C-NORMAL AND hypercyclically embedded EMBEDDED SUBGROUPS OF FINITE GROUPS[1]

Ning Su
(School of. Math., Sun Yatsen University, Guangzhou, 510275, P.R. China)
(E-mail: zsusuning@hotmail.com)

Yanming Wang
(Lingnan Coll. and Math. Dept., Sun Yatsen University, Guangzhou, 510275, China)
(Email: stswym@mail.sysu.edu.cn)

Zhouqing Xie [2]
(School of. Math., Sun Yatsen University, Guangzhou, 510275, P.R. China)
(E-mail: st03xzq@mail2.sysu.edu.cn)

Abstract Let p be a prime, E be a normal subgroup of a finite group G. In this paper, we will investigate the way E embedded in G under the assumption that some p-subgroups of E are c-normal in G. We pay more attention to the p-subgroups of E with given order p^d. We generalized several recent results of other scholars.

Keywords c-normal, hypercenter, p-nilpotent, supersolvable.
2010 MR Subject Classification 20D10

1 Introduction

All groups considered in this paper are finite. We use conventional notions and notation, as in [5]. G always denotes a finite group, $|G|$ is the order of G, and $\pi(G)$ denotes the set of all primes dividing $|G|$, G_p is a Sylow p-subgroup of G for a prime $p \in \pi(G)$.

In [13], Wang defined the c-normality of a subgroup as follows and prove that a finite group G is solvable if and only if every maximal subgroup of G is c-normal in G.

[1]Project supported by NSFC(11171353)
[2]The corresponding author

Definition 1.1 ([13, Definition 1.1]). *Let H be a subgroup of a finite group G. We call H is c-normal in G if there exists a normal subgroup N of G such that $HN = G$ and $H \cap N \leq H_G$.*

The basic properties of c-normality are as follows.

Lemma 1.2 (see [13, Lemma 2.1] and [9, Lemma 2.4]). *Let G be a group. Then*
(1) *If H is normal in G, then H is c-normal in G.*
(2) *If H is c-normal in G and $H \leq K \leq G$, then H is c-normal in K.*
(3) *Suppose that K is a normal subgroup of G and that H is c-normal in G. Then HK/K is c-normal in G/K when $K \leq H$ or $(|H|, |K|) = 1$.*

Let G be a finite group. Several authors successfully use the c-normal property of some subgroups of G of prime power order to determine the structure of G (see [1, 2, 8–12]). Many results in these previous papers have the following form: Let E be a normal subgroup of G and $\mathcal{F}$ be a saturated formation containing the class of all supersoluble groups. Suppose that G/E is in $\mathcal{F}$. If for each prime divisor p of $|E|$, some p-subgroups of E are c-normal G, then $G \in \mathcal{F}$. Actually, in a more general case, if we can get a criterion that E lies in the $\mathcal{F}$-hypercenter, then $G/E \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In order to get good results, many authors have to impose the c-normal hypotheses on all the prime divisors or the minimal or maximal divisor p of $|G|$ rather than any prime divisor. In this paper, we try to get more general results.

Now let p be a fixed prime. In this paper, we focus on how a normal subgroup E embedded in G provided every p-subgroup of E with some fix order is c-normal in G. For this purpose, we introduce the concept of p-hypercentrally embeded:

Definition 1.3. *Let G be a finite group. A normal subgroup E of G is said to be p-hypercentrally embedded in G if every p-chief factor of G below E is cyclic.*

The main result of this paper is the following theorem.

Theorem A. *Let p be a fixed prime and E be a normal subgroup of a finite group G. Suppose that E_p is a Sylow p-subgroup of E and p^d is a prime power such that $1 < p^d \leq \max(|E_p|/p, p)$. If all the subgroups of E_p with order p^d and $2p^d$ (if a quaternion group is involved in E_p) are c-normal in G, then E is p-hypercyclically embedded in G.*

2 Proof of the results

First we need some results on c-supplemented subgroups of finite groups. Following [3], a group H is said to be c-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$. It is clear from the definition that if a subgroup H of G is c-normal in G, then H is c-supplemented in G.

Lemma 2.1. *If N is a minimal abelian normal subgroup of G, then any proper subgroup of N is not c-supplement in G.*

Proof. Suppose that this Lemma is not true and let H be a proper subgroup of N such that H is c-supplemented in G. Obviously $H_G = 1$ since $H_G < N$ and N is a minimal normal subgroup of G. By the definition of c-supplemented subgroups, there exists a subgroup M of G such that $G = HM$ with $H \cap M \leq H_G = 1$. Hence $NM \geq HM = G$. Since N is abelian, we know that $N \cap M \trianglelefteq G$. Hence $N \cap M = 1$. Therefore we have $|G| = |NM| = |N||M| > |H||M| = |HM|$, a contradiction. □

For a saturated formation $\mathcal{F}$, the $\mathcal{F}$-hypercenter of a group G is denoted by $Z_{\mathcal{F}}(G)$ (see [5, p 389, Notation and Definitions 6.8(b)]). Let $\mathcal{U}$ denote the class of all supersolvable groups and let $\mathcal{N}$ denote the class of all nilpotent groups. Suppose that A is a normal subgroup of G. It is clear that $A \leq Z_{\mathcal{U}}(G)$ if and only if every chief factor of G below A is cyclic, and that $A \leq Z_{\mathcal{N}}(G)$ if and only if every chief factor of G below A is central. In [1], Asaad gave the following result: Let P be a nontrivial normal p-subgroup, where p is an odd prime. If every minimal subgroup of P is c-supplemented in G, then $P \leq Z_{\mathcal{U}}(G)$. It is helpful to give a result for $p = 2$. In fact, we have the following proposition:

Proposition 2.2. *Let P be a normal 2-subgroup of G. If all minimal subgroups of P and all cyclic subgroups of P with order 4 (if a quaternion group is involved in P) are c-supplemented in G, then $P \leq Z_{\mathcal{N}}(G)$.*

Proof. Let Q be a Sylow q-subgroup of G, where q is prime different from p. We claim that PQ is nilpotent. Suppose that the claim is not true and let H be a minimal non-nilpotent subgroup of PQ. Then $H = [H_2]H_q$, where $H_q \in Syl_q(H)$ and H_2 is a normal Sylow 2-subgroup of H. By Itô's result (see [4, Chapter 3, 5.2]), we have that $\exp(H_2) \leq 4$ and that H_q acts irreducibly on $H_2/\Phi(H_2)$. It is easy to see that $|H_2/\Phi(H_2)| \geq 4$. Clearly H_q acts nontrivially on H_2 but acts trivially on any proper H_q-invariant subgroup of H_2. It follows by the reduction theorem of Hall and Higmman (see [7, Chapter 5 Theorem 3.7]) that $H_2' = \Phi(H_2)$.

Case 1. Suppose first that a quaternion group is involved in H_2. Then $\exp(H_2) = 4$, and we may take a subgroup $\langle x \rangle \leq H_2$ of G of order 4 and $\langle x \rangle \not\leq \Phi(H_2)$. By hypotheses $\langle x \rangle$ is c-supplemented in G, and thus $\langle x \rangle \Phi(H_2)/\Phi(H_2) \neq 1$ is c-supplemented in $H/\Phi(H)$, which contradicts Lemma 2.1.

Case 2. Suppose that H_2 is quaternion free. Assume that H_2 is abelian. Then $H_2' = \Phi(H_2) = 1$ and thus H_2 is a minimal normal subgroup of H. It follows from Lemma 2.1 that $H_2 = 1$, a contradiction. Assume that H_2 is not abelian. Applying [6, Theorem 2.7], H_q acts on $H_2/\Phi(H_2)$ with at least one fixed point. This implies that $|H_2/\Phi(H_2)| = 2$, a contradiction.

The above proof shows that PQ is nilpotent as claimed. In particular, P centralizes all odd elements of G. Thus for any G-chief factor H/K of P, $G/C_G(H/K)$ is a 2-group.

By [5, A, Lemma 13.6], we have $\mathrm{O}_2(G/C_G(H/K)) = 1$. It follows that $G/C_G(H/K) = 1$ and thus $P \leq Z_{\mathcal{N}}(G)$. □

As an application of Proposition 2.2, we have:

Corollary 2.3. *If all minimal subgroups of G_2 and all cyclic subgroups of G_2 with order 4 (if a quaternion group is involved in G_2) is c-supplemented in G, then G is 2-nilpotent.*

Proof. Suppose that this corollary is not true and let G be a counterexample with minimal order. Obviously the hypotheses are inhered by all subgroups of G, hence G is a minimal non 2-nilpotent group. It follows that G_2 is a normal subgroup in G. Applying Proposition 2.2 to G_2, we get a contradiction. □

By combining [1, Theorem 1.1] and Proposition 2.2, we have:

Lemma 2.4. *Let P be a normal p-subgroup of G. If all cyclic subgroups of P with order p or 4 (if a quaternion group is involved in P) are c-supplemented in G, then $P \leq Z_{\mathcal{U}}(G)$.*

Next, we will show that if some class of p-subgroups of G is c-normal in G, then G is p-solvable.

Lemma 2.5. *If G_p is c-normal in G then G is p-solvable.*

Proof. Suppose that this Lemma is not true and consider G to be a counterexample with minimal order. By Lemma 1.2 (3), the hypothesis holds for both $G/\mathrm{O}_p(G)$ and $G/\mathrm{O}_{p'}(G)$, thus the minimal choice of G implies that $\mathrm{O}_p(G) = \mathrm{O}_{p'}(G) = 1$. By the definition of c-normality, there exists a normal subgroup H of G such that $G = G_pH$ and $H \cap G_p \leq (G_p)_G$. But $(G_p)_G = \mathrm{O}_p(G) = 1$, hence H is a normal p'-subgroup of G. The fact that $\mathrm{O}_{p'}(G) = 1$ then indicates that $H = 1$ and thus $G = G_p$, a contradiction. □

Lemma 2.6. *Let G be a finite group and p^d be a prime power such that $3 \leq p^d \leq |G|_p$. If all subgroups of G with order p^d are c-normal in G, then G is p-solvable.*

Proof. By Lemma 2.4, we may assume that $p^d < |G|_p$. Let D be any subgroup of order p^d. By the hypotheses, there exists a normal subgroup H of G such that $G = DH$ and $D \bigcap H \leq D_G$. Assume that $H < G$. Since G/H is a p-group, we may take a normal subgroup M of G such that $M \geq H$ and $|G : M| = p$. As $p^d < |G|_p$, $p^d \leq |M|_p$.

Clearly all subgroups of M with order p^d are c-normal in M. It follows by induction that M is p-solvable, and so is G. Hence we may assume that $H = G$ and then $D = D_G$ is normal in G. Assume that D is not a minimal normal subgroup of G. Let V be a minimal G-invariant subgroup of D and $|V| = p^e$. Then $p \leq |V| < p^d$ and all subgroups of G/V with order p^{d-e} are c-normal in G/V. By Induction G/V is p-solvable, and so is G. Hence we may assume that D is minimal normal in G whenever D is a subgroup of order p^d.

Note that if all subgroups of order p^d are contained in $Z(G)$, then G is p-nilpotent by a well known result of Itô (see [4, IV, 5.3]). Hence we may assume that there is a subgroup U of order p^d such that $U \not\leq Z(G)$. Suppose that $|U| = p^d \geq p^2$. Let K be a subgroup of order p^{d+1} such that $U < K$. Clearly U is not cyclic, and hence there is a maximal subgroup U_1 of K such that $U_1 \neq U$. Since U_1 is normal as assumed, we get that $U \cap U_1$ is a nontrivial G-invariant subgroup of U, and this contradicts the minimal normality of U. Hence $|U| = p$. Observe that $G/C_G(U)$ is a p'-group and that $C_G(U) < G$. It follows by induction that $C_G(U)$ is p-solvable, and so is G. □

Now, we will study the properties of p-hypercyclically embedding. Clearly, if a normal subgroup E is p-hypercyclically embedded in G, then E is p-solvable and every normal subgroup of G contained in E is also p-hypercyclically embedded in G. The following lemma shows that for a p-solvable normal subgroup E, we can deduce that E is p-hypercyclically embedded in G if the maximal p-nilpotent normal subgroup of E (denoted by $F_p(E)$) is p-hypercyclically embedded in G.

Lemma 2.7. *A p-solvable normal subgroup E is p-hypercyclically embedded in G if and only if $F_p(E)$ is p-hypercyclically embedded in G.*

Proof. We only need to prove the sufficiency. Suppose that the assertion is false and let (G, E) be a counterexample with $|G| + |E|$ minimal. We claim that $\mathrm{O}_{p'}(E) = 1$. Indeed, since $F_p(E/\mathrm{O}_{p'}(E)) = F_p(E)/\mathrm{O}_{p'}(E)$, it is easy to verify that the hypotheses hold for $(G/\mathrm{O}_{p'}(E), E/\mathrm{O}_{p'}(E))$. If $\mathrm{O}_{p'}(E) \neq 1$, then the the minimal choice of (G, E) implies that $E/\mathrm{O}_{p'}(E)$ is p-hypercyclically embedded in $G/\mathrm{O}_{p'}(E)$. Clearly $\mathrm{O}_{p'}(E)$ is p-hypercyclically embedded in G. Therefore E is p-hypercyclically embedded in G, a contradiction.

Let N be a minimal normal subgroup of G contained in E. N is an abelian normal p-subgroup since E is p-solvable and $\mathrm{O}_{p'}(E) = 1$. Consider the group $C_E(N)/N$. Let $L/N = \mathrm{O}_{p'}(C_E(N)/N)$ and K be a Hall p'-subgroup of L. Then $L = KN$. Since $K \leq L \leq C_E(N)$, we have $K = \mathrm{O}_{p'}(L) \leq \mathrm{O}_{p'}(G) = 1$. Consequently $\mathrm{O}_{p'}(C_E(N)/N) = 1$ and we have $F_p(C_E(N)/N) = \mathrm{O}_p(C_E(N)/N) \leq \mathrm{O}_p(E)/N = F_p(E)/N$. As a result, we know that the hypotheses hold for $(G/N, C_E(N)/N)$ and the minimal choice of (G, E) yields that $C_E(N)/N$ is p-hypercyclically embedded in G/N. But $N \leq F_p(G)$ and thus N is also p-hypercyclically embedded in G. It follows that $C_E(N)$ is p-hypercyclically embedded in G.

Since N is a normal p-subgroup that is p-hypercyclically embedded in G, $|N| = p$. It yields that $G/C_G(N)$ is a cyclic group. As a result, $EC_G(N)/C_G(N)$ is p-hypercyclically embedded in $G/C_G(N)$. Note that $E/C_E(N) = E/(E \cap C_G(N))$ is G-isomorphic to $EC_G(N)/C_G(N)$, $E/C_E(N)$ is p-hypercyclically embedded in $G/C_E(N)$. But $C_E(N)$ is p-hypercyclically embedded in G and thus E is or p-hypercyclically embedded in G, a final contradiction. □

Denote $\mathcal{A}(p-1)$ as the formation of all abelian groups of exponent divisible by $p-1$. The following proposition is well known:

Lemma 2.8 ([15, Theorem 1.4])**.** *Let H/K be a chief factor of G and p be a prime divisor of $|H/K|$. Then $|H/K| = p$ if and only if $G/C_G(H/K) \in \mathcal{A}(p-1)$.*

Let f be a formation function, and E be a normal subgroup of G. We say that G acts f-centrally on E if $G/C_G(H/K) \in f(p)$ for every chief factor H/K of G below E and every prime p dividing $|H/K|$ ([5], p.387, Definitions 6.2). Fixing a prime p, define a formation function g_p as follows:

$$g_p(q) = \begin{cases} \mathcal{A}(p-1) & (\text{if } q = p) \\ \text{all finite group} & (\text{if } q \neq p) \end{cases}$$

From Lemma 2.8, we can see that E is p-hypercyclically embedded in G if and only if G acts g_p-centrally on E. By applying [5, p.388, Theorem 6. 7], we get the following useful results:

Lemma 2.9. *A normal subgroup E of G is p-hypercyclically embedded in G if and only if $E/\Phi(E)$ is p-hypercyclically embedded in $G/\Phi(E)$.*

Then following lemma is evident.

Lemma 2.10. *Let K and L be two normal subgroup of G contained in E. If E/K is p-hypercyclically embedded in G/K and E/L is p-hypercyclically embedded in G/L, then $E/(L \cap K)$ is p-hypercyclically embedded in $G/(L \cap K)$.*

The following proposition indicates that Theorem A holds when $p^d = p$.

Proposition 2.11. *Let E be a normal subgroup of G. If all cyclic subgroups of E_p with order p and 4 (if a quaternion group is involved in E_p) are c-normal in G, then E is p-hypercyclically embedded in G.*

Proof. Note that E is p-solvable by Lemma 2.6. Suppose that $\mathrm{O}_{p'}(E) > 1$. Since the hypotheses hold for $G/\mathrm{O}_{p'}(E)$, we conclude by induction that $E/\mathrm{O}_{p'}(E)$ is p-hypercyclically embedded in $G/\mathrm{O}_{p'}(E)$ and thus E is p-hypercyclically embedded in G. Suppose that $\mathrm{O}_{p'}(E) = 1$. By Lemma 2.4, $\mathrm{O}_p(E) \leq Z_{\mathcal{U}}(G)$. As $\mathrm{O}_{p'}(E) = 1$, $F_p(E) = \mathrm{O}_p(E)$. It follows that E is p-hypercyclically embedded in G by Lemma 2.7. □

With the aid of the preceding results, we can now prove Theorem A.

Proof of Theorem A. Suppose that Theorem A is not true and let (G, E) be a counterexample such that $|G|+|E|$ is minimal. Then the minimal choice of (G, E) implies that $\mathrm{O}_{p'}(E) = 1$. If $|E_p| = p$, then E_p itself is c-normal in G and by Lemma 1.2, E_p is also c-normal in E. By Lemma 2.5 we know that E is p-solvable and consequently E is

p-hypercyclically embedded in G since $|E_p| = p$. Therefore we may assume that $|E_p| > p$ and $1 < p^d < |E_p|$. By Proposition 2.11, we may further assume that $p^d > p$. By Lemma 2.6, E is p-solvable. We derive a contradiction through the following steps.

(1) If N is a minimal G-invariant subgroup of E, then $|N| > p$.

Suppose that $|N| = p$, then $p^d > |N|$ by the assumption that $p^d > p$. Hence $(G/N, E/N)$ also satisfies the hypotheses of this Theorem and E/N is p-hypercyclically embedded in G/N by the choice of (G, E). Since $|N| = p$, E is p-hypercyclically embedded in G, a contradiction. Hence $|N| > p$.

(2) If N is a minimal G-invariant subgroup of E, then $p^d > |N|$.

By Lemma 2.1 we have $p^d \geq |N|$. Suppose that $p^d = |N|$. Since $p^d < |E_p|$ by our assumption, E_p has a subgroup H such that N is a maximal subgroup of H. By (1), N is not cyclic and so is H. Hence we can choose a maximal subgroup K of H other than N. Obviously we have $H = NK$. If $N \cap K = 1$, then $|N| = |H|/|K| = p$, a contradiction. Thus $N \cap K \neq 1$ and $|K : K \cap N| = |KN : N| = |H : N| = p$. Since $K_G \cap N \leq K \cap N < N$, we have $K_G \cap N = 1$. Assume that $K_G > 1$. Then $|K_G| = |K_G N/N| = |H/N| = p$ and this contradicts (1). Therefore we have $K_G = 1$. Since $|K| = |N| = p^d$, K is c-normal in G by the hypotheses of this theorem. So there exists a proper normal subgroup L of G such that $G = KL$ and $K \cap L \leq K_G = 1$. Since $K \cap N \neq 1$ and $K \cap L = 1$, we have $N \neq L$ and thus $N \cap L = 1$. Consequently $|NL| = |N||L| = |K||L| = |KL| = |G|$ and thus $G = NL$. Let M be a maximal subgroup of G containing L, then $|G : M| = p$ since G/L is a p-group. Obviously $G = NM$ and $N \cap M = 1$. But then $|N| = |G : M| = p$, a contradiction.

(3) $\Phi(E) = 1$ and E contains a unique minimal G-invariant subgroup, say N.

Let L be any minimal G-invariant subgroup of E. Since $p^d > |L|$ by (2), it is easy to verify that $(G/L, E/L)$ satisfies the hypotheses of this theorem. Thus the minimal choice of (G, E) implies that E/L is p-hypercyclically embedded in G/L. By Lemma 2.9, we have $\Phi(E) = 1$. By Lemma 2.10, we have that E contains a unique minimal G-invariant subgroup, say N.

(4) Final contradiction.

Since $\Phi(E) = 1$, E is split over N (see [5, Chapter A, 9.10]). Hence $E = [N]Y$ for some subgroup Y of E, and so $E_p = [N]Y_p$ for some $Y_p \in Syl_p(Y)$. Let U be a maximal subgroup of N such that U is E_p-invariant. Since $d_p < |E_p|$, we may take a subgroup $D = UV$ such that $V \leq Y_p$ and $|D| = p^d$. Assume that $D_G > 1$. Then $D \geq D_G \geq N$ because N is the unique minimal G-invariant subgroup of E, a contradiction. Hence $D_G = 1$. Now the hypotheses imply that $G = D[L]$ for some G-invariant subgroup L. Note that $E \cap L = 1$ contrary to $|D| = |E|$. Thus $E \cap L$ is a nontrivial normal subgroup of G, so $U < N \leq E \cap L \leq L$, but then $1 < U \leq D \cap L = 1$, a contradiction. □

Remark. The conclusion of Theorem A does not hold if we replace "c-normal" with "c-supplemented" in the hypothesis. One can take A_5 for a example. Obviously every subgroup of A_5 with order 5 is c-supplemented in A_5, but A_5 is not 5-hypercyclically embedded in itself.

Corollary 2.12. *Let p be a fixed prime and G_p be a Sylow p-subgroup of a finite group G. Suppose that p^d is a prime power such that $1 < p^d \leq \max(|G_p|/p, p)$. If all the subgroups of G_p with order p^d and $2p^d$ (if a quaternion group is involved in G_p) are c-normal in G, then G is p-supersolvable.*

Corollary 2.13. *Let p be a fixed prime and E be a normal subgroup of a finite group G. Suppose that E_p is a Sylow p-subgroup of E and p^d is a prime power such that $1 < p^d \leq \max(|E_p|/p, p)$. If G/E is p-supersolvable, and all the subgroups of E_p with order p^d and $2p^d$ (if a quaternion group is involved in E_p) are c-normal in G, then G is p-supersolvable.*

Corollary 2.14. *Let p be a fixed prime and E be a normal subgroup of a finite group G. Suppose that E_p is a Sylow p-subgroup of E and p^d is a prime power such that $1 < d \leq \max(|E_p|/p, p)$. Suppose that $N_G(E_p)$ is p-nilpotent. If either E_p is abelian or every subgroup of E_p with order p^d and $2p^d$ (if a quaternion group is involved in E_p) is c-normal in E, then G is p-nilpotent.*

Proof. If E_p is abelian, then E is p-nilpotent by Burnside's theorem. If E_p is not abelian, then E is p-supersolvable by Theorem A. In both cases we have that $E_p \operatorname{O}_{p'}(E)$ is a normal subgroup of G. By Frattini argument, $G = N_G(E_p) \operatorname{O}_{p'}(E)$. Note that $N_G(E_p)$ is p-nilpotent by hypotheses, we have that G is p-nilpotent, as wanted. □

Corollary 2.15. *Let p be a fixed prime and G_p be a Sylow p-subgroup of a finite group G. Suppose that p^d is a prime power such that $1 < p^d \leq \max(|G_p|/p, p)$. Suppose that $N_G(G_p)$ is p-nilpotent. If all the subgroups of G_p with order p^d and $2p^d$ (if a quaternion group is involved in G_p) are c-normal in G, then G is p-nilpotent.*

3 Some applications

In this section, we give some applications to show that we can apply our results to get some known results.

Corollary 3.1 ([2, Theorem 3.4]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If all minimal subgroups and all cyclic subgroups with order 4 of $G^{\mathcal{F}}$ are c-normal in G, then $G \in \mathcal{F}$.*

Proof. From Theorem A, we know that $G^{\mathcal{F}}$ is p-hypercentrally embedded in G for all $p \in \pi(G^{\mathcal{F}})$ and thus $G^{\mathcal{F}} \leq Z_{\mathcal{U}}(G)$. Since $\mathcal{F}$ is a saturated formation containing $\mathcal{U}$, we have that $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$. Consequently $G \in \mathcal{F}$ because $G/G^{\mathcal{F}} \in \mathcal{F}$ and $G^{\mathcal{F}} \leq Z_{\mathcal{U}}(G) \leq$

$Z_{\mathcal{F}}(G)$. □

The following lemma is evident.

Lemma 3.2. *Let G be a group and p be a prime such that $(p-1,|G|)=1$. Then G is p-nilpotent if and only if G is p-supersolvable.*

Corollary 3.3 ([12, Theorem 0.1])**.** *Let E be a normal subgroup of a group G of odd order such that G/E is supersolvable. Suppose that every non-cyclic Sylow subgroup P of E has a subgroup D such that $1<|D|<|P|$ and all subgroups H of P with order $|H|=|D|$ are c-normal in G. Then G is supersolvable.*

Proof. Let p be the minimal prime divisor of $|E|$. If E_p is cyclic, then E is p-nilpotent by [14, Lemma 2.8]. If E_p is not cyclic, then by Theorem A, E is p-supersolvable and thus p-nilpotent by Lemma 3.2. By repeating this argument we know that E has a Sylow-tower and therefore E is solvable. Let p be any prime divisor of $|E|$, If E_p is cyclic, then E is p-hypercentrally embedded in G since now E is p-solvable. If E_p is not cyclic, then E is also p-hypercentrally embedded in G by Theorem A. Therefore we have $E\le Z_{\mathcal{U}}(G)$. It follows that G is supersolvable since G/E is supersolvable and $E\le Z_{\mathcal{U}}(G)$. □

Corollary 3.4 ([9, Theorem 3.1])**.** *Let p be an odd prime dividing the order of a group G and P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

Corollary 3.5 ([9, Theorem 3.4])**.** *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

Proof. If $|P|=p$, then G is p-nilpotent by [14, Lemma 2.8]. If $|P|>p$, then by Corollary 2.12, G is p-supersolvable. Hence G is p-nilpotent by Lemma 3.2. □

Corollary 3.6 ([9, Theorem 3.6])**.** *Let p be the smallest prime dividing the order of group G and P be a Sylow p-subgroup of G. If every minimal subgroup of $P\cap G'$ is c-normal in G and when $p=2$, either every cyclic subgroup of $P\cap G'$ with order 4 is also c-normal in or P is quaternion-free, then G is p-nilpotent.*

Proof. By Theorem A, G' is p-hypercyclically embedded in G. Since G/G' is abelian, G is p-supersoluble. It then follows from Lemma 3.2 that G is p-nilpotent. □

Corollary 3.7 ([9, Corollary 3.9])**.** *Let p be an odd prime dividing the order of a group G and P be a Sylow p-subgroup of G. If every minimal subgroup of $P\cap G'$ is c-normal in G, then G is p-supersolvable.*

Acknowledgment

The authors would like to thank the referee for his/her careful corrections and valuable suggestions. In fact, the proofs of several results of this paper were modifies by the referee in order to make them more simple and clear.

References

[1] M. Asaad and M. Ramadan. Finite groups whose minimal subgroups are c-supplemented. *Comm. Algebra*, 36(3):1034–1040, 2008.

[2] A. Ballester-Bolinches and Yanming Wang. Finite groups with some C-normal minimal subgroups. *J. Pure Appl. Algebra*, 153(2):121–127, 2000.

[3] A. Ballester-Bolinches, Yanming Wang, and Guo Xiuyun. c-supplemented subgroups of finite groups. *Glasg. Math. J.*, 42(3):383–389, 2000.

[4] B.Huppert. *Endliche Gruppen, I.* Berlin–Heidelberg–New York: Springer, 1967.

[5] K. Doerk and T.O. Hawkes. *Finite soluble groups*, volume 4. Walter de Gruyter, 1992.

[6] Larry Dornhoff. M-groups and 2-groups. *Math. Z.*, 100:226–256, 1967.

[7] D. Gorenstein. *Finite groups*. Chelsea Pub. Co.(New York), 1980.

[8] Guo, Wenbin ; Feng, Xiuxian ; Huang, Jianhong. New characterizations of some classes of finite groups. *Bull. Malays. Math. Sci. Soc. (2).*, 34, no. 3, 575–589, 2011.

[9] Xiuyun Guo and K. P. Shum. On c-normal maximal and minimal subgroups of Sylow p-subgroups of finite groups. *Arch. Math. (Basel)*, 80(6):561–569, 2003.

[10] Liu, Jianjun; Guo, Xiuyun; Li, Shirong. The influence of CAP-subgroups on the solvability of finite groups. *Bull. Malays. Math. Sci. Soc. (2)*, 35, no. 1, 227–237, 2012.

[11] Jehad J. Jaraden and Alexander N. Skiba. On c-normal subgroups of finite groups. *Comm. Algebra*, 35(11):3776–3788, 2007.

[12] Alexander N. Skiba. A note on c-normal subgroups of finite groups. *Algebra Discrete Math.*, (3):85–95, 2005.

[13] Yanming Wang. c-normality of groups and its properties. *J. Algebra*, 180(3):954–965, 1996.

[14] Huaquan Wei and Yanming Wang. On c^*-normality and its properties. *J. Group Theory*, 10(2):211–223, 2007.

[15] M. Weinstein. *Between nilpotent and solvable*. New Jersey, 1982.