On Graded Second and Coprimary Modules and Graded Secondary Representations

Seçil ÇEKEN and Mustafa ALKAN
Akdeniz University Department of Mathematics Antalya, Turkey
secilceken@akdeniz.edu.tr, alkan@akdeniz.edu.tr

Abstract

In this paper we introduce and study the concepts of graded second (gr-second) and graded coprimary (gr-coprimary) modules which are different from second and coprimary modules over arbitrary graded rings. We list some properties and characterizations of gr-second and gr-coprimary modules and also study graded prime submodules of modules with gr-coprimary decompositions. We also deal with graded secondary representations for graded injective modules over commutative graded rings. By using the concept of σ-suspension $(\sigma)M$ of a graded module M, we prove that a graded injective module over a commutative graded Noetherian ring has a graded secondary representation.

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1 Introduction

Second submodules of modules over commutative rings were introduced in [16] as the dual notion of prime submodules. Recently this submodule class has been studied in detail by some authors (see [2], [3]). Second modules over arbitrary rings were defined in [1] and used as a tool for the study of attached prime ideals over noncommutative rings. In [6], second modules have been studied in detail in the noncommutative setting. In [4], the authors have introduced and studied graded second modules over commutative graded rings. Most of their results are related to reference [16] which have been proved for second submodules.

In [9], the authors introduced the concept of coprimary module which is a generalization of second modules. They gave some characterizations and properties of this module class and study coprimary decompositions of modules.

Secondary modules are generalizations of second modules over commutative rings. In [15], secondary modules were considered over commutative graded rings. In [15], Sharp defined graded secondary modules and used them as a tool for the study of asymptotic behavior of attached prime ideals.

In this paper we introduce and study graded second and graded coprimary modules over arbitrary graded rings. We also deal with graded secondary representations for graded injective modules over commutative graded rings.

Firstly we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [10] and [11] for these basic properties and more information on graded rings and modules. Throughout this paper, all rings are assumed to have identity elements and all modules are unital right modules unless otherwise stated. Let G be a multiplicative group and e denote the identity element of G. A ring R is called a graded ring (or G-graded ring) if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \oplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. If the inclusion is an equality, then the ring R is called strongly graded. The elements of R_g are called homogeneous of degree g and all the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \cup_{g\in G} R_g$. If $x \in R$, then x can be written uniquely as $\Sigma_{g\in G} x_g$, where x_g is called homogeneous component of x in R_g. Moreover, R_e is a subring of R and $1 \in R_e$. Also, if $r \in R_g$ and r is a unit, then $r^{-1} \in R_{g-1}$. A G-graded ring $R = \oplus_{g\in G} R_g$ is called a crossed product if R_g contains a unit for every $g \in G$. Note that a G-crossed product $R = \oplus_{g\in G} R_g$ is a strongly graded ring (see [11, 1.1.2. Remark]). For a G-graded ring R, $U^{gr}(R)$ denotes the set of units of R that are homogeneous, and $Z(R)$ denotes the set of central elements of R.

An ideal I of R is said to be a graded ideal if $I = \oplus_{g\in G}(I \cap R_g)$. Left and right graded ideals are defined analogously. A proper graded ideal P of a graded ring R is said to be a graded prime ideal (or gr-prime ideal) of R if whenever A and B are graded ideals of R such that $AB \subseteq P$, then either $A \subseteq P$ or $B \subseteq P$. A proper graded ideal P is a graded prime ideal of R if and only if whenever a and b are homogeneous elements of R such that $aRb \subseteq P$, then either $a \in P$ or $b \in P$. If 0 is a graded prime ideal of R, then R is said to be a graded prime (or gr-prime) ring.

Let R be a G-graded ring. A right R-module M is said to be a graded R-module (or G-graded R-module) if there exists a family of additive subgroups $\{M_g\}_{g\in G}$ of M such that $M = \oplus_{g\in G} M_g$ and $M_g R_h \subseteq M_{gh}$ for all $g, h \in G$. Also if an element of M belongs to $\cup_{g\in G} M_g = h(M)$, then it is called homogeneous. Note that M_g is a R_e-module for every $g \in G$.

Let $M = \oplus_{g\in G} M_g$ be a G-graded R-module and N be a submodule of M. Then N is called a graded submodule of M if $N = \oplus_{g\in G} N_g$, where $N_g = N \cap M_g$ for all $g \in G$. In this case, N_g is called the g-component of N. Moreover, M/N becomes a G-graded R-module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$.

Let N be an arbitrary submodule of a graded R-module M. Then by N^* we mean the graded submodule of M generated by all homogeneous elements $x \in N$. It is clear that N^* is the largest graded submodule contained in N. Note that $N^* = \oplus_{g\in G}(N \cap M_g)$.

Let M and M' be graded R-modules. Then an R-module homomorphism, $f : M \longrightarrow M'$ is called a graded homomorphism of degree g, if $f(M_h) \subseteq M'_{gh}$ for all $h \in G$.

Let R be a G-graded ring. One can form the category gr-R of graded right R-modules whose objects are graded right R-modules and whose morphisms are graded module homomorphisms of degree e. For $M \in$gr-R and $\sigma \in G$, the σ-suspension $(\sigma)M$ of M is defined to be the graded R-module obtained from M by putting $((\sigma)M)_\tau = M_{\sigma\tau}$ for all $\tau \in G$.

Let $R = \oplus_{g\in G} R_g$ be a G-graded ring. We define graded second (or gr-second) modules and list some properties of them. We prove that if G is an abelian group and R is a left graded fully bounded ring such that R/P is a left gr-Goldie ring for every gr-prime ideal P of R, then a graded right R-module M is a gr-second R-module if and only if $Q = ann_R(M)$ is a gr-prime ideal of R and M is a gr-divisible right (R/Q)-module (Theorem 2.7). We study the existence of gr-second factor modules of certain graded modules. We also prove that every non-zero gr-Artinian module contains only a finite number of maximal gr-second submodules (Theorem 2.11). After that we define the concept of graded coprimary (or gr-coprimary) module (which is a generalization of gr-second module) and study gr-coprimary decompositions of graded modules. In particular we prove that if M is a right module which has a gr-coprimary decomposition over a graded ring R such that for each homogeneous element a of R, the graded right ideal aR is generated by a central homogeneous element, then every graded prime submodule of M has a gr-coprimary decomposition (Theorem 3.5). We deal with gr-secondary representations for gr-injective modules over commutative graded rings. By using the concept of σ-suspension $(\sigma)M$ of a graded module M, we prove that a gr-injective module over a commutative gr-noetherian ring has a gr-secondary representation (Corollary 4.5). This result is the graded version of [14, Theorem 2.3].

2 Graded Second Modules

An R-module M is called a *second module* provided $M \neq 0$ and $ann_R(M) = ann_R(M/N)$ for every proper submodule N of M. By a *second submodule* of a module, we mean a submodule which is also a second module. In [6], it was proved that an R-module M is a second R-module if and only if $MI = M$ or $MI = 0$ for every ideal I of R.

Remark 1 *[12, Lemma 1] Let M be a graded R-module and let I be a graded ideal of R. Then MI and $(0 :_M I)$ are graded submodules of M and $ann_R(M)$ is a graded ideal of R.*

Definition 2.1 *Let R be a G-graded ring. A graded R-module M is said to be a graded second (or gr-second) R-module if $M \neq 0$ and $ann_R(M) = ann_R(M/N)$ for every proper graded submodule N of M.*

Let M be a graded R-module and K be a graded submodule of M. K is said to be a graded second submodule of M if it is a graded second module itself.

It can be easily checked that if M is a gr-second R-module, then $ann_R(M) = P$ is a gr-prime ideal of R. In this case M is called graded P-second (or gr-P-second) module.

Proposition 2.2 *Let R be a G-graded ring and M be a graded R-module. M is a gr-second R-module if and only if $MI = 0$ or $MI = M$ for every graded ideal I of R.*

Proof Use the similar arguments as in the ungraded case (see [6, Lemma 2.1]). $\square$

Note that a non-zero graded module M over a commutative graded ring R is gr-second if and only if $Mr = 0$ or $Mr = M$ for every $r \in h(R)$.

A graded R-module M is said to be graded simple (or gr-simple) if 0 and M are its only graded submodules. It is clear that every gr-simple R-module is gr-second.

Clearly every second graded module is a gr-second module. But the converse of this statement is not true in general. If $R = k[x, x^{-1}]$ is the ring of Laurent polynomials, where k is a field, then the right R-module R_R is a gr-second R-module but it is not a second R-module. (See also [4, Remark 2.1]).

Theorem 2.3 *Let R be a G-graded ring and $M = \oplus_{g\in G} M_g$ be a graded R-module. Then we have the following.*

(1) *If M is a gr-second R-module, then M_g is a second R_e-module for every $g \in G$ with $M_g \neq 0$.*

(2) *If R is a strongly graded ring and M_g is a second R_e-module for every $g \in G$, then M is a gr-second R-module.*

(3) *If R is a crossed product, $U^{gr}(R) \subseteq Z(R)$ and M_e is a second R_e-module, then M_g is a second R_e-module for every $g \in G$.*

(4) *If R is a graded integral domain, M is a torsion-free graded R-module and N is a second submodule of M such that N contains a nonzero homogeneous element, then N^* is a gr-second submodule of M.*

Proof (1) Let J be an ideal of R_e. Then $I = \oplus_{g\in G} R_g J$ is a graded ideal of R. Since M is gr-second, $MI = 0$ or $MI = M$. Let $g \in G$ with $M_g \neq 0$. If $MI = 0$, then $M_g J = M_g R_e J \subseteq M_g I = 0$ and so $M_g J = 0$. If $MI = M$, then we get that $M_g J = M_g$. Thus M_g is a second R_e-module.

(2) Clearly $M \neq 0$.Let $I = \oplus_{g\in G} I_g$ be a graded ideal of R. Then I_e is an ideal of R_e. Since R is strongly graded, $I = RI_e$ by [10, A-I.3.8. Corollary]. It follows that $MI = MRI_e = MI_e = \oplus_{g\in G}(M_g I_e)$. If $M_g I_e = 0$ for some $g \in G$, then $MI = 0$ by [10, A-I.3.7. Corollary]. If $M_g I_e \neq 0$ for every $g \in G$, then $M_g I_e = M_g$ and we get that $MI = M$.

(3) Since R is a strongly graded ring, $M_e = M_g R_{g^{-1}}$ and so $M_g \neq 0$ for every $g \in G$. Let I be an ideal of R_e and $g \in G$. Then $M_e I = 0$ or $M_e I = M_e$. Since R is crossed product, $R_{g^{-1}}$ contains a unit, say x. If $M_e I = 0$, then $M_g I = M_g x x^{-1} I \subseteq M_e x^{-1} I = M_e I x^{-1} = 0$ and so $M_g I = 0$. If $M_e I = M_e$, then $M_g = M_g x x^{-1} \subseteq M_e x^{-1} = M_e I x^{-1} = M_e x^{-1} I \subseteq M_g I$ and so $M_g = M_g I$. Thus M_g is a second R_e-module.

(4) $N^* \neq 0$, by the hypothesis. Let $0 \neq r \in h(R)$. Since M is torsion-free, $N^* r \neq 0$. Let $x \in N^*$. We can write $x = x_{g_1} + ... + x_{g_t}$ with $x_{g_i} \in N \cap M_{g_i}$, $x_{g_i} \neq 0$ for each $1 \leq i \leq t$. Since $Nr = N$, we can write $x_{g_i} = (n_{h_{i1}} + ... + n_{h_{it_i}})r$ with $n_{h_{ij}} \in h(M)$ and $n_{h_{i1}} + ... + n_{h_{it_i}} \in N$. Then $x_{g_i} = n_{h_{ij}} r$ for some $1 \leq j \leq t_i$ and $n_{h_{ik}} r = 0$ for $k \neq j$. Since M is torsion-free, $n_{h_{ik}} = 0$ for $k \neq j$. Thus $n_{h_{ij}} \in h(N)$ and so $x_{g_i} \in N^* r$ for each $1 \leq i \leq t$. This shows that $N^* r = N^*$ and hence N^* is a gr-second submodule of M. □

Proposition 2.4 *Let R be a G-graded ring, M be a graded R-module and A be a graded ideal of R such that $MA = 0$. Then, M is a gr-second R-module if and only if M is a gr-second (R/A)-module.*

Proof Use the similar arguments as in the ungraded case (see [6, Corollary 2.4]). □

Let R be a G-graded ring and M be a graded R-module. A graded submodule N of M is said to be a graded essential (or gr-essential) submodule of M, if for every non-zero graded submodule L of M we have $L \cap N \neq 0$.

Let N be a graded submodule of a graded module M. Then N is gr-essential in M if and only if N is essential in M by [11, 2.3.5 Proposition].

A graded prime ring R is said to be left graded bounded if each gr-essential left ideal contains a non-zero graded ideal. A graded ring R is said to be left graded fully bounded if the ring

R/P is left graded bounded for every graded prime ideal P of R. Right graded bounded and right graded fully bounded rings are defined analogously.

A left graded fully bounded ring need not be left fully bounded. For example, consider $R = \Delta[x, \varphi]$ where φ is an automorphism of the skewfield Δ, x is a variable and multiplication is given by $xa = \varphi(a)x$. R is a left graded fully bounded ring because every graded left ideal of R is two-sided. But R is not left fully bounded if φ is not an inner automorphism of Δ. (See [10, page 241]).

A graded ring R having finite Goldie dimension in the category of graded left R-modules and satisfying the ascending chain condition on graded left annihilators is called a left graded Goldie (or left gr-Goldie) ring. Right graded Goldie rings are defined analogously.

A left gr-Golide ring is not necessarily a left Goldie ring. Let k be a field and R be the polynomial ring $k[x, y]$ subject to the relation $xy = yx = 0$. Put $R_n = kx^n$ if $n \geq 0$ and $R_m = ky^m$ if $m < 0$. As a consequence of [10, C-I.1.1 Example], $R = \oplus_{n \in \mathbb{Z}} R_n$ is a left gr-Goldie ring but not a left Goldie ring.

Let R be a ring. An element c of R is called right regular provided $cr \neq 0$ for every non-zero element r in R. There is an analogous definition of left regular elements. An element c of R is called regular provided it is right and left regular.

Let R be a prime, right Goldie ring. Then every essential right ideal of R contains a regular element of R by a theorem of Goldie. In [7], Goodearl and Stafford proved the graded version this theorem.

Theorem 2.5 *[7, Theorem 4] Let G be an abelian group and R be a G-graded, gr-prime, right gr-Goldie ring. Then, any essential, graded right ideal I of R contains a homogeneous regular element.*

Let R be a G-graded ring and M be a graded R-module. Following [10], we say that M is a graded divisible (or gr-divisible) R-module if $M = Mc$ for every homogeneous regular element c in R. (See also [10, Page 179]).

Clearly every divisible graded module is gr-divisible. But a gr-divisible module is not necessarily divisible as the following example shows.

Example 2.6 *If $R = k[x, x^{-1}]$ is the ring of Laurent polynomials, where k is a field, then the right R-module R_R is a gr-divisible R-module but not a divisible R-module.*

Theorem 2.7 *Let G be an abelian group and R be a G-graded ring.*

(1) *If R is a gr-prime, right or left gr-Goldie ring, then every non-zero gr-divisible right R-module is 0-gr-second.*

(2) *Let R be a left graded fully bounded ring such that R/P is a left gr-Goldie ring for every gr-prime ideal P of R. Then a graded right R-module M is a gr-second R-module if and only if $Q = ann_R(M)$ is a gr-prime ideal of R and M is a gr-divisible right (R/Q)-module.*

Proof (1) Let X be a non-zero gr-divisible R-module and $A = ann_R(X)$. Suppose that $A \neq 0$. Then A is a gr-essential right (and left) ideal of R. A is an essential right (and left) ideal of R by [11, Proposition 2.3.5]. A contains a homogeneous regular element c, by Theorem 2.5. But this implies that $X = Xc \subseteq XA = 0$, a contradiction. Therefore $A = 0$. Let B be a non-zero graded ideal of R. Since R is a gr-prime ring, B is an essential graded ideal of R. So B contains a homogeneous regular element d by Theorem 2.5. It follows that $X = Xb \subseteq XB$ and hence $X = XB$. Thus X is a 0-gr-second R-module.

(2) Suppose that M is a gr-second R-module and $Q = ann_R(M)$. Then Q is a gr-prime ideal of R. Let $\overline{R}$ denote the left gr-bounded left gr-Goldie ring R/Q, and $\overline{c}$ be a homogeneous regular element of $\overline{R}$. Then the graded essential left ideal $\overline{R}\overline{c}$ contains non-zero two-sided graded ideal $\overline{A}$ of $\overline{R}$. There exists a graded ideal A of R such that $\overline{A} = A/Q$. $M = MA \subseteq M(Rc+Q) = Mc$ and hence $M = Mc$. It follows that $M\overline{c} = M$ for the $\overline{R}$-module M. Thus M is a gr-divisible right $\overline{R}$-module. The converse follows from Propositon 2.4 and (1). □

Lemma 2.8 *Let R be a G-graded ring such that R satisfies ascending chain condition on gr-prime ideals and for every proper graded ideal I of R there exists a finite collection of gr-prime ideals $Q_i (1 \leq i \leq n)$ such that $Q_1...Q_n \subseteq I \subseteq Q_1 \cap ... \cap Q_n$. Let M be a non-zero graded R-module.*

(1) *M is a gr-second R-module if and only if, for each gr-prime ideal P of R, either $MP = 0$ or $M = MP$.*

(2) *There exists a gr-second factor module of M.*

Proof (1) The necessity is clear. Conversely, suppose that $M = MP$ or $MP = 0$ for every gr-prime ideal P of R. Let I be any proper graded ideal of R. By the hypothesis, there exists a finite family of gr-prime ideals $Q_i\,(1 \leq i \leq n)$ such that $Q_1 \ldots Q_n \subseteq I \subseteq Q_1 \cap \cdots \cap Q_n$. If $MQ_i = 0$ for some $1 \leq i \leq n$ then $MI = 0$. Otherwise $M = MQ_i\,(1 \leq i \leq n)$ and hence

$$M = MQ_n = MQ_{n-1}Q_n = \cdots = MQ_1 \ldots Q_n \subseteq MI \subseteq M.$$

Thus $M = MI$. It follows that $MI = 0$ or $M = MI$ for every graded ideal I of R and so M is a gr-second R-module.

(2) By hypothesis there exists a finite family of gr-prime ideals $P_i\,(1 \leq i \leq t)$ such that $P_1...P_n \subseteq ann_R(M) \subseteq \cap_{i=1}^{n} P_i$. This implies that $MP_1...P_n = 0$. If $MP_i = M$ for every $1 \leq i \leq n$, then $0 = MP_1...P_n = ... = MP_n = M$, a contradiction. Thus $M \neq MP_i$ for some $1 \leq i \leq t$. Let P be a gr-prime ideal of R maximal in the collection of gr-prime ideals Q of R such that $M \neq MQ$. Note that $M \neq MP$. Let T be any gr-prime ideal of R properly containing P. By the choice of P, we have $M = MT$. Thus $M/MP = (M/MP)(T/P)$. By (1), M/MP is a gr-second (R/P)-module. Then Proposition 2.4 gives that the R-module M/MP is a gr-second R-module. □

By [8, Proposition 1.1], the conditions on the graded ring R in Lemma 2.8 are satisfied when R is a graded ring which satisfies ascending chain condition on graded ideals.

Let R be a G-graded ring, M be a graded R-module and K be a graded submodule of M. K is called graded small (or gr-small) submodule of M if whenever L is a graded submodule of M such that $K + L = M$ we must have $L = M$.

Clearly every small graded submodule of a graded module is gr-small. But a gr-small submodule need not be a small submodule. For example, if k is a field and $R = k[x]$ is the polynomial ring, then (x) is a gr-small submodule of R_R but not a small submodule of R_R.

We say that M is a graded hollow (or gr-hollow) module if $M \neq 0$ and every proper graded submodule of M is gr-small.

Clearly every hollow graded module is gr-hollow. The following example shows that the converse of this statement is not true in general.

Example 2.9 *Let k be a field and $R = k[x]$ be the polynomial ring. Consider the right R-module R_R. Every proper graded ideal of R is of the form (x^n) for some $n \in \mathbb{Z}^+$. So R_R is a gr-hollow module but it is not a hollow module.*

Theorem 2.10 *Let R be a G-graded ring and M be a gr-hollow module.*

(1) There exists at most one gr-prime ideal P of R such that M/N is a gr-P-second R-module for some graded submodule N of M.

(2) If R satisfies ascending chain condition on gr-prime ideals and for every proper graded ideal I of R there exists a finite collection of gr-prime ideals $Q_i (1 \leq i \leq n)$ such that $Q_1 ... Q_n \subseteq I \subseteq Q_1 \cap ... \cap Q_n$, then there exists only one gr-prime ideal P of R such that M/N is a gr-P-second R-module for some graded submodule N of M, where $P = \{\Sigma_{g \in G} r_g \in R : M r_g R \neq M$ for every $g \in G\}$.

Proof (1) Let P_1 and P_2 be gr-prime ideals of R such that M/N_1 is a gr-P_1-second R-module and M/N_2 is a gr-P_2-second R-module for some graded submodules N_1, N_2 of M. As M is gr-hollow, $N_1 + N_2$ is a proper graded submodule of M. $M/(N_1 + N_2) \simeq (M/N_i)/(N_1 + N_2/N_i)$ is a non-zero graded factor module of the gr-second module M/N_i for each $i = 1, 2$. So $M/(N_1 + N_2)$ must be a gr-seond R-module and $P_1 = P_2$.

(2) We know that there exists only one gr-prime ideal Q of R such that M/N is a gr-Q-second module for some graded submodule N of M, by Lemma 2.8 and (1). We must show that $Q = P$. If $x = \Sigma_{g \in G} x_g \in Q$, then $x_g \in Q$ for every $g \in G$ as Q is a graded ideal. So $M x_g R \subseteq N \neq M$ for every $g \in G$ and this shows that $x \in P$. Conversely, if $x = \Sigma_{g \in G} x_g \in P$, then $M x_g R \neq M$ for every $g \in G$. Since M is gr-hollow, we have $M x_g R + N \neq M$ for every $g \in G$, and since M/N is graded Q-second, we have that $M/(M x_g R + N)$ is a graded Q-second module for every $g \in G$. Thus $x_g \in ann_R(M/M x_g R + N) = Q$ for every $g \in G$. This implies that $x \in Q$. $\square$

Let R be a graded ring and M be a non-zero graded R-module. By a maximal gr-second submodule of M we mean a gr-second submodule L of M such that L is not properly contained in another gr-second submodule of M. Let $(N_i)_{i \in I}$ be a chain of gr-second submodules of M. We can prove that $\cup_{i \in I} N_i$ is a gr-second submodule of M by using similar arguments as in the ungraded case. (See [6, Proposition 4.2]). By using this result and Zorn's Lemma, we can prove that every gr-second submodule of M is contained in a maximal gr-second submodule of M.

Theorem 2.11 *Let R be a G-graded ring and M be a non-zero graded Artinian R-module. Then M contains only a finite number of maximal gr-second submodules.*

Proof Use the similar arguments as in the ungraded case (see [6, Theorem 4.4]). $\square$

3 Graded Coprimary Modules

In [9], the authors defined a coprimary module as follows: Let R be a ring. Given a prime ideal P of R, a non-zero R-module M is called P-coprimary if

(i) $(N : M) \subseteq P$ for every proper submodule N of M, and

(ii) $P^h \subseteq ann_R(M)$ for some positive integer h.

M is called coprimary if it is P-coprimary for some prime ideal P of R.

In this section we introduce and study the notion of graded coprimary module which is a generalization of the notion of gr-second module.

Definition 3.1 *Let R be a graded ring and P be a gr-prime ideal R. A non-zero graded R-module M is called graded P-coprimary (or gr-P-coprimary) provided there exists a positive integer n such that*

$$P^n \subseteq (0 :_R M) \subseteq (N :_R M) \subseteq P,$$

for every proper graded submodule N of M. The graded module M is called graded coprimary (or gr-coprimary) if it is graded P-coprimary for some gr-prime ideal P.

A non-zero graded R-module M has a gr-coprimary decomposition if there exist a positive integer n and graded submodules $M_i (1 \leq i \leq n)$ of M such that $M = M_1 + ... + M_n$, and M_i is gr-coprimary for each $1 \leq i \leq n$. If M has a gr-coprimary decomposition, then we say that M has a normal gr-coprimary decomposition if there exist a positive integer n, distinct gr-prime ideals $P_i (1 \leq i \leq n)$ of R, and gr-P_i-coprimary submodules $M_i (1 \leq i \leq n)$ of M such that

(i) $M = M_1 + ... + M_n$, and

(ii) $M \neq M_1 + ... + M_{i-1} + M_{i+1} + ... + M_n$ for all $1 \leq i \leq n$.

In this case the set $\{P_1, ..., P_n\}$ is called graded attached primes of M and denoted by $Att^*(M)$.

It is clear that every gr-second module is gr-coprimary. Also it is easy to see that every graded factor module of a gr-coprimary module is gr-coprimary.

Lemma 3.2 *Let R be a graded ring and P be a gr-prime ideal of R. Then a non-zero graded R-module M is gr-P-coprimary if and only if, for every graded ideal A of R, $M = MA$ if $A \nsubseteq P$ and there exists a positive integer h such that $MA^h = 0$ if $A \subseteq P$.*

Proof This is straightforward. □

Definition 3.3 *Let R be a graded ring, M be a graded R-module and N be a graded submodule of M. N is called a graded pure submodule of M if $NI = MI \cap N$ for every graded ideal I of R.*

Proposition 3.4 *Let R be a graded ring, P be a gr-prime ideal of R, M be a graded R-module and N be a non-zero proper graded pure submodule of M. M is a gr-P-coprimary module if and only if N and M/N are gr-P-coprimary modules.*

Proof Suppose that M is gr-P-coprimary. Then $P^h \subseteq ann_R(M)$ for some $h \in \mathbb{Z}^+$. Let A be graded ideal of R. If $A \subseteq P$, then $NA^h = 0$. If $A \nsubseteq P$, then $NA = MA \cap N = M \cap N = N$. Thus N is gr-P-coprimary. It is clear that M/N is gr-P-coprimary. Conversely suppose that N and M/N are gr-P-coprimary modules. Then $P^{h_1} \subseteq ann_R(N)$ and $P^{h_2} \subseteq ann_R(M/N)$ for some $h_1, h_2 \in \mathbb{Z}^+$. Let $h = \max(h_1, h_2)$. Then we have $MP^h \subseteq N$ and $0 = NP^h = MP^h \cap N = MP^h$. Let A be a graded ideal of R. If $A \subseteq P$, then $MA^h = 0$. If $A \nsubseteq P$, then $NA = N$ and

$MA + N = M$. It follows that $MA + NA = MA + (MA \cap N) = MA = M$. Thus M is gr-P-coprimary. □

In [8], the authors defined a graded prime module as follows. A graded R-module M is called a graded prime module provided that $ann_R(N) = ann_R(M)$ for all non-zero graded R-submodules N of M. A graded submodule K of M is called a graded prime (or gr-prime) submodule of M, if M/K is a graded prime module. In this case $P = (K : M)$ is a gr-prime ideal of R and K is called graded P-prime submodule of M. (See [5], [8], [12] for more details about graded prime submodules).

Theorem 3.5 *Let R be a G-graded ring such that for each $a \in h(R)$ the graded right ideal aR is generated by a central homogeneous element and let M be a graded R-module.*

(1) *If M is gr-coprimary and N is a nonzero graded P-prime submodule of M, then N is gr-P-coprimary.*

(2) *If N is a gr-P-coprimary submodule of M and K is a graded prime sumodule of M, then $N \cap K$ is gr-P-coprimary.*

(3) *If M has a gr-coprimary decomposition and N is a graded prime submodule of M, then N has a gr-coprimary decomposition.*

Proof (1) Let M be gr-Q-coprimary module. Then $(N : M) = P \subseteq Q$ and $Q^h \subseteq ann_R(M) \subseteq P$ for some $h \in \mathbb{Z}^+$. So we get that $Q = P$.

There exists a positive integer h such that $P^h \subseteq ann_R(M) \subseteq ann_R(N)$. Let A be a graded ideal of R. If $A \subseteq P$, then $NA^h = 0$. Assume that $A \nsubseteq P$. Let $a \in A \backslash P$ be a homogeneous element. By hypothesis, $aR = bR = Rb$ for some $b \in h(R) \cap Z(R)$ and hence $M = M(RaR) = M(Rb) = Mb$. Let $n \in N$. Then $n = mb$ for some $m = \Sigma_{i=1}^{t} m_{g_i} \in M, (m_{g_i} \neq 0)$. Since N is graded $m_{g_i}b \in N$ for every $1 \leq i \leq t$. Let $i \in \{1, ..., t\}$. Since $b \in Z(R)$, $m_{g_i}bR = m_{g_i}Rb \subseteq N$ and so $b \in ann_R(N + m_{g_i}R/N)$. If $N + m_{g_i}R \neq N$, then $ann_R(N + m_{g_i}R/N) = P$ and so $b \in P$. But this implies that $a \in P$, a contradiction. Thus $N + m_{g_i}R = N$ and we have that $m_{g_i} \in N$. This shows that $n \in Nb \subseteq N(RaR) \subseteq NA$. Therefore we get that $N = NA$.

(2) It can be easily shown that $N \cap K$ is a graded prime submodule of N. So the result follows from (1).

(3) Let $M = \Sigma_{i=1}^{k} S_i$ be a normal gr-coprimary decomposition of M and $Att^*(M) = \{P_1, ..., P_k\}$. Let N be a graded P-prime submodule of M. Then $S_i \nsubseteq N$ for some S_i, say S_1. We show that $P = P_1$. There exists a homogeneous element $y_h \in S_1 \backslash N$. Also there exists a positive integer n_1 such that $P_1^{n_1} \subseteq ann_R(S_1)$. Since $y_h P_1^{n_1} = 0 \subseteq N$ and N is graded P-prime, we get that $P_1 \subseteq P$. For the other containment, suppose that there exists a homogeneous element $c \in P \backslash P_1$. Since $RcR \nsubseteq P_1$ and S_1 is gr-P_1-coprimary, we get that $S_1 = S_1(RcR) \subseteq M(RcR) = M(cR) \subseteq N$ which is a contradiction. Therefore $P_1 = P$. Similarly, if $S_j \nsubseteq N$ for $j \neq 1$, then $P = P_1 = P_j$, a contradiction. Thus $S_j \subseteq N$ for every $2 \leq j \leq n$. It follows that $N = N \cap (S_1 + \Sigma_{j=2}^{n} S_j) = \Sigma_{j=2}^{n} S_j + (N \cap S_1)$. Now the result follows from (2). □

A G-graded ring R is said to be gr-regular if for every homogeneous element $x \in h(R)$ there exists $y \in R$ such that $x = xyx$. By [10, C-I.5.1. Proposition], a G-graded ring R is gr-regular if and only if every principal left (right) graded ideal is generated by a homogeneous idempotent element. A gr-regular ring R is said to be gr-abelian regular if all homogeneous idempotent elements of R are central.

Clearly every regular (resp. abelian regular) graded ring is gr-regular (resp. gr-abelian regular). But the converse of this statement is not true in general. Let k be a field and consider the first Weyl algebra $A_1(k)$ that is the algebra generated by the elements of k together with x and y, which commute with the elements of k and satisfy the equation $xy - yx = 1$. Put $S = A_1(k)$, $\deg(x) = 1$ and $\deg(y) = -1$. Then S is a graded ring such that $S_0 = k[xy]$. By [10, C-I.5.24. Example], the total graded ring of fractions of S, $Q^g(S)$ is a gr-abelian regular ring but not a regular ring and hence not an abelian regular ring.

Corollary 3.6 *Let R be a gr-abelian regular ring and M be a graded R-module which has a gr-coprimary decomposition. Then every gr-prime submodule of M has a gr-coprimary decomposition.*

Proof By [10, C-I.5.1 Proposition], the conditions on the graded ring R in Theorem 3.5 are satisfied when R is a gr-Abelian regular ring. Thus the result follows. □

4 Graded Secondary Representations For Graded Injective Modules

In this section we deal with graded secondary representations for graded injective modules over commutative graded rings.

Let R be a commutative G-graded ring and I be a graded ideal of R. The graded radical of I (in abbreviation "$Gr(I)$") is the set of all $x = \Sigma_{g\in G} x_g \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element of R, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$.

Let R be a commutative G-graded ring. In [15], Sharp defined graded secondary modules as follows: A graded R-module M is said to be graded secondary (or gr-secondary) if $M \neq 0$ and, for each homogeneous element r of R, the endomorphism of M given by multiplication by r is either surjective or nilpotent. In this case $Gr(ann_R(M)) = P$ is a gr-prime ideal of R, and M is said to be graded P-secondary. M is said to have a gr-secondary representation if it can be written as a sum $M = M_1 + ... + M_k$ with each M_i gr-secondary.

Clearly every gr-second module over a commutative graded ring is gr-secondary. Also note that, when R is a commutative graded Noetherian ring, M is gr-coprimary if and only if M is gr-secondary.

Proposition 4.1 *Let R be a graded integral domain and M be a torsionfree graded R-module which has a secondary representation. Then M has a gr-secondary representation.*

Proof Firstly we show that if N_1 and N_2 are submodules of M, then $(N_1 + N_2)^* = N_1^* + N_2^*$. Clearly $N_1^* + N_2^* \subseteq (N_1 + N_2)^*$. Let $x \in h((N_1 + N_2)^*)$. $x = n_1 + n_2$ for some $n_1 \in N_1$, $n_2 \in N_2$. Since x is homogeneous, n_1 and n_2 must be homogeneous of the same degree with x. Hence $n_1 \in N_1^*$, $n_2 \in N_2^*$ and so $x \in N_1^* + N_2^*$.

Let $M = N_1 + ... + N_k$ be a secondary representation of M with N_i a secondary submodule of M for $1 \leq i \leq k$. Then we have $M = N_1^* + ... + N_k^*$. It can be proved that $N_i^* = 0$ or N_i^* is a gr-secondary submodule of M for $i = 1, .., k$, as in the proof of Theorem 2.3-(4). This shows that M has a gr-secondary representation. $\square$

Le R be a commutative graded ring. Following [13], we say that I is a graded primary ideal of R (in abbreviation, "G-primary ideal") if $I \neq R$ and whenever $a, b \in h(R)$ with $ab \in I$ then $a \in I$ or $b \in Gr(I)$. In this case $Gr(I) = P$ is a gr-prime ideal of R and I is called G-P-primary. Let I be a proper graded ideal of R. In [13], a graded primary G-decomposition of I is defined as an intersection of finitely many graded primary ideals of R. Such a graded primary G-decomposition $I = Q_1 \cap ... \cap Q_n$ with $Gr(Q_i) = P_i$ for $i = 1, ..., n$ of I is said to be a minimal graded primary G-decomposition of I precisely when

(i) $P_1, ..., P_n$ are different gr-prime ideals of R, and

(ii) $Q_j \not\supseteq \cap_{\substack{i=1 \\ i \neq j}}^{n} Q_i$ for all $j = 1, ..., n$.

I is said to be a G-decomposable graded ideal of R precisely when it has a graded primary G-decomposition. Note that every G-decomposable graded ideal of R has a minimal graded primary G-decomposition.

Let R be a G-graded ring and E be a graded R-module. Following [10], we say that E is a gr-injective R-module if E is an injective object in gr-R. In [10] it was shown that every injective graded module is gr-injective but a gr-injective module need not be injective. (See [10, A-I.2.5. Corollary and A-I.2.6. Remark]).

By using the notion of σ-suspension $(\sigma)M$ of a graded module M, we obtain the following two lemmas which are the graded versions of [14, Lemma 2.1 and 2.2].

Lemma 4.2 *Let R be a commutative G-graded ring, Q be a graded P-primary ideal of R and E be a gr-injective R-module. Then $(0 :_E Q)$ is zero or graded P-secondary submodule of E.*

Proof Suppose that $(0 :_E Q) \neq 0$. Let $a \in R$ be a homogeneous element of degree σ.

If $a \in P$, then $a^n \in Q$ for some positive integer n, so that $(0 :_E Q)a^n = 0$.

If $a \notin P$, then we see that $(0 :_E Q) = (0 :_E Q)a$ as follows. Let $x \in (0 :_E Q)$ be a homogeneous element of degree δ. Define the map $\phi : (\delta^{-1})(R/Q) \longrightarrow E$ for which $\phi(b + Q) = xb$ for all $b + Q \in (\delta^{-1})(R/Q)$. Clearly ϕ is an R-module homomorphism. Let $b + Q \in (\delta^{-1})(R/Q)_\tau$ for $\tau \in G$. Then $b = b_{\delta^{-1}\tau} + q$ for some $b_{\delta^{-1}\tau} \in R_{\delta^{-1}\tau}$ and $q \in Q$. We have $xb = x(b_{\delta^{-1}\tau} + q) = xb_{\delta^{-1}\tau} \in E_\tau$. So ϕ is a graded R-module homomorphism.

Let $g_a : (\delta^{-1})(R/Q) \longrightarrow (\sigma\delta^{-1})(R/Q)$ be the map defined by $g_a(y + Q) = ya + Q$ for all $y + Q \in R/Q$. Clearly g_a is an R-module homomorphism. If $y + Q \in (\delta^{-1})(R/Q)_\tau$ for $\tau \in G$, then $y = y_{\delta^{-1}\tau} + q'$ for some $y_{\delta^{-1}\tau} \in R_{\delta^{-1}\tau}$ and $q' \in Q$. We have $ya + Q = (y_{\delta^{-1}\tau} + q')a + Q = ay_{\delta^{-1}\tau} + Q \in (\sigma\delta^{-1})(R/Q)_\tau$. Thus g_a is a graded R-module homomorphism.

If $y + Q \in \ker(g_a)$, where $y = \Sigma_{i=1}^{m} y_{g_i}$, $y_{g_i} \neq 0$, then $ya = \Sigma_{i=1}^{m} y_{g_i} a \in Q$. Since Q is a graded ideal, $y_{g_i} a \in Q$ for all $1 \leq i \leq m$. We get that $y_{g_i} \in Q$ for all $1 \leq i \leq m$, as Q is graded P-primary. Therefore $y \in Q$, so that g_a is a monomorphism in gr-R.

The diagram

$$\begin{array}{ccccc} & & E & & \\ & & \phi \uparrow & & \\ 0 & \longrightarrow & (\delta^{-1})(R/Q) & \xrightarrow{g_a} & (\sigma\delta^{-1})(R/Q) \end{array}$$

has exact row in gr-R. Since E is a gr-injective module, this diagram can be completed with a graded R-module homomorphism $\psi : (\sigma\delta^{-1})(R/Q) \longrightarrow E$ such that $\psi g_a = \phi$. Thus $x =$

$\phi(\bar{1}) = \psi g_a(\bar{1}) = \psi(\bar{1}a) = \psi(\bar{1})a$. Since $\psi(\bar{1}) \in (0 :_E Q)$, we have $x \in (0 :_E Q)a$. As $(0 :_E Q)$ is generated by homogeneous elements, we get that $(0 :_E Q) = (0 :_E Q)a$, and the result follows. □

Lemma 4.3 *Let R be a commutative graded ring, $I_1, ..., I_n$ be graded ideals of R and E be a gr-injective R-module. Then*

$$\sum_{i=1}^{n}(0 :_E I_i) = (0 :_E \bigcap_{i=1}^{n} I_i)$$

Proof Let $x \in (0 :_E \cap_{i=1}^{n} I_i)$ be a homogeneous element of degree σ. Let $\pi : (\sigma^{-1})R \longrightarrow (\sigma^{-1})(R/\cap_{i=1}^{n} I_i)$ and, for each $i = 1, ..., n$, $\pi_i : (\sigma^{-1})R \longrightarrow (\sigma^{-1})(R/I_i)$, be the natural graded homomorphisms. There is an R-monomorphism $f : (\sigma^{-1})(R/\cap_{i=1}^{n} I_i) \longrightarrow \oplus_{i=1}^{n}(\sigma^{-1})(R/I_i)$ for which $f(\pi(a)) = (\pi_1(a), ..., \pi_n(a))$ for all $a \in R$. If $\pi(a) = a + \cap_{i=1}^{n} I_i \in (\sigma^{-1})(R/\cap_{i=1}^{n} I_i)_\tau$ for $\tau \in G$, then $a = r_{\sigma^{-1}\tau} + y$ for some $r_{\sigma^{-1}\tau} \in R_{\sigma^{-1}\tau}$ and $y \in \cap_{i=1}^{n} I_i$. We have $(\pi_1(a), ..., \pi_n(a)) = (r_{\sigma^{-1}\tau} + y + I_1, ..., r_{\sigma^{-1}\tau} + y + I_n) = (r_{\sigma^{-1}\tau} + I_1, ..., r_{\sigma^{-1}\tau} + I_n) \in \oplus_{i=1}^{n}(\sigma^{-1})(R/I_i)_\tau = (\oplus_{i=1}^{n}(\sigma^{-1})(R/I_i))_\tau$. Thus f is a graded R-module homomorphism.

Also, there is an R-module homomorphism $g : (\sigma^{-1})(R/\cap_{i=1}^{n} I_i) \longrightarrow E$ for which $g(\pi(a)) = xa$ for all $a \in R$. If $a + \cap_{i=1}^{n} I_i \in (\sigma^{-1})(R/\cap_{i=1}^{n} I_i)_\tau$ for $\tau \in G$, then $a = s_{\sigma^{-1}\tau} + z$ for some $s_{\sigma^{-1}\tau} \in R_{\sigma^{-1}\tau}$ and $z \in \cap_{i=1}^{n} I_i$. We have $xa = x(s_{\sigma^{-1}\tau} + z) = xs_{\sigma^{-1}\tau} \in E_\tau$. Thus g is a graded R-module homomorphism.

As E is gr-injective, the diagram

$$\begin{array}{ccccc} & & E & & \\ & & g \uparrow & & \\ 0 & \longrightarrow & (\sigma^{-1})(R/\cap_{i=1}^{n} I_i) & \xrightarrow{f} & \oplus_{i=1}^{n}(\sigma^{-1})(R/I_i) \end{array}$$

can be completed with a graded homomorphism $h : \oplus_{i=1}^{n}(\sigma^{-1})(R/I_i) \longrightarrow E$ such that $hf = g$. Now $x = g(\pi(1)) = hf(\pi(1)) \in \mathrm{Im}(h)$, and it is clear that $\mathrm{Im}(h) \subseteq \Sigma_{i=1}^{n}(0 :_E I_i)$. It follows that $(0 :_E \cap_{i=1}^{n} I_i) \subseteq \Sigma_{i=1}^{n}(0 :_E I_i)$. Since the reverse inclusion is clear the result follows. □

Theorem 4.4 *Let R be a commutative graded ring and the zero ideal of R have a graded primary G-decomposition. If E is a gr-injective R-module, then E has a gr-secondary representation.*

More precisely, let $0 = Q_1 \cap ... \cap Q_n$ be a minimal graded primary G-decomposition of the zero ideal of R, with Q_i a graded G-P_i-primary ideal for $i = 1, ..., n$. Then

$E = (0 :_E Q_1) + ... + (0 :_E Q_n)$, and $(0 :_E Q_i)$ is either zero or graded P_i-secondary for $i = 1, ..., n$.

Proof $(0 :_E Q_i)$ is either zero or gr-secondary for each $1 \leq i \leq n$, by Lemma 4.2. Lemma 4.3 shows that $E = (0 :_E 0) = (0 :_E \cap_{i=1}^{n} Q_i) = \Sigma_{i=1}^{n}(0 :_E Q_i)$, where $(0 :_E Q_i)$ is either zero or graded P_i-secondary. Thus E has a gr-secondary representation. □

In [14, Theorem 2.3], it was proved that every injective module over a commutative Noetherian ring has a secondary representation. In the following corollary we get the graded version of this result by using the concept of σ-suspension $(\sigma)M$ of a graded module M.

Corollary 4.5 *Let R be a commutative graded Noetherian ring and E be a gr-injective R-module. Then E has a gr-secondary representation.*

Proof Since R is commutative graded Noetherian, every proper graded ideal of R has a graded primary G-decomposition by [13, Corollary 2.16]. So E has a gr-secondary representation by Theorem 4.4. □

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