MAXIMAL ENERGY OF SUBDIVISIONS OF GRAPHS WITH A FIXED CHROMATIC NUMBER

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ABSTRACT. The energy of a simple graph G, denoted by $E(G)$, is defined as the sum of the absolute values of eigenvalues of G. In this paper, we show that, among all subdivisions of graphs with n vertices and chromatic number k, the subdivision of the Turán graph $T(n,k)$ has the maximal energy.

1. INTRODUCTION

Let G be a simple graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{e_1, e_2, \ldots, e_m\}$. Suppose $S(G)$ is the subdivision of G, which is obtained from G by replacing each edge with a path with three vertices (i.e. inserting a new vertex to each edge of G). For the graph G in Figure 1(a), the subdivision $S(G)$ of G is illustrated in Figure 1(b). Hence $S(G)$ is a bipartite graph with $m+n$ vertices and $2m$ edges. Let $\Delta(G)$ be the diagonal matrix of G whose i-th diagonal entry is the degree of the vertex v_i $(1 \leq i \leq n)$. The adjacency matrix $A(G)$ of G is the square matrix $A(G) = (a_{ij})$ of order n, where $a_{ij} = 1$ if v_i and v_j are adjacent and 0 otherwise. Let $Q(G) = \Delta(G) + A(G)$ be the signless Laplacian matrix of G. The eigenvalues of $Q(G)$ are called the signless Laplacian eigenvalues of G. Let $K_{a_1,a_2,\ldots,a_k}$ be the complete multipartite graph with $n = \sum_{i=1}^{k} a_i$ vertices, whose vertex set is partitioned into k parts: $V_1, V_2, \ldots, V_k$, of cardinalities $a_1, a_2, \ldots, a_k$, and an edge joins two vertices if and only if they belong to different parts. Let n and k be two positive integers satisfying $n = rk + s$ and $r > 0, 0 \leqslant s < k$. The complete multipartite graph $K_{\underbrace{r,\ldots,r}_{k-s},\underbrace{r+1,\ldots,r+1}_{s}}$ is called the Turán graph, denoted by $T(n,k)$.

Gutman [8] define the energy of a graph G with n vertices, denoted by $E(G)$, as

$$E(G) = \sum_{i=1}^{n} |\lambda_i(G)|,$$

where $\lambda_i(G)$'s are the eigenvalues of the adjacency matrix of G.

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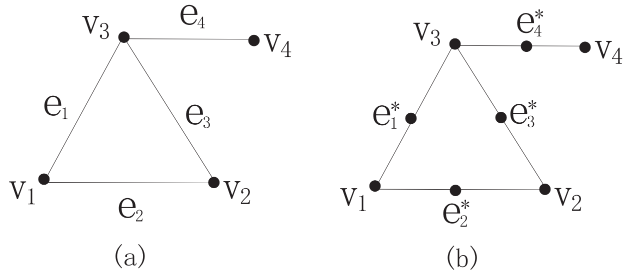

FIGURE 1. (a). A graph G. (b). The subdivision $S(G)$ of G.

Historically chemists used the model in which the experimental heats of formation of conjugated hydrocarbons are closely related to the total π-electron energy. Today such a model is over simplistic, but nevertheless HMO has some value as it points to that part of the experimental heats of formation of conjugated hydrocarbons that can be viewed as due to molecular connectvtiz (molecular topology). The calculation of the total π-electron energy in a conjugated hydrocarbon can be reduced (within the framework of the HMO approximation [10]) to $\mathcal{E}(G)$ of the corresponding graph G. The energy of graphs has been studied extensively (see for example [1, 7, 11, 12, 13, 14, 16, 19]).

In general, let X_k be any square matrix of order k and let I_k be the unit matrix of order k. The characteristic polynomial of X_k is defined as

$$\sigma(X_k, x) = det[xI_k - X_k],$$

where det[ ] is used to denote the determinant of a square matrix.

It well know [3] that if G is a bipartite with n vertices then the characteristic polynomial $\sigma(G,x)$ of G has the following form:

$$\sigma(G,x) = \det(xI_n - A(G)) = \sum_{t=0}^{[n/2]} (-1)^t b(G,t) x^{n-2t}, \tag{1.1}$$

where $b(G,0) = 1$ and $b(G,t) \geq 0$ for all $t = 1, 2, \ldots, [n/2]$. This expression for $\sigma(G,x)$ induces a quasi-order relation (i.e. reflexive and transitive relation) on the set of all bipartite graphs with n vertices: If G_1 and G_2 are bipartite graphs with characteristic polynomials in the form (1.1)

$$G_1 \succeq G_2 \Longleftrightarrow b(G_1,t) \geq b(G_2,t) \text{ for all } t = 0, 1, \ldots, [n/2].$$

If $G_1 \succeq G_2$ and there exists k such that $b(G_1,k) > b(G_2,k)$, then we write $G_1 \succ G_2$.

Gutman [5] introduced this quasi-order relation in order to compare the energies of a pair of graphs. It is well known that if G is a bipartite graph, then the energy of G can be expressed by means of the Coulson integral formula [6, 10]

$$E(G) = \frac{2}{\pi}\int_0^\infty x^{-2}\ln[1+\sum_{t=1}^{[n/2]} b(G,t)x^{2t}]d_x, \tag{1.2}$$

which implies: $G_1 \succeq G_2 \Longrightarrow E(G_1) \geq E(G_2)$ and $G_1 \succ G_2 \Longrightarrow E(G_1) > E(G_2)$.

This increasing property of E has been successfully applied in the study of the extremal values of the energy over a significant class of graphs. See for example the papers [15, 16, 20, 21].

In this paper, we compute the signless Laplacian eigenvalues of the complete multipartite graphs $K_{a_1,a_2,\ldots,a_k}$ in the next section. In Section 3, we prove that, among all graphs with n vertices and chromatic number k, the Turán graph $T(n,k)$ has the maximal coefficients of signless Laplacian characteristic polynomial, which implies immediately that, among all subdivisions of graphs with n vertices and chromatic number k, the subdivision of the Turán graph $T(n,k)$ has the maximal energy.

2. The signless Laplacian eigenvalues of complete multipartite graphs

Delorme [4] determined the eigenvalues of the adjacency matrix of the complete multipartite graphs. In this section, we use a similar method to compute the signless Laplacian eigenvalues of complete multipartite graphs.

Theorem 2.1. *The characteristic polynomial of signless Laplacian matrix of complete multipartite graph $G = K_{a_1,a_2,\ldots,a_k}$ with $\sum_{i=1}^k a_i = n$ is*

$$\sigma(Q(G),x) = \prod_{i=1}^k (x-n+a_i)^{a_i-1}\left(\prod_{i=1}^k (x-n+2a_i) - \sum_{j=1}^k a_j \prod_{i=1,i\neq j}^k (x-n+2a_i)\right).$$

Proof. For the convenience, we use Q, A and Δ to denote matrices $Q(G)$, $A(G)$ and $\Delta(G)$.

Note that if vertices v and w are in the same part of G, the transpose of the row vector β_i whose coordinates on v, w and elsewhere are respectively 1, -1 and 0 is an eigenvector for the eigenvalue $n-a_i$ of the signless Laplacian matrix Q, and there are a_i-1 eigenvectors for the eigenvalue $n-a_i$ $(1 \leq i \leq k)$. So we can find $\sum_{i=1}^k (a_i-1) = n-k$ linearly independent eigenvectors of matrix Q which generate a linear subspace U of dimension $n-k$. Now we choose an orthogonal basis of the orthogonal complement of U. It is constituted by the transposes of k row vectors γ_i $(1 \leq i \leq k)$, where γ_i is the vector whose coordinates on vertices $v \in V_i$ are 1 and elsewhere are 0 , that is, $\gamma_i = (0,\ldots,0,\overbrace{1,\ldots,1}^{a_i},0,\ldots,0)$. It is easy to find that $Q(\gamma_1^T,\gamma_2^T,\ldots,\gamma_k^T) = (\gamma_1^T,\gamma_2^T,\ldots,\gamma_k^T)N_k$, where $N_k = (n_{ij})$ is a $k \times k$

matrix such that $n_{ij} = n - a_i$ if $i = j$ and $n_{ij} = a_j$ if $i \neq j$. It is not difficult to prove the following claim.

Claim. The characteristic polynomial of N_k is given by

$$\det(xI_k - N_k) = \prod_{i=1}^{k}(x - n + 2a_i) - \sum_{j=1}^{k} a_j \prod_{i=1, i\neq j}^{k}(x - n + 2a_i).$$

Let

$$X_n = (\beta_1^T, \beta_2^T, \ldots, \beta_{n-k}^T, \gamma_1^T, \gamma_2^T, \ldots, \gamma_k^T).$$

Then $QX_n = X_n M_n$, where the block diagonal matrix

$$M_n = diag((n - a_1)I_{a_1-1}, (n - a_2)I_{a_2-1}, \ldots, (n - a_k)I_{a_k-1}, N_k).$$

Hence $Q = X_n M_n X_n^{-1}$ has the same eigenvalues as M_n. Note that, by the the claim above,

$$|xI_n - M_n| = \prod_{i=1}^{k}(x - n + a_i)^{a_i-1}\left(\prod_{i=1}^{k}(x - n + 2a_i) - \sum_{j=1}^{k} a_j \prod_{i=1, i\neq j}^{k}(x - n + 2a_i)\right).$$

The theorem has thus proved. □

Remark 2.2. By Theorem 2.1, the signless Laplacian eigenvalues of $K_{a_1,a_2,\ldots,a_k}$ ($\sum_{i=1}^{k} a_i = n$) are $n - a_i$ with multiplicity $a_i - 1$ ($i = 1, 2, \ldots, k$) and the roots of the polynomial $\prod\limits_{i=1}^{k}(x - n + 2a_i) - \sum\limits_{j=1}^{k} a_j \prod\limits_{i=1, i\neq j}^{k}(x - n + 2a_i)$.

3. Main results

First, we define the relation $\succ$ ($\prec, \succeq, \preceq$) as follows.

Definition 3.1 ([17])**.** We say p is partial larger than q if $|p| > |q|$, denoted by $p \succ q$. Similarly, we have $p \prec q, p \succeq q, p \preceq q$.

Definition 3.2 ([17])**.** Let $p(x) = \sum_{i=0}^{n} p_i x^i$ and $q(x) = \sum_{i=0}^{n} q_i x^i$. If $|p_i| \geqslant |q_i|$ (resp. $|p_i| \leqslant |q_i|$) for each $0 \leqslant i \leqslant n$, then we call $p(x) \succeq q(x)$ (resp. $p(x) \preceq q(x)$). If $p(x) \succeq q(x)$ (resp. $p(x) \preceq q(x)$), and there exists a $j \in \{0, 1, \cdots, n\}$ such that $p_j \succ q_j$ (resp. $p_j \prec q_j$), we call $p(x) \succ q(x)$ (resp. $p(x) \prec q(x)$).

By the definition above, the following result is immediate.

Lemma 3.3. *Suppose $a_i \geqslant b_i \geqslant 0$ for $i = 1, 2, \cdots, n$. Then*

$$\prod_{i=1}^{n}(x - a_i) \succeq \prod_{i=1}^{n}(x - b_i),$$

furthermore, if there exists a $j \in \{1, 2, \cdots, n\}$ *such that* $a_j > b_j$, *then*

$$\prod_{i=1}^{n}(x-a_i) \succ \prod_{i=1}^{n}(x-b_i).$$

Lemma 3.4. *Let* n, a *and* b *be three positive integers and* $a-b \geqslant 2, a \leqslant n$. *Then*

$$(x-n+a-1)^{a-2}(x-n+b+1)^{b} \succ (x-n+a)^{a-1}(x-n+b)^{b-1}.$$

Proof. Note that

$$(x-n+a-1)(x-n+b+1) = x^2-2nx+(a+b)x-n(a+b)+ab+n^2+a-b-1$$

and

$$(x-n+a)(x-n+b) = x^2-2nx+(a+b)x-n(a+b)+ab+n^2.$$

Since $a-b-2 \geq 0$, by Lemma 3.3,

$$(x-n+a-1)^{b-1}(x-n+b+1)^{b-1} \succ (x-n+a)^{b-1}(x-n+b)^{b-1}. \tag{3.1}$$

Again, by Lemma 3.3,

$$(x-n+a-1)^{a-b-1}(x-n+b+1)^{1} \succ (x-n+a)^{a-b}. \tag{3.2}$$

The result follows from (3.1) $\times$ (3.2). $\square$

Lemma 3.5 ([9])**.** *Let* G_1 *and* G_2 *be two bipartite graphs with* n *vertices. Then*

$$\sigma(G_1,x) \succeq \sigma(G_2,x) \Rightarrow E(G_1) \succeq E(G_2);$$

$$\sigma(G_1,x) \succ \sigma(G_2,x) \Rightarrow E(G_1) \succ E(G_2).$$

Lemma 3.6 ([18])**.** *Let* G *be a graph with* n *vertices and* m *edges. Then*

$$\sigma(S(G),x) = x^{m-n}\sigma(Q(G),x^2) = x^{m-n}\det(x^2 I_n - Q(G)).$$

A subgraph H of a graph G is called a TU-subgraph if each component of H is a tree or a unicyclic graph whose cycle has an odd number of vertices. Suppose all components of a TU-subgraph H are $T_1, T_2, \ldots, T_s, U_1, U_2, \ldots, U_t$, where T_i's are trees and U_j's are unicyclic graph, and n_i is the number of vertices of T_i for $i = 1, 2, \ldots, s$. Define:

$$\phi(H) = 4^t \prod_{i=1}^{s} n_i.$$

Lemma 3.7 ([2]). *Let G be a simple graph of order n with the signless Laplacian characteristic polynomial of form*

$$\sigma(Q(G),x)=det(xI_n-Q(G))=x^n+q_1(G)x^{n-1}+q_2(G)x^{n-2}+\ldots+q_{n-1}(G)x+q_n(G).$$

Then, for $0\leqslant i\leqslant n$,

$$(-1)^i q_i(G)=\sum_H \phi(H),$$

where the summation is over all TU-subgraph of G with i edges.

We use $\overline{G}$ to denote the complement of a graph G. For any $e=uv\in E(\overline{G})$, i.e., $e=uv$ is not an edge in G. We use $G+e$ to denote the graph obtained by adding e to G. Similarly, for any set W of vertices (edges), $G-W$ and $G+W$ are the graphs obtained by deleting the vertices (edges) in W from G and by adding the vertices (edges) in W to G, respectively. By Lemma 3.7, we obtain immediately the following result.

Lemma 3.8. *Let G be a non-complete connected graph of order n and $e\in E(\overline{G})$. Then*

$$\sigma(Q(G+e),x)=\det(xI_n-Q(G+e))\succ\det(xI_n-Q(G))=\sigma(Q(G),x).$$

Lemma 3.9. *Let n, a_i, k and b_i be positive integers and $a_1\geqslant a_2\geqslant,\ldots,\geqslant a_k$, and $a_1-a_2\geqslant 2$, where $n=\sum_{i=1}^k a_i$. If $a_i=b_i$ $(3\leqslant i\leqslant k), b_1=a_1-1, b_2=a_2+1$, then*

$$\prod_{i=1}^k(x-n+2b_i)-\sum_{j=1}^k b_j\prod_{i=1,i\neq j}^k(x-n+2b_i)\succ\prod_{i=1}^k(x-n+2a_i)-\sum_{j=1}^k a_j\prod_{i=1,i\neq j}^k(x-n+2a_i).$$

Proof. Note that

$$(x-n+2a_1)(x-n+2a_2)=x^2-2nx+2(a_1+a_2)x-2(a_1+a_2)n+n^2+4a_1a_2$$

and

$$(x-n+2b_1)(x-n+2b_2)=x^2-2nx+2(b_1+b_2)x-2(b_1+b_2)n+n^2+4b_1b_2$$

$$=x^2-2nx+2(a_1+a_2)x-2(a_1+a_2)n+n^2+4a_1a_2+4(a_1-a_2)-4.$$

So

$$(x-n+2b_1)(x-n+2b_2)-(x-n+2a_1)(x-n+2a_2)=4(a_1-a_2)-4. \tag{3.3}$$

Again,

$$(-a_1)(x-n+2a_2)+(-a_2)(x-n+2a_1)=-(a_1+a_2)x+(a_1+a_2)n-4a_1a_2$$

and

$$(-b_1)(x-n+2b_2)+(-b_2)(x-n+2b_1) = -(b_1+b_2)x+(b_1+b_2)n-4b_1b_2$$

$$= -(a_1+a_2)x+(a_1+a_2)n-4a_1a_2-4(a_1-a_2)+4.$$

Thus

$$-b_1(x-n+2b_2)-b_2(x-n+2b_1)-[-a_1(x-n+2a_2)-a_2(x-n+2a_1)] = -4(a_1-a_2)+4. \tag{3.4}$$

Hence, by (3.3) and (3.4),

$$\prod_{i=1}^{k}(x-n+2b_i)-\prod_{i=1}^{k}(x-n+2a_i) = [4(a_1-a_2)-4]\prod_{j=3}^{k}(x-n+2a_j), \tag{3.5}$$

(3.6)

$$-\sum_{j=1}^{2} b_j \prod_{i=1,i\neq j}^{k}(x-n+2b_i)+\sum_{j=1}^{2} a_j \prod_{i=1,i\neq j}^{k}(x-n+2a_i) = -[4(a_1-a_2)-4]\prod_{j=3}^{k}(x-n+2a_j).$$

For $3 \leqslant i \leqslant n$,

$$-a_i(x-n+2a_1)(x-n+2a_2) = -a_i[x^2-2nx+2(a_1+a_2)x-2(a_1+a_2)n+n^2+4a_1a_2]$$

and

$$-b_i(x-n+2b_1)(x-n+2b_2) = -b_i[x^2-2nx+2(a_1+a_2)x-2(a_1+a_2)n+n^2+4a_1a_2+4(a_1-a_2)-4].$$

Therefore,

$$-b_i(x-n+2b_1)(x-n+2b_2)-[-a_i(x-n+2a_1)(x-n+2a_2)] = a_i(-4(a_1-a_2)+4). \tag{3.7}$$

By (3.6) and (3.7),

$$-\sum_{j=1}^{k} b_j \prod_{i=1,i\neq j}^{k}(x-n+2b_i)+\sum_{j=1}^{k} a_j \prod_{i=1,i\neq j}^{k}(x-n+2a_i) \tag{3.8}$$

$$= -[4(a_1-a_2)-4]\prod_{j=3}^{k}(x-n+2a_j)+\sum_{i=3}^{k}\prod_{j=3,j\neq i}^{k}(x-n+2a_j)a_i[-4(a_1-a_2)+4)].$$

Hence, by (3.5) and (3.8),

$$\left[\prod_{i=1}^{k}(x-n+2b_i)-\sum_{j=1}^{k} b_j \prod_{i=1,i\neq j}^{k}(x-n+2b_i)\right]-\left[\prod_{i=1}^{k}(x-n+2a_i)-\sum_{j=1}^{k} a_j \prod_{i=1,i\neq j}^{k}(x-n+2a_i)\right]$$

$$= \sum_{i=3}^{k}\prod_{j=3,j\neq i}^{k}(x-n+2a_j)a_i[-4(a_1-a_2)+4)].$$

Thus, by Definition 3.2

$$\prod_{i=1}^{k}(x-n+2b_i)-\sum_{j=1}^{k}b_j\prod_{i=1,i\neq j}^{k}(x-n+2b_i)\succ\prod_{i=1}^{k}(x-n+2a_i)-\sum_{j=1}^{k}a_j\prod_{i=1,i\neq j}^{k}(x-n+2a_i).$$

Hence we have finished the proof of the lemma. □

Theorem 3.10. *Let G be a connected graph of order n with chromatic number k. Then*

$$\sigma(Q(G),x)\preceq(x-n+r)^{(r-1)(k-s)}(x-n+r+1)^{rs}[(x-n+2r)^{k-s}(x-n+2r+2)^{s}$$

$$-(k-s)r(x-n+2r)^{k-s-1}(x-n+2r+2)^{s}-s(r+1)(x-n+2r)^{k-s}(x-n+2r+2)^{s-1}].$$

The equality holds if and only if $G\cong K_{\underbrace{r,\ldots,r}_{k-s},\underbrace{r+1,\ldots,r+1}_{s}}$, where r and s are integers with $n=rk+s$ and $0\leq s<k$.

Proof. Let G^* be a graph having the maximum coefficients of the signless Laplacian characteristic polynomial among all connected graphs of order n with chromatic number k. Then $V(G^*)$ can be partitioned into k color classes, say $V_1,V_2,\ldots,V_k$. Let $|V_i|=a_i$ for $i=1,2,\ldots,k$. Then $\sum_{i=1}^{k}a_i=n$. Lemma 3.8 implies that $G^*\cong K_{a_1,a_2,\ldots,a_k}$. Assume that $a_1\geqslant a_2\geqslant\ldots\geqslant a_k$. By theorem 2.1,

$$\sigma(Q(G),x)=\prod_{i=1}^{k}(x-n+a_i)^{a_i-1}\left(\prod_{i=1}^{k}(x-n+2a_i)-\sum_{j=1}^{k}a_j\prod_{i=1,i\neq j}^{k}(x-n+2a_i)\right).$$

If $a_p-a_q\geq 2$, Lemma 3.4 implies that (let $a_p=a,a_q=b$ in Lemma 3.4)

$$(x-n+a_p-1)^{a_p-2}(x-n+a_q+1)^{a_q}\succ(x-n+a_p)^{a_p-1}(x-n+a_q)^{a_q-1}.$$

Hence

$$\prod_{i=1}^{k}(x-n+b_i)^{b_i-1}\succ\prod_{i=1}^{k}(x-n+a_i)^{a_i-1}, \tag{3.9}$$

where $b_p=a_p-1,b_q=a_q+1,b_i=a_i$ for $1\leq i\leq k,i\neq p,q$. By Lemma 3.9,

$$\prod_{i=1}^{k}(x-n+2b_i)-\sum_{j=1}^{k}b_j\prod_{i=1,i\neq j}^{k}(x-n+2b_i))\succ\prod_{i=1}^{k}(x-n+2a_i)-\sum_{j=1}^{k}a_j\prod_{i=1,i\neq j}^{k}(x-n+2a_i).$$

Thus, by (3.9),

$$\prod_{i=1}^{k}(x-n+b_i)^{b_i-1}\left(\prod_{i=1}^{k}(x-n+2b_i)-\sum_{j=1}^{k}b_j\prod_{i=1,i\neq j}^{k}(x-n+2b_i)\right)$$

$$\succ\prod_{i=1}^{k}(x-n+a_i)^{a_i-1}\left(\prod_{i=1}^{k}(x-n+2a_i)-\sum_{j=1}^{k}a_j\prod_{i=1,i\neq j}^{k}(x-n+2a_i)\right).$$

Hence, replacing any pair (a_i, a_j) satisfying $a_i - a_j \geq 2$ with the pair $(a_i - 1, a_j + 1)$ in product

$$\prod_{i=1}^{k}(x-n+a_i)^{a_i-1}\left(\prod_{i=1}^{k}(x-n+2a_i)-\sum_{j=1}^{k}a_j\prod_{i\neq j}^{k}(x-n+2a_i)\right)=:f(x)$$

will increase the coefficients. By repeating this process, we find the coefficients of $f(x)$ with $\sum_{i=1}^{k} a_i = n$ and $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_k$ is maximum if and only if $a_1 = a_2 = \ldots = a_{k-s} = r$ and $a_{k-s+1} = \ldots = a_k = r+1$, where r, s are integers with $n = rk + s$ and $0 \leqslant s < k$. Then $G^* \cong K_{\underbrace{r,\ldots,r}_{k-s},\underbrace{r+1,\ldots,r+1}_{s}}$, which is called the Turán graph. It is not difficult to prove that if $G^* \cong K_{\underbrace{r,\ldots,r}_{k-s},\underbrace{r+1,\ldots,r+1}_{s}}$, then

$$\sigma(Q(G^*),x) = (x-n+r)^{(r-1)(k-s)}(x-n+r+1)^{rs}[(x-n+2r)^{k-s}(x-n+2r+2)^{s}$$

$$-(k-s)r(x-n+2r)^{k-s-1}(x-n+2r+2)^{s} - s(r+1)(x-n+2r)^{k-s}(x-n+2r+2)^{s-1}].$$

The theorem thus follows. □

Remark 3.11. Theorem 3.10 implies that the Turán graph $K_{\underbrace{r,\ldots,r}_{k-s},\underbrace{r+1,\ldots,r+1}_{s}}$ has the maximum coefficients of signless Laplacian characteristic polynomials among all graphs of order n with chromatic number k, where $n = rk + s$ and $0 \leq s \leq k$.

The following lemma is immediate from Lemmas 3.5.

Lemma 3.12. *Let G_1 and G_2 be two bipartite graphs with n_1 and n_2 vertices, respectively. For any two positive integers p_1 and p_2 satisfying $n_1 + p_1 = n_2 + p_2$, then*

$$x^{p_1}\sigma(G_1,x) \succeq x^{p_2}\sigma(G_2,x) \Rightarrow E(G_1) \succeq E(G_2);$$

$$x^{p_1}\sigma(G_1,x) \succ x^{p_2}\sigma(G_2,x) \Rightarrow E(G_1) \succ E(G_2).$$

By Lemmas 3.6 and 3.12 and Theorem 3.10, the following result is obvious.

Theorem 3.13. *Let G be a simple graph of order n whose chromatic number is k, where $n = rk + s$ and $0 \leqslant s < k$. Then*

$$E(S(G)) \leqslant E(S(K_{\underbrace{r,\ldots,r}_{k-s},\underbrace{r+1,\ldots,r+1}_{s}}))$$

with equality if and only if G is the complete multipartite graph $K_{\underbrace{r,\ldots,r}_{k-s},\underbrace{r+1,\ldots,r+1}_{s}}$, where $S(G)$ denotes the subdivision of G.

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