η_i, $i = 1,\ 2,\ \cdots,\ n$ be chosen to satisfy (2.1) and (2.2) in Lemma 2.1, and let $\eta_0 = 1 - \sum_{i=1}^{n}\eta_i$. Employing the arithmetic-geometric mean inequality in [35],

$$\sum_{i=0}^{n}\eta_i u_i \ge \prod_{i=0}^{n} u_i^{\eta_i}, \quad u_i \ge 0,$$

we see that, for $t \in [a_1, b_1)$,

$$\begin{aligned}&|e(t)|x^{-\gamma}(t) + \sum_{i=1}^{n} q_i(t)\delta_{i,1}^{\alpha_i}(t)x^{\alpha_i-\gamma}(t)\\ &= \eta_0(\eta_0^{-1}|e(t)|x^{-\gamma}(t)) + \sum_{i=1}^{n}\eta_i(\eta_i^{-1}q_i(t)\delta_{i,1}^{\alpha_i}(t)x^{\alpha_i-\gamma}(t))\\ &\ge (\eta_0^{-1}|e(t)|x^{-\gamma}(t))^{\eta_0}\prod_{i=1}^{n}(\eta_i^{-1}q_i(t)\delta_{i,1}^{\alpha_i}(t)x^{\alpha_i-\gamma}(t))^{\eta_i}\\ &= (\eta_0^{-1}|e(t)|)^{\eta_0}\prod_{i=1}^{n}(\eta_i^{-1}q_i(t)\delta_{i,1}^{\alpha_i}(t))^{\eta_i}.\end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), we get

$$\omega'(t) \le -P_{1,1}(t) + \left(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\right)\omega(t) - \frac{\gamma\omega^{\frac{\gamma+1}{\gamma}}(t)}{(\rho(t)r(t))^{\frac{1}{\gamma}}}. \tag{3.6}$$

Multiplying (3.6) by $u^{\gamma+1}(t)$ and integrating from a_1 to b_1, we obtain

$$\int_{a_1}^{b_1} u^{\gamma+1}(t)\omega'(t)\mathrm{d}t \le -\int_{a_1}^{b_1} u^{\gamma+1}(t)P_{1,1}(t)\mathrm{d}t$$

$$+\int_{a_1}^{b_1}\left(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\right)u^{\gamma+1}(t)\omega(t)\mathrm{d}t-\int_{a_1}^{b_1}\frac{\gamma u^{\gamma+1}(t)}{(\rho(t)r(t))^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(t)\mathrm{d}t.$$

Using integration by parts on the first integral, we have

$$-\int_{a_1}^{b_1}(\gamma+1)u^{\gamma}(t)u'(t)\omega(t)\mathrm{d}t\le-\int_{a_1}^{b_1}u^{\gamma+1}(t)P_{1,1}(t)\mathrm{d}t$$

$$+\int_{a_1}^{b_1}\left(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\right)u^{\gamma+1}(t)\omega(t)\mathrm{d}t-\int_{a_1}^{b_1}\frac{\gamma u^{\gamma+1}(t)}{(\rho(t)r(t))^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(t)\mathrm{d}t.$$

Thus

$$\int_{a_1}^{b_1}u^{\gamma+1}(t)P_{1,1}(t)\mathrm{d}t\le-\int_{a_1}^{b_1}\frac{\gamma u^{\gamma+1}(t)}{(\rho(t)r(t))^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(t)\mathrm{d}t$$

$$+\int_{a_1}^{b_1}\left[(\gamma+1)u'(t)+u(t)\left(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\right)\right]u^{\gamma}(t)\omega(t)\mathrm{d}t. \tag{3.7}$$

Set

$$F(v)=P(t)u^{\gamma}(t)v-\frac{\gamma u^{\gamma+1}(t)}{(\rho(t)r(t))^{\frac{1}{\gamma}}}v^{\frac{\gamma+1}{\gamma}},$$

where P is defined as in Theorem 3.1. By simple calculation, we find that, F has the maximum

$$F_{\max}=\left(\frac{1}{\gamma+1}\right)^{\gamma+1}P^{\gamma+1}(t)\rho(t)r(t),\quad\text{when}\quad v=\frac{P^{\gamma}(t)\rho(t)r(t)}{(\gamma+1)^{\gamma}u^{\gamma}(t)}. \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$\int_{a_1}^{b_1}u^{\gamma+1}(t)P_{1,1}(t)\mathrm{d}t\le\int_{a_1}^{b_1}\frac{\rho(t)r(t)}{(\gamma+1)^{\gamma+1}}P^{\gamma+1}(t)\mathrm{d}t,$$

which contradicts (3.1). The proof when x is eventually negative follows the same arguments using the interval $[G_1(a_2),G_2(b_2))$ instead of $[G_1(a_1),G_2(b_1))$, where we use $q(t)\ge 0$ and $e(t)\ge 0$ on $[G_1(a_2),G_2(b_2))$. This completes the proof.

Remark 3.1 *If $p(t)\equiv 0$, then Theorem 3.1 reduces to Theorem 2.5 in [33]. Furthermore, if we take $g_i(t)=t$, $i=0,\ 1,\ 2,\ \cdots,\ n$, then Theorem 3.1 reduces to Theorem 2.1 in [33].*

Theorem 3.2 *Suppose that for any $T\ge t_0$, there exist constants $a_k,\ b_k\in[T,\infty)$ such that $a_k<b_k$, $k=1,\ 2$, with*

$$q_i(t)\ge 0,\quad\textit{for}\quad t\in[G_1(a_1),G_2(b_1))\cup[G_1(a_2),G_2(b_2)),\quad i=0,\ 1,\ 2,\ \cdots,\ n$$

and

$$(-1)^k e(t)\ge 0,\quad\textit{for}\quad t\in[G_1(a_k),G_2(b_k)),\quad k=1,\ 2,$$

where G_1 and G_2 are defined as in Lemma 2.3. Furthermore, assume that there exist functions $\rho\in C^1([t_0,\infty),\mathbb{R}^+)$ and $u\in\xi(a_k,b_k)$, $k=1,\ 2$, such that

$$\int_{a_k}^{b_k}\left[P_{2,k}(t)u^{\gamma+1}(t)-\frac{\rho(t)r(t)}{(\gamma+1)^{\gamma+1}}P^{\gamma+1}(t)\right]\mathrm{d}t>0,\quad k=1,\ 2, \tag{3.9}$$

where

$$P_{2,k}(t)=\rho(t)\left(q_0(t)\delta_{0,k}^{\gamma}(t)+\prod_{i=1}^{n}(\eta_i^{-1}q_i(t)\delta_{i,k}^{\alpha_i}(t))^{\eta_i}\right),$$

$\eta_i>0$, $i=1,\ 2,\ \cdots,\ n$ satisfy (2.1) and (2.3) of Lemma 2.1, P is defined as in Theorem 3.1 and $\delta_{i,k}$, $i=0,\ 1,\ 2,\ \cdots,\ n$ and $k=1,\ 2$ are defined as in Lemma 2.3. Then every solution of Eq. (1.1) is oscillatory.

Proof. Suppose that Eq. (1.1) has a nonoscillatory solution x on $[t_0,\infty)$. Without loss of generality, we assume that there exists a $t_1 \ge t_0$, such that $x(t) > 0$, $x(g_i(t)) > 0$, $i = 0,\ 1,\ 2,\ \cdots,\ n$, for all $t \ge t_1$. By assumption, we can choose $b_1 > a_1 > t_1$, such that $q_i(t) \ge 0$ and $e(t) \le 0$ on the interval $[G_1(a_1), G_2(b_1))$. We define the function ω as in the proof of Theorem 3.1. Proceeding as in the proof of Theorem 3.1, we have

$$\omega'(t) \le -\rho(t)q_0(t)\delta_{0,1}^{\gamma}(t) - \rho(t)\sum_{i=1}^{n} q_i(t)\delta_{i,1}^{\alpha_i}(t)x^{\alpha_i-\gamma}(t) + \left(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\right)\omega(t) - \frac{\gamma\omega^{\frac{\gamma+1}{\gamma}}(t)}{(\rho(t)r(t))^{\frac{1}{\gamma}}}. \quad (3.10)$$

Corresponding to the exponents α_i, $i = 1,\ 2,\ \cdots,\ n$ in Eq. (1.1), let η_i, $i = 1,\ 2,\ \cdots,\ n$ be chosen to satisfy (2.1) and (2.3) in Lemma 2.1. Employing the arithmetic-geometric mean inequality in [35],

$$\sum_{i=1}^{n} \eta_i u_i \ge \prod_{i=1}^{n} u_i^{\eta_i}, \quad u_i \ge 0,$$

we get, for $t \in [a_1, b_1)$,

$$\sum_{i=1}^{n} q_i(t)\delta_{i,1}^{\alpha_i}(t)x^{\alpha_i-\gamma}(t) \ge \prod_{i=1}^{n} \eta_i^{-\eta_i}(q_i(t)\delta_{i,1}^{\alpha_i}(t))^{\eta_i}. \quad (3.11)$$

Combining (3.10) and (3.11), we obtain

$$\omega'(t) \le -P_{2,1}(t) + \left(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\right)\omega(t) - \frac{\gamma\omega^{\frac{\gamma+1}{\gamma}}(t)}{(\rho(t)r(t))^{\frac{1}{\gamma}}}.$$

The remainder of the proof is similar to that of Theorem 3.1, so is omitted. Then the theorem is proved.

Remark 3.2 *If $p(t) \equiv 0$, then Theorem 3.2 reduces to Theorem 2.6 in [33]. Furthermore, if we take $g_i(t) = t$, $i = 0,\ 1,\ 2,\ \cdots,\ n$, then Theorem 3.2 reduces to Theorem 2.2 in [33].*

Theorem 3.3 *Suppose that for any $T \ge t_0$, there exist constants $a_k,\ b_k \in [T,\infty)$ such that $a_k < b_k$, $k = 1,\ 2$, with*

$$q_i(t) \ge 0, \quad \text{for} \quad t \in [G_1(a_1), G_2(b_1)) \cup [G_1(a_2), G_2(b_2)), \quad i = 0,\ 1,\ 2,\ \cdots,\ n$$

and

$$(-1)^k e(t) \ge 0, \quad \text{for} \quad t \in [G_1(a_k), G_2(b_k)), \quad k = 1,\ 2,$$

where G_1 and G_2 are defined as in Lemma 2.3. Furthermore, assume that there exist functions $\rho \in C^1([t_0,\infty), \mathbb{R}^+)$ and $u \in \xi(a_k, b_k)$, $k = 1,\ 2$, such that

$$\int_{a_k}^{b_k} \left[P_{3,k}(t)u^{\gamma+1}(t) - \frac{\rho(t)r(t)}{(\gamma+1)^{\gamma+1}}P^{\gamma+1}(t)\right] \mathrm{d}t > 0, \quad k = 1,\ 2, \quad (3.12)$$

where

$$P_{3,k}(t) = \rho(t)q_0(t)\delta_{0,k}^{\gamma}(t) + \rho(t)\sum_{i=1}^{n} \alpha_i \left(\left(\frac{q_i(t)\delta_{i,k}^{\alpha_i}(t)}{\gamma}\right)^{\gamma}\left(\frac{\lambda_i|e(t)|}{\alpha_i-\gamma}\right)^{\alpha_i-\gamma}\right)^{\frac{1}{\alpha_i}}$$

λ_i are positive numbers with $\sum_{i=1}^{n}\lambda_i = 1$, P is defined as in Theorem 3.1 and $\delta_{i,k}$, $i = 0,\ 1,\ 2,\ \cdots,\ n$ and $k = 1,\ 2$ are defined as in Lemma 2.3. Then every solution of Eq. (1.1) is oscillatory.

Proof. Suppose that Eq. (1.1) has a nonoscillatory solution x on $[t_0,\infty)$. Without loss of generality, we assume that there exists a $t_1 \ge t_0$, such that $x(t) > 0$, $x(g_i(t)) > 0$, $i = 0,\ 1,\ 2,\ \cdots,\ n$, for all $t \ge t_1$. By assumption, we can choose $b_1 > a_1 > t_1$, such that $q_i(t) \ge 0$ and $e(t) \le 0$ on the interval $[G_1(a_1), G_2(b_1))$. We define ω as in the proof of Theorem 3.1. Then from (3.4), we find that

$$\omega'(t) = -\rho(t)q_0(t)\delta_{0,1}^{\gamma}(t) - \rho(t)\sum_{i=1}^{n}\left[q_i(t)\delta_{i,1}^{\alpha_i}(t)x^{\alpha_i-\gamma}(t) + \lambda_i|e(t)|x^{-\gamma}(t)\right]$$

$$+\left(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\right)\omega(t)-\frac{\gamma\omega^{\frac{\gamma+1}{\gamma}}(t)}{(\rho(t)r(t))^{\frac{1}{\gamma}}}. \tag{3.13}$$

From (2.5), we get, for $t\in(a_1,b_1)$ and $i=0,\ 1,\ 2,\ \cdots,\ m$,

$$q_i(t)\delta_{i,1}^{\alpha_i}(t)x^{\alpha_i-\gamma}(t)+\lambda_i|e(t)|x^{-\gamma}(t)\geq\alpha_i\left(\left(\frac{q_i(t)\delta_{i,k}^{\alpha_i}(t)}{\gamma}\right)^{\gamma}\left(\frac{\lambda_i|e(t)|}{\alpha_i-\gamma}\right)^{\alpha_i-\gamma}\right)^{\frac{1}{\alpha_i}}. \tag{3.14}$$

Combining (3.13) and (3.14), we obtain

$$\omega'(t)\leq-P_{3,k}(t)+\left(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\right)\omega(t)-\frac{\gamma\omega^{\frac{\gamma+1}{\gamma}}(t)}{(\rho(t)r(t))^{\frac{1}{\gamma}}}.$$

The remainder of the proof is similar to that of Theorem 3.1, so is omitted. The proof is complete.

Next, let us introduce the class of functions Y, which will be extensively used in the sequel.

Let $\mathbb{D}_0=\{(t,s):t_0\leq s<t<\infty\}$ and $\mathbb{D}=\{(t,s):t_0\leq s\leq t<\infty\}$. We say that the function $H\in C(\mathbb{D},\mathbb{R})$ belongs to the class Y, denoted by $H\in Y$, if

(i) $H(t,t)=0$, $t\geq t_0$, $H(t,s)>0$ on $\mathbb{D}_0$;

(ii) H has continuous partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on $\mathbb{D}$ such that

$$\frac{\partial H(t,s)}{\partial t}=h_1(t,s)H^{\frac{\gamma}{\gamma+1}}(t,s)\quad\text{and}\quad\frac{\partial H(t,s)}{\partial s}=-h_2(t,s)H^{\frac{\gamma}{\gamma+1}}(t,s),$$

where h_1 and h_2 are locally integrable functions.

Theorem 3.4 *Suppose that for any $T\geq t_0$, there exist constants $a_k,\ b_k\in[T,\infty)$ such that $a_k<b_k$, $k=1,\ 2$, with*

$$q_i(t)\geq0,\quad for\quad t\in[G_1(a_1),G_2(b_1))\cup[G_1(a_2),G_2(b_2)),\quad i=0,\ 1,\ 2,\ \cdots,\ n$$

and

$$(-1)^ke(t)\geq0,\quad for\quad t\in[G_1(a_k),G_2(b_k)),\quad k=1,\ 2,$$

where G_1 and G_2 are defined as in Lemma 2.3. Furthermore, assume that there exist functions $\rho\in C^1([t_0,\infty),\mathbb{R}^+)$ such that for some $H\in Y$ and $c_k\in(a_k,b_k)$,

$$\frac{1}{H(c_k,a_k)}\int_{a_k}^{c_k}\left[H(s,a_k)P_{1,k}(s)-\frac{\rho(s)r(s)}{(\gamma+1)^{\gamma+1}}K_1^{\gamma+1}(s,a_k)\right]\mathrm{d}s$$
$$+\frac{1}{H(b_k,c_k)}\int_{c_k}^{b_k}\left[H(b_k,s)P_{1,k}(s)-\frac{\rho(s)r(s)}{(\gamma+1)^{\gamma+1}}K_2^{\gamma+1}(b_k,s)\right]\mathrm{d}s>0,\quad k=1,\ 2, \tag{3.15}$$

where

$$K_1(s,a_k)=h_1(s,a_k)+H^{\frac{1}{\gamma+1}}(s,a_k)\left(\frac{\rho'(s)}{\rho(s)}-\frac{p(s)}{r(s)}\right),$$
$$K_2(b_k,s)=H^{\frac{1}{\gamma+1}}(b_k,s)\left(\frac{\rho'(s)}{\rho(s)}-\frac{p(s)}{r(s)}\right)-h_2(b_k,s),$$

$\eta_i>0$, $i=1,\ 2,\ \cdots,\ n$ satisfy (2.1) and (2.2) of Lemma 2.1, $P_{1,k}$, $k=1,\ 2$ are defined as in Theorem 3.1 and $\delta_{i,k}$, $i=0,\ 1,\ 2,\ \cdots,\ n$ and $k=1,\ 2$ are defined as in Lemma 2.3. Then every solution of Eq. (1.1) *is oscillatory.*

Proof. To arrive at a contradiction, suppose that Eq. (1.1) has a nonoscillatory solution x on $[t_0,\infty)$. Without loss of generality, we assume that there exists a $t_1\geq t_0$, such that $x(t)>0$, $x(g_i(t))>0$, $i=0,\ 1,\ 2,\ \cdots,\ n$, for all $t\geq t_1$. By assumption, we can choose $b_1>a_1>t_1$, such that $q_i(t)\geq0$ and $e(t)\leq0$ on the interval $[G_1(a_1),G_2(b_1))$. Proceeding as in the proof of Theorem

3.1, we get (3.6). Multiplying both sides of (3.6) by $H(s,t)$, and integrating with respect to s from t to c_1, for $t \in (a_1, c_1]$, we have

$$\begin{aligned}\int_t^{c_1} H(s,t)P_{1,1}(s)\mathrm{d}s \le -\int_t^{c_1} H(s,t)\omega'(s)\mathrm{d}s + \int_t^{c_1} H(s,t)\left(\frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{r(s)}\right)\omega(s)\mathrm{d}s \\ - \int_t^{c_1} H(s,t)\frac{\gamma}{(\rho(s)r(s))^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(s)\mathrm{d}s. \qquad (3.16)\end{aligned}$$

In view of (i) and (ii), we see that

$$\int_t^{c_1} H(s,t)\omega'(s)\mathrm{d}s = H(c_1,t)\omega(c_1) - \int_t^{c_1} h_1(s,t)H^{\frac{\gamma}{\gamma+1}}(s,t)\omega(s)\mathrm{d}s. \qquad (3.17)$$

Then, using (3.17) in (3.16), we get

$$\begin{aligned}\int_t^{c_1} H(s,t)P_{1,1}(s)\mathrm{d}s \le -H(c_1,t)\omega(c_1) - \int_t^{c_1} H(s,t)\frac{\gamma}{(\rho(s)r(s))^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(s)\mathrm{d}s \\ + \int_t^{c_1}\left(h_1(s,t)H^{\frac{\gamma}{\gamma+1}}(s,t) + H(s,t)\left(\frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{r(s)}\right)\right)\omega(s)\mathrm{d}s. \qquad (3.18)\end{aligned}$$

Set

$$G(v) = K_1(s,t)H^{\frac{\gamma}{\gamma+1}}(s,t)v - H(s,t)\frac{\gamma}{(\rho(s)r(s))^{\frac{1}{\gamma}}}v^{\frac{\gamma+1}{\gamma}},$$

where K_1 is defined as in Theorem 3.4. By simple calculation, we find that, G has the maximum

$$G_{\max} = \frac{\rho(s)r(s)}{(\gamma+1)^{\gamma+1}}K_1^{\gamma+1}(s,t), \quad \text{when} \quad v = \frac{K_1^{\gamma}(s,t)\rho(s)r(s)}{(\gamma+1)^{\gamma}}H^{-\frac{\gamma}{\gamma+1}}(s,t). \qquad (3.19)$$

From (3.18) and (3.19), we obtain

$$\int_t^{c_1} H(s,t)P_{1,1}(s)\mathrm{d}s \le -H(c_1,t)\omega(c_1) + \int_t^{c_1}\frac{\rho(s)r(s)}{(\gamma+1)^{\gamma+1}}K_1^{\gamma+1}(s,t)\mathrm{d}s.$$

Letting $t \to a_1^+$ in the above inequality and dividing it by $H(c_1, a_1)$, we have

$$\frac{1}{H(c_1,a_1)}\int_{a_1}^{c_1}\left[H(s,a_1)P_{1,1}(s) - \frac{\rho(s)r(s)}{(\gamma+1)^{\gamma+1}}K_1^{\gamma+1}(s,a_1)\right]\mathrm{d}s \le -\omega(c_1). \qquad (3.20)$$

Similarly, multiplying both sides of (3.6) by $H(t,s)$, and integrating with respect to s from c_1 to t, for $t \in [c_1, b_1)$, we get

$$\begin{aligned}\int_{c_1}^{t} H(t,s)P_{1,1}(s)\mathrm{d}s \le -\int_{c_1}^{t} H(t,s)\omega'(s)\mathrm{d}s + \int_{c_1}^{t} H(t,s)\left(\frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{r(s)}\right)\omega(s)\mathrm{d}s \\ - \int_{c_1}^{t} H(t,s)\frac{\gamma}{(\rho(s)r(s))^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(s)\mathrm{d}s \\ \le H(t,c_1)\omega(c_1) - \int_{c_1}^{t} H(t,s)\frac{\gamma}{(\rho(s)r(s))^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(s)\mathrm{d}s \\ + \int_{c_1}^{t}\left(H(t,s)\left(\frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{r(s)}\right) - h_2(t,s)H^{\frac{\gamma}{\gamma+1}}(t,s)\right)\omega(s)\mathrm{d}s. \qquad (3.21)\end{aligned}$$

Let

$$\widetilde{G}(v) = K_2(t,s)H^{\frac{\gamma}{\gamma+1}}(t,s)v - H(t,s)\frac{\gamma}{(\rho(s)r(s))^{\frac{1}{\gamma}}}v^{\frac{\gamma+1}{\gamma}},$$

where K_2 is defined as in Theorem 3.4. By simple calculation, we find that, G has the maximum

$$\widetilde{G}_{\max} = \frac{\rho(s)r(s)}{(\gamma+1)^{\gamma+1}}K_2^{\gamma+1}(t,s), \quad \text{when} \quad v = \frac{K_2^{\gamma}(t,s)\rho(s)r(s)}{(\gamma+1)^{\gamma}}H^{-\frac{\gamma}{\gamma+1}}(t,s). \qquad (3.22)$$

From (3.21) and (3.22), we obtain

$$\int_{c_1}^{t} H(t,s)P_{1,1}(s)\mathrm{d}s \le H(t,c_1)\omega(c_1) + \int_{c_1}^{t} \frac{\rho(s)r(s)}{(\gamma+1)^{\gamma+1}} K_2^{\gamma+1}(t,s)\mathrm{d}s.$$

Letting $t \to b_1^-$ in the above inequality and dividing it by $H(b_1, c_1)$, we have

$$\frac{1}{H(b_1,c_1)} \int_{c_1}^{b_1} \left[H(b_1,s)P_{1,1}(s) - \frac{\rho(s)r(s)}{(\gamma+1)^{\gamma+1}} K_2^{\gamma+1}(b_1,s) \right] \mathrm{d}s \le \omega(c_1). \tag{3.23}$$

Adding (3.20) and (3.23), we get a contradiction to (3.15). This completes the proof.

Remark 3.3 *when $\gamma = 1$, Theorem 3.4 reduces to the main results in [23].*

Particularly, when $g_i(t) = t$, $i = 0,\ 1,\ 2,\ \cdots,\ n$, Eq. (1.1) reduces to the following equations

$$(r(t)(x'(t))^{\gamma})' + p(t)(x'(t))^{\gamma} + \sum_{i=0}^{n} q_i(t)\,|x(t)|^{\alpha_i}\,\mathrm{sgn}\ x(t) = e(t). \tag{3.24}$$

We can also remove the sign condition imposed on the coefficients of the half-linear terms to obtain interval oscillation criterion for Eq. (3.24) which is applicable for the case when some or all of the functions q_i, $i = m+1,\ \cdots,\ n$ are nonpositive. The results is as follows.

Theorem 3.5 *Suppose that for any $T \ge t_0$, there exist constants $a_k,\ b_k \in [T,\infty)$ such that $a_k < b_k$, $k = 1,\ 2$, with*

$$q_i(t) \ge 0, \quad \text{for} \quad t \in [a_1, b_1) \cup [a_2, b_2), \quad i = 0,\ 1,\ 2,\ \cdots,\ m$$

and

$$(-1)^k e(t) > 0, \quad \text{for} \quad t \in [a_k, b_k), \quad k = 1,\ 2.$$

Furthermore, assume that there exist functions $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ and $u \in \xi(a_k, b_k)$, $k = 1,\ 2$, such that

$$\int_{a_k}^{b_k} \left[Q(t)u^{\gamma+1}(t) - \frac{\rho(t)r(t)}{(\gamma+1)^{\gamma+1}} P^{\gamma+1}(t) \right] \mathrm{d}t > 0, \quad k = 1,\ 2, \tag{3.25}$$

where

$$Q(t) = \rho(t)q_0(t) + \rho(t)\sum_{i=1}^{m} \alpha_i \left(\left(\frac{q_i(t)}{\gamma}\right)^{\gamma} \left(\frac{\lambda_i |e(t)|}{\alpha_i - \gamma}\right)^{\alpha_i-\gamma} \right)^{\frac{1}{\alpha_i}}$$

$$-\rho(t) \sum_{i=m+1}^{n} \alpha_i \left(\left(\frac{(q_i(t))_-}{\gamma}\right)^{\gamma} \left(\frac{\gamma - \alpha_i}{\lambda_i |e(t)|}\right)^{\gamma-\alpha_i} \right)^{\frac{1}{\alpha_i}},$$

λ_i are positive numbers with $\sum_{i=1}^{n} \lambda_i = 1$, $(q_i(t))_- = \max\{-q_i(t),\ 0\}$, $i = m+1,\ m+2,\ \cdots,\ n$ and P is defined as in Theorem 3.1. Then every solution of Eq. (1.1) is oscillatory.

Proof. To arrive at a contradiction, suppose that Eq. (1.1) has a nonoscillatory solution x on $[t_0,\infty)$. Without loss of generality, we assume that there exists a $t_1 \ge t_0$, such that $x(t) > 0$, $x(g_i(t)) > 0$, $i = 0,\ 1,\ 2,\ \cdots,\ n$, for all $t \ge t_1$. By assumption, we can choose $b_1 > a_1 > t_1$, such that $q_i(t) \ge 0$ and $e(t) < 0$ on the interval $[a_1, b_1)$. We define the function ω as in the proof of Theorem 3.1. Similarly to the proof of Theorem 3.1, we have

$$\omega'(t) = -\rho(t)q_0(t) - \rho(t)\sum_{i=1}^{m} \left[q_i(t)x^{\alpha_i-\gamma}(t) + \lambda_i|e(t)|x^{-\gamma}(t) \right]$$

$$-\rho(t)\sum_{i=m+1}^{n} \left[q_i(t)x^{\alpha_i-\gamma}(t) + \lambda_i|e(t)|x^{-\gamma}(t) \right]$$

$$+\left(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\right)\omega(t)-\frac{\gamma\omega^{\frac{\gamma+1}{\gamma}}(t)}{(\rho(t)r(t))^{\frac{1}{\gamma}}}. \tag{3.26}$$

From (2.5), we get, for $t\in(a_1,b_1)$ and $i=0,\ 1,\ 2,\ \cdots,\ m$,

$$q_i(t)x^{\alpha_i-\gamma}(t)+\lambda_i|e(t)|x^{-\gamma}(t)\geq\alpha_i\left(\left(\frac{q_i(t)}{\gamma}\right)^{\gamma}\left(\frac{\lambda_i|e(t)|}{\alpha_i-\gamma}\right)^{\alpha_i-\gamma}\right)^{\frac{1}{\alpha_i}}. \tag{3.27}$$

From (2.4), we obtain, for $t\in(a_1,b_1)$ and $i=m+1,\ m+2,\ \cdots,\ n$,

$$\begin{aligned}q_i(t)x^{\alpha_i-\gamma}(t)+\lambda_i|e(t)|x^{-\gamma}(t)&\geq\lambda_i|e(t)|x^{-\gamma}(t)-(q_i(t))_-x^{\alpha_i-\gamma}(t)\\&\geq-\alpha_i\left(\left(\frac{(q_i(t))_-}{\gamma}\right)^{\gamma}\left(\frac{\gamma-\alpha_i}{\lambda_i|e(t)|}\right)^{\gamma-\alpha_i}\right)^{\frac{1}{\alpha_i}}.\end{aligned} \tag{3.28}$$

Combining (3.26)–(4.2), we have

$$\omega'(t)\leq-Q(t)+\left(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\right)\omega(t)-\frac{\gamma\omega^{\frac{\gamma+1}{\gamma}}(t)}{(\rho(t)r(t))^{\frac{1}{\gamma}}}.$$

The remainder of the proof is similar to that of Theorem 3.1, so we omit it. Then the theorem is proved.

Remark 3.4 *If $p(t)\equiv 0$, $g_i(t)=t$, $i=0,\ 1,\ 2,\ \cdots,\ n$, then Theorem 3.5 reduces to Theorem 2.3 in [33].*

4 Examples

In this section, we will present the applications of our interval oscillation criteria in three examples. In particular, we will show a real life application problem of our results.

Firstly, we give an application of Theorem 3.1 on damped simple harmonic motion

$$x''(t)+\beta x'(t)+\omega_0^2x(t)=0, \tag{4.1}$$

where $\beta>0$ is the damping constant. Here

$$\gamma=1, r(t)=1,\quad p(t)=\beta,\quad q_0(t)=\omega_0^2, e(t)=0, g_0(t)=t.$$

Let $\eta_0=\eta_1=\eta_2=1/3$, and

$$a_h=\frac{(h-1)\pi}{\omega_0},\quad b_h=a_{h+1}=\frac{h\pi}{\omega_0},\quad b_{h+1}=\frac{(h+1)\pi}{\omega_0},\quad h=1,\ 2,\ \cdots,$$

such that (2.1) and (2.2) in Lemma 2.1 are satisfied, and

$$q_0(t)\geq 0\ \text{ on }\ [0,\frac{\pi}{\omega_0})\cup[\frac{\pi}{\omega_0},\frac{2\pi}{\omega_0}),$$

and

$$(-1)^ke(t)\geq 0,\quad t\in[\frac{(k-1)\pi}{\omega_0},\frac{k\pi}{\omega_0)},\quad k=1,\ 2.$$

Setting $\rho(t)=e^{\beta t}$ and $u(t)=\sin\omega_0 t$, we have $P_{1,1}(t)=P_{1,2}=t\omega_0^2, P(t)=2\omega_0\cos\omega t$ and

$$\begin{aligned}\int_{a_1}^{b_1}\left[P_{1,1}(t)u^{\gamma+1}(t)-\tfrac{\rho(t)r(t)}{(\gamma+1)^{\gamma+1}}P^{\gamma+1}(t)\right]\mathrm{d}t&=\int_0^{\frac{\pi}{\omega_0}}\left[t\omega_0^2\sin^2(\omega_0t)-\omega_0^2e^{\beta t}\cos^2(\omega_0t))\right]\mathrm{d}t\\&=\tfrac{\pi^2}{4}+\tfrac{\omega_0^2}{2\beta}(1-e^{\frac{\beta\pi}{\omega_0}}),\end{aligned}$$

and

$$\int_{a_2}^{b_2}\left[P_{1,2}(t)u^{\gamma+1}(t)-\frac{\rho(t)r(t)}{(\gamma+1)^{\gamma+1}}P^{\gamma+1}(t)\right]\mathrm{d}t \quad = \int_{\frac{\pi}{\omega_0}}^{\frac{2\pi}{\omega_0}}\left[t\omega_0^2\sin^2(\omega_0 t)-\omega_0^2 e^{\beta t}\cos^2(\omega_0 t))\right]\mathrm{d}t$$

$$= \frac{3\pi^2}{4}+\frac{\omega_0^2}{2\beta}(e^{\frac{\beta\pi}{\omega_0}}-e^{\frac{2\beta\pi}{\omega_0}}).$$

Then by Theorem 3.1, every solution of Eq. (4.1) is oscillatory if

$$\frac{\pi^2}{4}+\frac{\omega_0^2}{2\beta}(1-e^{\frac{\beta\pi}{\omega_0}})>0, \frac{3\pi^2}{4}+\frac{\omega_0^2}{2\beta}(e^{\frac{\beta\pi}{\omega_0}}-e^{\frac{2\beta\pi}{\omega_0}})>0. \tag{4.2}$$

In particular, take $\beta=\frac{1}{4}, \omega_0=\frac{\pi}{4}$. Then two inequalities in (4.2) hold. Hence every solution of

$$x''(t)+\frac{1}{4}x'(t)+(\frac{\pi}{4})^2x(t)=0 \tag{4.3}$$

is oscillatory. See Figure below for damped simple harmonic motion equation (4.2).

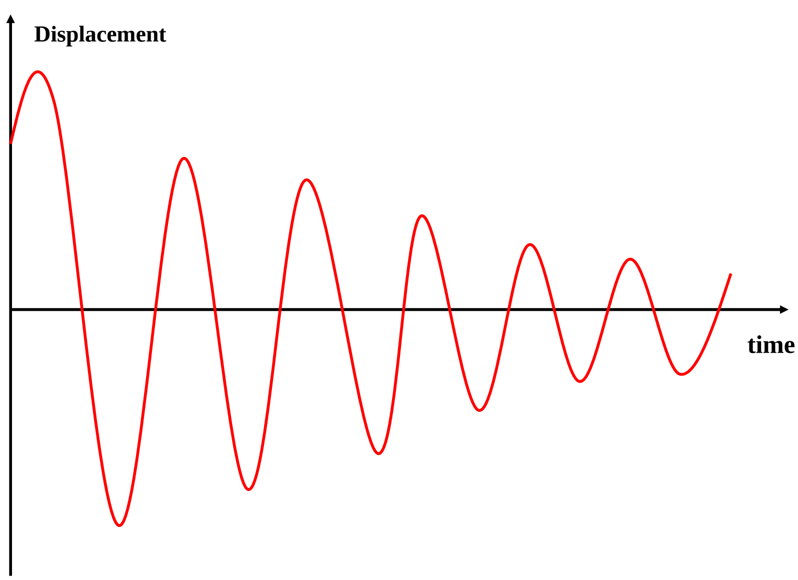

Figure 1: damped simple harmonic motion

Next, we will give another example to illustrate Theorem 3.1.

Example 4.2 Consider the following second order damped differential equations with mixed nonlinearities.

$$\left(\frac{\sin 8t+2}{t}(x'(t))^{\gamma}\right)'+\frac{\sin 8t+2}{t^2}(x'(t))^{\gamma}+c_0\sin^2 8t\ x^{\gamma}(t)+4c_1\cos 2t|x(t)|^{\frac{5}{2}\gamma}\mathrm{sgn}\ x(t)$$

$$+c_2\sin 2t|x(t)|^{\frac{\gamma}{2}}\mathrm{sgn}\ x(t)=-\cos 4t, \quad t\geq 1, \tag{4.4}$$

where γ is a quotient of odd positive integer, c_0, c_1 and c_2 are positive constants. Here

$$r(t)=\frac{\sin 8t+2}{t}, \quad p(t)=\frac{\sin 8t+2}{t^2}, \quad q_0(t)=c_0\sin^2 8t, \quad q_1(t)=4c_1\cos 2t, \quad q_2(t)=c_2\sin 2t,$$

$$e(t)=\cos 4t,\quad g_i(t)=t,\quad i=0,\ 1,\ 2,\quad \alpha_1=\frac{5}{2}\gamma,\quad \alpha_2=\frac{\gamma}{2}.$$

Let $\eta_0=\eta_1=\eta_2=1/3$, and

$$a_h=2h\pi,\quad b_h=a_{h+1}=2h\pi+\frac{\pi}{8},\quad b_{h+1}=2h\pi+\frac{\pi}{4},\quad h=0,1,\ 2,\ \cdots,$$

such that (2.1) and (2.2) in Lemma 2.1 are satisfied, and

$$q_i(t)\geq 0\ \text{ on }\ [2h\pi,2h\pi+\frac{\pi}{8})\cup[2h\pi+\frac{\pi}{8},2h\pi+\frac{\pi}{4}),\quad i=0,\ 1,\ 2,$$

and

$$(-1)^k e(t)\geq 0,\quad t\in[2h\pi,2h\pi+\frac{\pi}{8})\cup[2h\pi+\frac{\pi}{8},2h\pi+\frac{\pi}{4}),\quad k=1,\ 2.$$

Setting $\rho(t)=t$ and $u(t)=\sin 8t$, we have

$$P_{1,1}(t)=t\left(c_0\sin^2 8t+3\sqrt[3]{2c_1c_2\sin 4t|\cos 4t|}\right),$$

and

$$\int_{a_1}^{b_1}\left[P_{1,1}(t)u^{\gamma+1}(t)-\frac{\rho(t)r(t)}{(\gamma+1)^{\gamma+1}}P^{\gamma+1}(t)\right]\mathrm{d}t$$

$$=\int_0^{\frac{\pi}{8}}\left[t\left(c_0\sin^2 8t+3\sqrt[3]{c_1c_2\sin 8t}\right)\sin^{\gamma+1}8t-\frac{\sin 8t+2}{(\gamma+1)^{\gamma+1}}((\gamma+1)8\cos 8t)^{\gamma+1}\right]\mathrm{d}t$$

$$\geq\int_0^{\frac{\pi}{8}}\left[2\pi\left(c_0\sin^2 8t+3\sqrt[3]{c_1c_2\sin 8t}\right)\sin^{\gamma+1}8t-2^{3\gamma+3}(2\cos^{\gamma+1}8t+\sin 8t\cos^{\gamma+1}8t)\right]\mathrm{d}t$$

$$=\frac{\pi\sqrt{\pi}}{4}\left[\frac{c_0\gamma(\gamma+2)}{(\gamma+3)(\gamma+1)}\frac{\Gamma(\frac{\gamma}{2})}{\Gamma(\frac{\gamma+1}{2})}+\frac{3\sqrt[3]{c_1c_2}(3\gamma+1)}{3\gamma+4}\frac{\Gamma(\frac{3\gamma+1}{6})}{\Gamma(\frac{3\gamma+4}{6})}\right]-8^\gamma\left[\frac{\gamma\sqrt{\pi}}{\gamma+1}\frac{\Gamma(\frac{\gamma}{2})}{\Gamma(\frac{\gamma+1}{2})}+\frac{1}{\gamma+2}\right],$$

where Γ is the Gamma function. Then by Theorem 3.1, every solution of Eq. (4.4) is oscillatory if

$$\frac{\pi\sqrt{\pi}}{4}\left[\frac{c_0\gamma(\gamma+2)}{(\gamma+3)(\gamma+1)}\frac{\Gamma(\frac{\gamma}{2})}{\Gamma(\frac{\gamma+1}{2})}+\frac{3\sqrt[3]{c_1c_2}(3\gamma+1)}{3\gamma+4}\frac{\Gamma(\frac{3\gamma+1}{6})}{\Gamma(\frac{3\gamma+4}{6})}\right]>8^\gamma\left[\frac{\gamma\sqrt{\pi}}{\gamma+1}\frac{\Gamma(\frac{\gamma}{2})}{\Gamma(\frac{\gamma+1}{2})}+\frac{1}{\gamma+2}\right].$$

Finally, we will give an example to illustrate Theorem 3.5.

Example 4.3 Consider the following second order damped differential equations with mixed nonlinearities

$$\left(\frac{\sin 2t+2}{t}(x'(t))^\gamma\right)'+\frac{\sin 2t+2}{t^2}(x'(t))^\gamma+c_0\cos^{2\gamma}2t\ x^\gamma(t)+c_1\sin 2t|x(t)|^{2\gamma}\mathrm{sgn}\ x(t)$$

$$-c_2\cos^{\gamma+1}2t|x(t)|^{\frac{\gamma}{2}}\mathrm{sgn}\ x(t)=-\cos 2t,\quad t\geq 1,\tag{4.5}$$

where γ is a quotient of odd positive integer, c_0, c_1 and c_2 are positive constants. Here

$$r(t)=\frac{\sin 2t+2}{t},\quad p(t)=\frac{\sin 2t+2}{t^2},\quad q_0(t)=c_0\cos^{2\gamma}2t,\quad q_1(t)=c_1\sin 2t,$$

$$q_2(t)=-c_2\cos^{\gamma+1}2t,\quad e(t)=\cos 2t,\quad \alpha_1=2\gamma,\quad \alpha_2=\frac{\gamma}{2},\quad c_0\geq\frac{c_2^2}{2}.$$

Let

$$a_1=2h\pi,\quad b_1=a_2=2h\pi+\frac{\pi}{4},\quad b_2=2h\pi+\frac{\pi}{2},\quad h=1,\ 2,\ \cdots,$$

such that

$$q_i(t)\geq 0\ \text{ on }\ [2h\pi,2h\pi+\frac{\pi}{4})\cup[2h\pi+\frac{\pi}{4},2h\pi+\frac{\pi}{2}),\quad i=0,\ 1,\ 2,$$

and

$$(-1)^k e(t) \geq 0, \quad t \in [2h\pi, 2h\pi + \frac{\pi}{4}) \cup [2h\pi + \frac{\pi}{4}, 2h\pi + \frac{\pi}{2}), \quad k = 1,\ 2.$$

Setting $\rho(t) = t$, $\lambda_1 = \lambda_2 = 1/2$ and $u(t) = \sin 2t$, we get

$$Q(t) = t\Big(c_0 \cos^{2\gamma} 2t + \sqrt{2c_1 \sin 2t |\cos 2t|} - \frac{c_2^2 \cos^{2(\gamma+1)} 2t}{2|\cos 2t|}\Big)$$

and

$$\begin{aligned}
&\int_{a_1}^{b_1} \Big[Q(t)u^{\gamma+1}(t) - \frac{\rho(t)r(t)}{(\gamma+1)^{\gamma+1}} P^{\gamma+1}(t)\Big]\mathrm{d}t \\
= &\int_0^{\frac{\pi}{4}} \Big[t\Big(c_0 \cos^{2\gamma} 2t + \sqrt{2c_1 \sin 2t \cos 2t} - \frac{c_2^2}{2} \cos^{2\gamma+1} 2t\Big) \sin^{\gamma+1} 2t \\
&-\frac{\sin 2t + 2}{(\gamma+1)^{\gamma+1}}((\gamma+1)2\cos 2t)^{\gamma+1}\Big]\mathrm{d}t \\
\geq &\int_0^{\frac{\pi}{4}} \Big[2\pi\Big(\Big(c_0 - \frac{c_2^2}{2}\Big)\cos^{2\gamma+1} 2t + \sqrt{2c_1 \sin 2t \cos 2t}\Big)\sin^{\gamma+1} 2t \\
&-2^{\gamma+1}(\sin 2t \cos^{\gamma+1} 2t + 2\cos^{\gamma+1} 2t)\Big]\mathrm{d}t \\
= &\frac{\pi}{2}\Big(c_0 - \frac{c_2^2}{2}\Big)\frac{\Gamma(\frac{\gamma}{2}+1)\Gamma(\gamma+1)}{\Gamma(\frac{3}{2}\gamma+2)} + \frac{\sqrt{2c_1}}{2}\frac{\Gamma(\frac{2\gamma+5}{4})\Gamma(\frac{3}{4})}{\Gamma(\frac{\gamma}{2}+2)} - 2^{\gamma+1}\Big(\frac{1}{\gamma+2} + \frac{\gamma\sqrt{\pi}}{\gamma+1}\frac{\Gamma(\frac{\gamma}{2})}{\Gamma(\frac{\gamma+1}{2})}\Big),
\end{aligned}$$

where Γ is the Gamma function. Then by Theorem 3.5, every solution of Eq. (4.5) is oscillatory if

$$\frac{\pi}{2}\Big(c_0 - \frac{c_2^2}{2}\Big)\frac{\Gamma(\frac{\gamma}{2}+1)\Gamma(\gamma+1)}{\Gamma(\frac{3}{2}\gamma+2)} + \frac{\sqrt{2c_1}}{2}\frac{\Gamma(\frac{2\gamma+5}{4})\Gamma(\frac{3}{4})}{\Gamma(\frac{\gamma}{2}+2)} > 2^{\gamma+1}\Big(\frac{1}{\gamma+2} + \frac{\gamma\sqrt{\pi}}{\gamma+1}\frac{\Gamma(\frac{\gamma}{2})}{\Gamma(\frac{\gamma+1}{2})}\Big).$$

5 Conclusions

In this paper, new interval oscillation criteria for certain classes of second order nonlinear differential equations with mixed nonlinearities and with delayed or advanced arguments. Our results do not require that the functions p, q_i and e, $i = 0,\ 1,\ 2,\ \cdots,\ n$ are of definite sign, and these criteria are different from most known ones in the sense that they are based on the information only on a sequence of subintervals of $[t_0, \infty)$, rather than on the whole half-line. Moreover, our results improve and extend the main results of [16–19, 21, 24, 26, 27], for example, if $p(t) \equiv 0$, then Theorems 3.1 and 3.2 reduce to Theorems 2.5 and 2.6 in [33]. Furthermore, if we take $g_i(t) = t$, $i = 0,\ 1,\ 2,\ \cdots,\ n$, then Theorems 3.1, 3.2 and 3.5 reduce to Theorems 2.1, 2.2 and 2.3 in [33]. When $\gamma = 1$, Theorem 3.4 reduces to the main results in [23]. The method can be applied on the second-order Emden-Fowler neutral differential equation

$$(r(t)(x'(t) + p(t)x(\tau(t)))^{\gamma})' + p(t)(x'(t))^{\gamma} + \sum_{i=0}^{n} q_i(t)\,|x(g_i(t))|^{\alpha_i}\,\mathrm{sgn}\ x(g_i(t)) = e(t), \quad t \geq t_0.$$

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