INTERSECTION GRAPHS OF S-ACTS

H. RASOULI* AND A. TEHRANIAN

Abstract. Let S be a semigroup. The *intersection graph* of an S-act A, denoted by $G(A)$, is the undirected simple graph obtained by setting all non-trivial subacts of A to be the vertices and defining two distinct vertices to be adjacent if and only if their intersection is non-empty. It is investigated the interplay between the algebraic properties of A and the graph-theoretic properties of $G(A)$. Also some characterization results regarding connectivity, completeness, diameter, and girth of $G(A)$ are presented.

1. Introduction

In algebra and theoretical computer science, an action or act of a semigroup on a set is a rule which associates to each element of the semigroup a transformation of the set in such a way that the product of two elements of the semigroup is associated with the composite of the two corresponding transformations. From an algebraic perspective, a semigroup action is a generalization of the notion of a group action in group theory (see [8]). Acts over semigroups appeared and were used in a variety of applications like graph theory, combinatorial problems, algebraic automata theory, mathematical linguistics, theory of machines, and theoretical computer science (see [4, 7]).

In graph theory, an intersection graph is a graph that represents the pattern of intersections of a family of sets. More precisely, an intersection graph is a graph whose vertices are sets and in which two distinct vertices are adjacent if and only if they have a non-empty intersection. It is known that every graph can be represented as an intersection graph (cf. [12]). Therefore, it would be more considerable to study the intersection graphs $G(F)$ of families F of sets when the members of F have an algebraic structure.

The investigation on the interplay between the algebraic properties of algebraic structures and the graph-theoretic properties of their intersection graphs has been of interest to several authors. The first step in this direction, the intersection graphs of semigroups, was taken by Bosák [1]. Then Csákány and Pollák [3] studied the graphs of subgroups of finite groups. In [12], Zelinka continued the work on intersection graphs of non-trivial subgroups of finite abelian groups (see also [9, 10, 13, 14]). Recently, the intersection graph of ideals of a ring was considered by Chakrabarty et al. in [2]. Motivated by these interdisciplinary studies on the intersection graph of

2000 *Mathematics Subject Classification.* 20M30, 05C25, 05C40, 58E40.

Key words and phrases. S-act, Intersection graph.

* Corresponding author.

algebraic structures, in this paper, we define the intersection graphs of S-acts for a semigroup S. Our purpose is to study the connection between the algebraic properties of S-acts and the graph theoretic properties of their intersection graphs.

The following natural questions arise about the intersection graphs of S-acts:

1. Given a graph G, does there exist an S-act A whose intersection graph is isomorphic to G? ~~If no,~~ under which conditions does it hold?

~~2. For every S-acts A and B, if their intersection graphs are isomorphic, are A and B the same? Otherwise, under which conditions do they coincide?~~

3. Find some characterization of S-acts for which the intersection graphs are connected. Also determine some graph-theoretic characters of the graphs of ~~S-acts, such~~ as the diameter and girth.

~~Using~~ the graphs of cyclic semigroups, we give a positive answer to the first question for the class of all complete graphs with countable ~~verices.~~ For the second question, it is shown that the answer is ~~negative in general;~~ but we do not know any condition ~~under which it holds.~~ In Section 3, we characterize all S-acts A for which the intersection graphs are connected (see Theorem 3.3). In this regard, the connectivity of intersection graphs of some classes of acts such as cyclic, free, projective, and cofree is investigated. In Section 4, the diameter and girth of the graphs of S-acts are determined. Finally, we study the graphs of S-acts where S is considered as a group.

2. Basic notation and properties

In this section, a brief account of some basic definitions about S-acts and their intersection graphs is given.

Recall that, for a semigroup S, a set A is a (*left*) *S-act* if there is an action $\lambda : S \times A \to A$, denoting $\lambda(s, a)$ by sa, satisfying $(st)a = s(ta)$ and, if S is a monoid with 1, $1a = a$, for all $a \in A$ and $s, t \in S$. A subset B of A is called a (*left*) *subact* of A if $sb \in B$, for every $b \in B$ and $s \in S$. By a *non-trivial* subact of an S-act A, we mean a non-empty proper subact of A. Each semigroup S can be considered as an S-act with the action given by its operation, and so subacts of S are exactly all its (semigroup) ideals. An element θ of an S-act is called a *zero* element if $s\theta = \theta$ for all $s \in S$. For more details on this basic concept we refer the reader to [5].

Also an element z of a semigroup S is said to be a *left zero* if $zs = z$ for all $s \in S$. A semigroup S whose elements are a left zero is called a *left zero semigroup*. Analogously, a *right zero semigroup* is defined.

Throughout this paper S stands for a semigroup unless otherwise stated. It is convenient to allow the empty set to be an S-act.

A non-empty S-act is said to be *simple* if it has no non-trivial subacts. Obviously, a monoid is simple if and only if it is a group. Also an S-act is called *completely reducible* if it is a disjoint union of simple subacts. By a *decomposable* S-act, we mean an S-act which is a disjoint union of two non-empty subacts. Otherwise, it is called *indecomposable*.

The coproduct of a family $\{A_i \mid i \in I\}$ of S-acts, denoted by $\coprod_{i\in I} A_i$, is their disjoint union, with natural action. In fact, $\coprod_{i\in I} A_i = \bigcup_{i\in I}(A_i \times \{i\})$ and $s(a,i) = (sa,i)$ for every $s \in S, a \in A_i, i \in I$.

Next we recall some preliminary definitions from graph theory needed in the sequel. By convention, all graphs are undirected and simple.

Let G be a graph with the vertex set $V(G)$. By *order* of G, denoted by $|G|$, we mean the number of vertices of G. For distinct elements x and y of $V(G)$, an x,y-path (or $x-y$) is a path with starting vertex x and ending vertex y, and the least length of x,y-path is denoted by $d(x,y)$. If G has no such a path, then $d(x,y)=\infty$. The *diameter* of G, $diam(G)$, is the supremum of the set $\{d(x,y) : x,y \in V(G), x \neq y\}$. The *girth* of a graph is the length of its shortest cycle. A graph with no cycle has infinite girth. By a *null graph*, we mean a graph with no edges. A *complete graph* is a graph in which every pair of distinct vertices are adjacent. We denote the complete graph with n vertices by $K_n, n \in \mathbb{N}$. For more information about graphs, see [11].

The *intersection graph* of an S-act A, denoted by $G(A)$, is defined to be the undirected simple graph whose vertices are in a one-to-one correspondence with all non-trivial subacts of A and two distinct vertices are adjacent if and only if the corresponding subacts of A have a non-empty intersection. So, $G(S)$ is the intersection graph of S, where S is regarded as an S-act. Clearly, $G(A)$ has no vertices for simple S-acts A.

We now state a useful lemma which determines a large number of complete graphs of semigroups.

Lemma 2.1. *Let S be a commutative or containing a left zero element semigroup. Then the graph $G(S)$ is complete.*

Proof. Let I and J be two non-trivial left ideals of S, $a \in I$ and $b \in J$. If S is commutative, $ab = ba \in I \cap J$. If $z \in S$ is a left zero, then $z = za = zb \in I \cap J$. Thus the assertion holds. □

Example 2.2. (i) The graphs of the commutative semigroups $(\mathbb{N},\cdot)$, $(\mathbb{N},+)$, $(\mathbb{N},max)$, $(\mathbb{N}^\infty,min)$, $(\mathbb{Q},+)$, $(\mathbb{Q},\cdot)$, and $(Mat_n\mathbb{R},+)$, where $Mat_n\mathbb{R}$ is the set of all $n\times n$ matrices over $\mathbb{R}$, are complete.

(ii) Each of the below non-commutative semigroups containing a left zero element has a complete graph:

- $(Mat_n\mathbb{R},\cdot)$;
- $(\beta(X),\circ)$, where $\beta(X) = \mathcal{P}(X\times X)$ is the set of binary relations on a set X, with the composition of relations, i.e. $\rho\circ\sigma = \{(x,z) \in X\times X \mid (\exists y \in X)\ x\sigma y, y\rho z\}$, and the empty relation as a left zero element;
- $(\mathcal{T}(X),\circ)$, the *full transformation monoid* of a non-empty set X with composition of mappings, i.e. $(f,g) \mapsto f\circ g$, and constant maps as left zero elements.

(iii) The *bicyclic monoid* $B = \langle u, v \mid uv = 1\rangle = \{v^m u^n : m, n \geq 0\}$ has a complete graph. To see this, let I be a non-trivial left ideal of B. We show that there exists a positive integer n such that $u^n \in I$. Let $v^m u^n \in I$ for some non-negative integers m and n. Then $u^n = u^m v^m u^n \in I$ and $n > 0$. Now let I and J be two non-trivial left ideals of B. Then there exist positive integers n and m, with $u^n \in I$ and $u^m \in J$. Hence, $u^{n+m} \in I \cap J$. Consequently, the graph $G(B)$ is complete.

Remark 2.3. Every set can be made into an S-act whose graph contains no vertices. In fact, a (non-empty) set X becomes an S-act by defining $fx = f(x)$ for all $f \in S, x \in X$, where $S = \mathcal{T}(X)$ is the full transformation monoid of X. Moreover, X is a simple S-act so that the graph $G(X)$ has no vertices. To see this, let Y be a non-trivial subact of X. Then there exist $a \in X \setminus Y$ and $b \in Y$. Take $f \in S$, mapping each element of X to a. This yields that $a = f(b) = fb \in Y$ which is a contradiction.

The following result gives a positive answer to the first question for the class of all complete graphs.

Let S be a *cyclic* (*monogenic*) *semigroup* of order n, that is, $S = \{s, s^2, s^3, ..., s^n\}$. Then $s^{n+1} = s^k$ for some $1 \leq k \leq n$. In fact, each such k gives a distinct semigroup of order n, and every cyclic semigroup of order n is isomorphic to one of these.

Proposition 2.4. *For each complete graph K_n, $n \in \mathbb{N}$, there exists a semigroup S whose intersection graph $G(S)$ is isomorphic to K_n.*

Proof. Consider the cyclic semigroup S of order $n+1$ with a generating element $s \in S$, where $s^{n+2} = s^{n+1}$. Since S is commutative, the graph $G(S)$ is complete by Lemma 2.1. It can be easily shown that all distinct non-trivial ideals of S form the chain:

$$\langle s^n\rangle \subset \langle s^{n-1}\rangle \subset ...\langle s^2\rangle \subset \langle s\rangle,$$

where $\langle s^k\rangle = \{s^i \mid k+1 \leq i \leq n+1\}$, for every $1 \leq k \leq n$. So, $G(S)$ is a complete graph with n distinct vertices which is clearly isomorphic to the complete graph K_n. □

Remark 2.5. The non-trivial ideals of the commutative semigroup $S = (\mathbb{N}, +)$ are exactly the sets $n + N = \{n + k \mid k \in \mathbb{N}\}$, where $n \in \mathbb{N}$. Further, $m + N \subset n + N$ if and only if $m > n$, for every $m, n \in \mathbb{N}$. Then the complete graph $G(S)$ consists of a countably infinite vertices. Therefore, Proposition 2.4 is also true for the complete graph with countably infinite vertices.

Proposition 2.4 states that every complete graph is the intersection graph of an S-act. The following example (Fig. 1) shows that the converse is not generally true.

Example 2.6. Consider the semigroup $S = \{0, s\}$, where s is an idempotent element, and $A = \{a, b, c\}$ with the action defined by: $0c = a$, $sc = b$, and a, b are zero elements. Then all non-trivial subacts of A are the sets $\{a\}, \{b\}$, and $\{a, b\}$. Hence, the graph $G(A)$ is not complete:

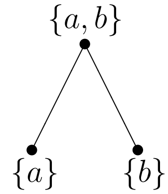

Figure 1. A non-complete intersection graph of acts

In the following, we give a condition on an S-act A under which the graph $G(A)$ is complete. It is obvious that $G(A)$ is a complete graph if and only if every non-empty subact of A is indecomposable. Hence, non-simple completely reducible acts do not have complete intersection graphs.

Recall that an S-act A is *Artinian* if every descending chain of subacts of A terminates. It can be easily seen that every non-empty subact of an Artinian S-act contains a minimal subact.

Proposition 2.7. *Let A be an Artinian S-act. Then $G(A)$ is complete if and only if A contains a unique minimal subact.*

Proof. Since A is Artinian, A has at least one minimal subact. Moreover, every non-empty subact of A contains a minimal subact. Therefore, if A possesses a unique minimal subact, say B, then B is contained in every non-empty subact of A. This implies that $G(A)$ is complete. The converse is straightforward. □

It is easy to see that, for given S-acts A and B, if $A \cong B$ as acts, then $G(A) \cong G(B)$ as graphs. But the converse is not true even for complete intersection graphs. This point is illustrated in Fig. 2 of the following example.

Example 2.8. Consider the cyclic semigroup $S = \{s, s^2, s^3, s^4\}$, with $s^5 = s^4$. By Proposition 2.4, S contains exactly three non-trivial ideals $I_k = \langle s^k \rangle = \{s^i \mid k+1 \leq i \leq 4\}$ for $k = 1, 2, 3$. Also the S-act $A = \{a, b, c\}$, with the action $xy = b$ for every $x \in S$ and $y \in A$, has exactly three non-trivial subacts $J_1 = \{b\}, J_2 = \{a, b\}$, and $J_3 = \{b, c\}$. So, $G(S) \cong G(A)$ whereas S and A are non-isomorphic S-acts:

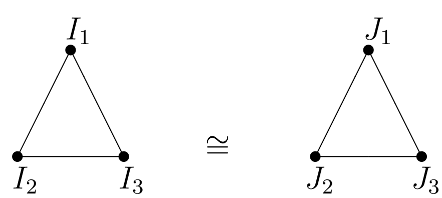

Figure 2. Some isomorphic graphs of non-isomorphic acts

3. Connectivity of $G(A)$

In this section, ~~we characterize~~ all S-acts A for which the intersection graphs are connected. Also the connectivity of some special S-acts is investigated.

Theorem 3.1. *Let A be an S-act. Then $G(A)$ is disconnected if and only if $G(A)$ is a null graph with $|G(A)| = 2$.*

Proof. Suppose that $G(A)$ is disconnected and $|G(A)| \geq 3$. Let B and C be two distinct non-trivial subacts of A. We claim that there is a path from B to C. For this, assume that B and C are not adjacent. If $A \neq B \cup C$, then $B - B \cup C - C$ is a B, C-path. Now let $A = B \cup C$. Since $|G(A)| \geq 3$, A contains a non-trivial subact D, with $D \neq B, C$. This clearly gives that $B \cap D \neq \emptyset$ or $C \cap D \neq \emptyset$. The first case implies that $B - C \cup D - C$ is a path between B and C. Also the second case gives the B, C-path $B - B \cup D - C$. So, $G(A)$ is connected which is a contradiction. Then $|G(A)| \leq 2$. Using the assumption, this implies that $G(A)$ is a null graph with two vertices. The converse is obvious. □

Theorem 3.1 states that the only possible case for a disconnected graph to be the intersection graph of an S-act~~, is~~ the following:

Figure 3. The only disconnected intersection graph of acts

Proposition 3.2. *Let A be an S-act. Then the following are equivalent:*

(i) *the graph $G(A)$ is disconnected.*
(ii) *the only non-trivial subacts of A are two disjoint subacts.*
(iii) *the only non-trivial subacts of A are two minimal (as well as maximal) subacts.*

Proof. (i) $\Leftrightarrow$ (ii) Follows from Theorem 3.1.

(ii) $\Rightarrow$ (iii) Consider the only non-trivial subacts of A, say, B and C, where $B \cap C = \emptyset$. We show that B is a minimal subact of A. Let D be a non-empty subact of A, with $D \subset B$. By the assumption, this implies that $D = C$ which is a contradiction. Similarly, C is also minimal. Now, to prove that B is a maximal subact of A, let D be a subact of A, with $B \subset D$. Then, using hypothesis, $D = C$ which is a contradiction. By the same way, C is also maximal.

(iii) $\Rightarrow$ (ii) Assume that A contains exactly two non-trivial subacts, say, B and C where they are minimal (maximal) subacts of A. Let $B \cap C \neq \emptyset$. Since $B \cap C \neq A$, we get $B \cap C$ is a non-trivial subact of A. Then $B \cap C = B$ or $B \cap C = C$. So, minimality (maximality) of B and C implies that $B = C$ which is a contradiction. □

The next characterization result regarding connectivity of intersection graphs of S-acts is obtained.

Theorem 3.3. *Let A be an S-act. Then the graph $G(A)$ is disconnected if and only if A is a coproduct of two simple S-acts.*

Proof. Let $G(A)$ be disconnected. Using Proposition 3.2, A contains exactly two non-trivial subacts, namely, B and C where they are disjoint. Clearly, $A = B \sqcup C$. If B is not simple, then there exists a non-trivial subact D of B. Thus D is a non-trivial subact of A and $D \neq B, C$ which is a contradiction. Then B and, by the same way, C are simple.

Conversely, let $A = B \sqcup C$, where B and C are simple S-acts. Without loss of generality, one may assume that B and C are disjoint. We show that the only non-trivial subacts of A are the disjoint subacts B and C. Let D be a non-trivial subact of A, with $D \neq B, C$. Then $B \cap D \neq \emptyset$ or $C \cap D \neq \emptyset$. The first case implies that $B \subseteq D$ by minimality of B. Then $C \cap D \neq \emptyset$ and so $C \subseteq D$ by minimality of C. Hence, $A = B \cup C \subseteq D$ which is a contradiction. By the same way, the second case gives a contradiction. Now Proposition 3.2 gives the result.

□

Remark 3.4. (i) For an S-act A, if $G(A)$ is connected, then every pair of maximal subacts of A have a non-empty intersection. To see this, assume that B and C are maximal subacts of A and $B \cap C = \emptyset$. Since $B \subset B \cup C$, maximality of B implies that $A = B \cup C$. We show that B and C are minimal subacts of A. Let D be a non-empty subact of A and $D \subseteq B$. Then $C \cap D = \emptyset$ and $C \subset C \cup D$ whence $A = C \cup D$ by maximality of C. This implies $B = D$, showing that B and similarly C are simple S-acts. Hence, $G(A)$ is disconnected by Theorem 3.3 which is a contradiction.

(ii) We know that every non-empty set A can be made into an S-act with *trivial action*: $sa = a$ for all $s \in S$ and $a \in A$. In this case, non-trivial subacts of A are exactly all proper non-empty subsets of A. Thus, by Proposition 3.2, $G(A)$ is disconnected if and only if $|A| = 2$. For instance, let S be a right zero semigroup. Then S, as an S-act, has trivial action and so $G(S)$ is disconnected if and only if $|S| = 2$. Also note that every left zero semigroup S is simple and so $V(G(S)) = \emptyset$.

Our next results involve the connectivity characterization of graphs of some special S-acts and so we recall some notions here.

Take an S-act A. The power set $\mathcal{P}(A) = \{X \mid X \subseteq A\}$ turns into an S-act by setting $sX = \{sx \mid x \in X\}$, for every $s \in S$ and $X \in \mathcal{P}(A)$.

Now let S be a commutative semigroup and A a non-empty S-act. Then it can be easily seen that the set $Sub(A)$ of all (possibly empty) subacts of A is an S-act by defining $sB = \{sb \mid b \in B\}$, for every $s \in S$ and $B \in Sub(A)$.

In the next result, under some conditions, the connectivity of intersection graphs of the S-acts $\mathcal{P}(A)$ and $Sub(A)$ is characterized.

Proposition 3.5. *Let A be a non-empty S-act. Then the following assertions hold:*

(i) *If A is finite, then $G(\mathcal{P}(A))$ is disconnected if and only if $|A| = 1$.*

(ii) *If S is a commutative semigroup, then $G(Sub(A))$ is disconnected if and only if A is simple.*

Proof. (i) Let A be finite and $G(\mathcal{P}(A))$ be disconnected. If $|A| \neq 1$, then $|\mathcal{P}(A)| \geq 4$. Clearly, the set $H = \{\emptyset\}$ is a non-trivial subact of $\mathcal{P}(A)$. Applying Proposition 3.2, we get H and $K = \mathcal{P}(A) \setminus H$ are all non-trivial subacts of $\mathcal{P}(A)$. This implies that $A \in K$. Also $K \neq \{A\}$ because $|\mathcal{P}(A)| \geq 4$. We show that the non-empty proper subset $L = K \setminus \{A\}$ is a non-trivial subact of $\mathcal{P}(A)$. For this, let $X \in L$ and $s \in S$. Then $sX \in K$ and $X \neq A$. If $sX = A$, we have $|A| = |sX| \leq |X|$. But $X \subseteq A$ and A is finite by hypothesis. So, $X = A$, a contradiction. Hence, $sX \in L$, as required. Since L is different from H and K, we get a contradiction. Therefore, $|A| = 1$. Conversely, if $|A| = 1$, then $|\mathcal{P}(A)| = 2$ and the actions in $\mathcal{P}(A)$ are trivial. This implies that $G(\mathcal{P}(A))$ is disconnected by Remark 3.4(ii).

(ii) Suppose that S is commutative and $G(Sub(A))$ is disconnected. If A is non-simple, then $|Sub(A)| \geq 3$. It is obvious that the set $H = \{\emptyset\}$ is a non-trivial subact of $Sub(A)$. Using Proposition 3.2, we get H and $K = Sub(A) \setminus H$ are all non-trivial subacts of $Sub(A)$. This gives that $A \in K$. Also $K \neq \{A\}$ because $|Sub(A)| \geq 3$. We show that the non-empty proper subset $L = K \setminus \{A\}$ is a non-trivial subact of $Sub(A)$. To see this, let $B \in L$ and $s \in S$. Then $sB \in K$ and $B \neq A$. If $sB = A$, then $A = sB \subseteq B$ and so $A = B$ which is a contradiction. Hence, $sB \in L$, as required. Since L is different from H and K, we get a contradiction. For the converse, let A ~~is simple.~~ Then $Sub(A) = \{\emptyset, A\}$. Simplicity of A implies that $\{A\}$ is a subact of $Sub(A)$. Thus $G(Sub(A))$ contains only two vertices $\{\emptyset\}$ and $\{A\}$, showing that $G(Sub(A))$ is disconnected. □

In the following, we study the connectivity of intersection graphs of some important S-acts such as cyclic, free, projective, and cofree acts.

Let S be a monoid. By a *cyclic* S-act, we mean an S-act A generated by an element $a \in A$. The definitions of *free* and *projective* acts in their categorical forms are well-known and can be found, for example, in [5].

Here it is shown that the intersection graph of every cyclic S-act is connected. Also we characterize all free S-acts for which the intersection graphs are connected.

Theorem 3.6. *Let S be a monoid. Then the following assertions hold:*

(i) *For every cyclic S-act A, $G(A)$ is connected. In particular, $G(S)$ is connected.*

(ii) *Let A be a free S-act. Then $G(A)$ is disconnected if and only if $A \cong S \sqcup S$ and S is a group.*

Proof. (i) Consider a cyclic S-act A with $G(A)$ disconnected. By Theorem 3.3, A is a disjoint union of two simple S-acts. Then A is completely reducible. Note that a cyclic S-act is completely reducible if and only if it is simple (see [5, Lemma I.5.32]). Thus this implies that A is simple and hence $V(G(A)) = \emptyset$ which is a contradiction.

(ii) Suppose that A is a free S-act and $G(A)$ is disconnected. Since A is free, $A \cong \bigsqcup_{x\in X} S$ where X is a basis of A (see [5, Theorem I.5.15]). By (i), $|X| \neq 1$. If $|X| > 2$, then the non-trivial subact $S \sqcup S$ of A is not minimal which contradicts Proposition 3.2. Then $|X| = 2$ and so $A \cong S \sqcup S$. Using again Proposition 3.2, S is a group. The converse follows from Theorem 3.3. □

Remark 3.7. Every connected S-act is not necessarily cyclic so that the converse of Theorem 3.6(i) is not true. For this, consider a free S-act A of rank at least three which is connected by Theorem 3.6(ii). But, A is not clearly a cyclic S-act.

In the following, we give a characterization of projective acts for which the graphs are connected.

Theorem 3.8. *Let S be a monoid and A be a projective S-act. Then $G(A)$ is disconnected if and only if $A \cong Se \sqcup Sf$ for some idempotents $e, f \in S$, where the cyclic S-acts Se and Sf are simple.*

Proof. Notice that an S-act A is projective if and only if $A \cong \bigsqcup_{i\in I} Se_i$, for some idempotents $e_i \in S, i \in I \neq \emptyset$ (cf. [6]). Now, analogously to the proof of Theorem 3.6(ii), the assertion holds. □

Corollary 3.9. *Let S be a commutative monoid and A be a projective S-act. Then $G(A)$ is disconnected if and only if $A \cong Se \sqcup Se$ for an idempotent $e \in S$, where the cyclic S-act Se is simple.*

Proof. If $G(A)$ is disconnected, then Theorem 3.8 implies that $A \cong Se \sqcup Sf$ for two idempotents $e, f \in S$ and simple S-acts Se and Sf. We claim that $e = f$. For this, we first show that for every $s \in S$, there exists $t \in S$ such that $e = tse$. Let $s \in S$. Then Sse is a non-empty subact of Se. Since Se is simple, $Sse = Se$. This implies that $e \in Sse$ and hence $e = tse$ for some $t \in S$. The same way can be applied for f. Thus there exist elements $u, v \in S$ such that $e = ufe$ and $f = vef$. Now, since S is commutative, we have $e = uvefe = uvef$ and $f = vufef = uvef$. Hence, $e = f$, as required. The converse follows from Theorem 3.3. □

Recall from [5] that a *cofree* S-act A is isomorphic to an S-act of the form X^S, the set of all maps from S to a non-empty set X, with the action given by $(sf)(t) = f(ts)$ for $s, t \in S$ and $f \in X^S$. The set X is called a *cobasis* for A.

Here we characterize all cofree S-acts for which the intersection graphs are connected. To this end, first we ~~bring~~ the following useful result:

Lemma 3.10. *Let A be an S-act containing at least two zero elements and $|A| \geq 3$. Then $G(A)$ is connected.*

Proof. Let $G(A)$ be disconnected. By hypothesis, there exist two zero elements θ_1 and θ_2 in A. Since $|A| \geq 3$, the sets $\{\theta_1\}, \{\theta_2\}$ and $\{\theta_1, \theta_2\}$ are distinct non-trivial subacts of A, which contradicts Proposition 3.2. □

Theorem 3.11. *Let A be a cofree S-act. Then $G(A)$ is disconnected if and only if $|A| = 2$.*

Proof. Let $G(A)$ be disconnected. If $|A| = 1$, then $G(A)$ has no vertices and so we get a contradiction. Now let $|A| \geq 3$. Since A is cofree, $A = X^S$ for a non-empty set X, where $|X| > 1$. Clearly, every constant map in A is a zero element and there exist exactly $|X|$ constant maps in A. Thus A contains at least two zero elements. Using Lemma 3.10, this implies that $G(A)$ is connected which is a contradiction. Conversely, let A be a cofree S-act, with $|A| = 2$. Then $A = X^S$, where $|X| = 2$ and $|S| = 1$. So, A consists of two zero elements, say θ_1 and θ_2, and then the only non-trivial subacts of A are the disjoint subacts $\{\theta_1\}, \{\theta_2\}$. Hence, by Proposition 3.2, $G(A)$ is disconnected.

□

4. Diameter and girth of $G(A)$

This section is devoted to investigate the diameter and the girth of intersection graphs of S-acts. In particular, we study these graph-theoretic characters for free, projective, and cofree acts. Also some characterization results such as completeness and connectivity of graphs of S-acts, for a group S, are presented.

First we determine the diameter and girth of $G(A)$, as follows:

Theorem 4.1. *Let A be an S-act. Then the following assertions hold:*
(i) *If $G(A)$ is connected, then diam$(G(A)) \leq 2$.*
(ii) *If $G(A)$ contains a cycle, then girth$(G(A)) = 3$.*

Proof. (i) See the proof of Theorem 3.1.

(ii) On the contrary, assume that girth$(G(A)) \geq 4$. This implies that every pair of distinct non-trivial subacts of A with a non-empty intersection should be comparable, otherwise, $G(A)$ will have a cycle of length 3 which is a contradiction. Now, since girth$(G(A)) \geq 4$, $G(A)$ contains a path of length 3, say $B - C - D - E$. Since every two subacts in this path are comparable and every chain of non-trivial subacts of length 2 induces a cycle of length 3 in $G(A)$, the only two possible cases are $B \subseteq C, D \subseteq C, D \subseteq E$, or $C \subseteq B, C \subseteq D, E \subseteq D$. The first case yields $D \subseteq C \cap E$ and hence $C \cap E \neq \emptyset$. Thus (C, D, E) is a cycle of length 3 in $G(A)$ which is a contradiction. In the second case, we have $C \subseteq B \cap D$ and so (B, C, D) is a cycle of length 3 in $G(A)$, which again yields a contradiction. This completes the proof. □

Question 1. Does there exist a connected graph with the diameter 2 which is not the intersection graph of any S-act?

In what follows, we determine diameter and girth of some free, projective, and cofree acts.

Lemma 4.2. *Let A be an S-act containing at least three pairwise disjoint non-trivial subacts. Then girth$(G(A)) = 3$.*

Proof. Let B, C, and D be some pairwise disjoint non-trivial subacts of A. Then $(B, B \cup C, B \cup D)$ is a cycle of length 3 in $G(A)$, as desired. □

Theorem 4.3. *Let A be a non-cyclic S-act, where S is a monoid. If A is free or projective, then $G(A)$ is connected ~~if and only~~ $|G(A)| \geq 6$. In this case, diam$(G(A)) = 2$ and girth$(G(A)) = 3$.*

Proof. The first part follows from Theorems 3.1 and 3.6(ii). Let A be a non-cyclic free S-act, with $G(A)$ connected. Using Theorem 3.6(ii), we get rank$(A) \geq 3$, or rank$(A) = 2$ and S is non-simple. Each case implies that $G(A)$ is not complete and then diam$(G(A)) = 2$ by Theorem 4.1(i). The first case, in view of Lemma 4.2, gives that girth$(G(A)) = 3$. For the second case, since S is non-simple, there exists a non-trivial left ideal I of S. Thus $(I, S, I \sqcup S)$ is a cycle of length 3 in $G(A)$. Hence, girth$(G(A)) = 3$. If A is projective, using Theorems 3.8, 4.1(i) and Lemma 4.2, a same argument can be applied to get the result. □

Now, in light of Theorems 3.11, 4.1(i) and Lemma 4.2, the following result is immediate:

Corollary 4.4. *Let A be a cofree S-act with a cobasis X, where $|X| \geq 3$. Then diam$(G(A)) = 2$ and girth$(G(A)) = 3$.*

Question 2. What about the diameter and girth of a ~~connected cofree~~ S-act with a cobasis X, where $|X| = 2$?

Finally, we study intersection graphs of S-acts, where S is a group. Acts over groups are appearing frequently in combinatorial problems (for example, in connection with the Burnside problem and with MacMahon's master theorem, see [7]).

Here we show that, in the case S is a group, there is no S-act whose graph is complete with a non-empty vertex set.

Proposition 4.5. *Let S be a group and A be an S-act. Then the following are equivalent:*
(i) *$G(A)$ is complete.*
(ii) *A is indecomposable.*
(iii) *A is simple.*

Proof. (iii) ⇒ (i) ⇒ (ii) Obvious.

(ii) ⇒ (i) Let A be indecomposable. If $G(A)$ is not complete, then there exist disjoint non-trivial subacts B and C of A. Note that $A \neq B \cup C$ because A is indecomposable. Since S is a group, the non-empty proper subset $D = A \setminus (B \cup C)$ is a non-trivial subact of A. Now we have $A = (B \cup C) \cup D$ and $(B \cup C) \cap D = \emptyset$ showing that A is decomposable which is a contradiction.

(ii) ⇒ (iii) Let A be indecomposable. If A is not simple, then there exists a non-trivial subact B of A. Since S is a group, $A \setminus B$ is a non-trivial subact of A. So, $A = B \cup (A \setminus B)$ and hence A is decomposable which is a contradiction. □

Now, being S a group, we prove that there is no connected intersection graph of S-acts with less than six distinct vertices. Also we determine their diameter and girth.

Theorem 4.6. *Let S be a group and A be a non-simple S-act. The graph $G(A)$ is connected if and only if $|G(A)| \geq 6$. In this case, girth$(G(A)) = 3$ and diam$(G(A)) = 2$.*

Proof. Let $G(A)$ be connected. Since A is non-simple, $G(A)$ is non-empty. In view of Theorem 3.1 and Proposition 4.5, $|G(A)| > 2$. So, there exist two non-trivial subacts B and C of A with a non-empty intersection. The possible cases are $B \subset C$, $C \subset B$, or $B \nsubseteq C$ and $C \nsubseteq B$. Since S is a group, the first case yields $B, C, C \setminus B, A \setminus B, A \setminus C$ and $(A \setminus C) \cup B$ are six distinct non-trivial subacts of A and $(C, C \setminus B, A \setminus B)$ is a cycle of length 3 in $G(A)$. Similarly, in the second case we get $|G(A)| \geq 6$ and $G(A)$ contains a cycle of length 3. The third case implies that the six distinct vertices $B, C, B \cap C, B \setminus C, C \setminus B$ and $A \setminus (B \cap C)$ belong to $G(A)$ and $(B, B \setminus C, A \setminus (B \cap C))$ is a cycle of length 3 in $G(A)$. Hence, $|G(A)| \geq 6$ and girth$(G(A)) = 3$. The converse follows from Theorem 3.1. Now let $G(A)$ be connected. Thus $|G(A)| \geq 6$ and then $G(A)$ is non-complete by Proposition 4.5. Hence, Theorem 4.1(i) implies that diam$(G(A)) = 2$. □

There are some natural examples of acts over groups. For instance, the set $\mathcal{U}(S)$ of all subgroups of a group S becomes an S-act by setting $s \cdot T =: sTs^{-1}$, for every $s \in S, T \in \mathcal{U}(S)$. Clearly, the intersection graph $G(\mathcal{U}(S))$ has no vertices if and only if $S = \{1\}$.

In the following, we characterize all groups S for which the graphs $G(\mathcal{U}(S))$ are connected.

Proposition 4.7. *Let S be a non-trivial group. Then the following assertions hold:*

(i) *If S is abelian, then $G(\mathcal{U}(S))$ is connected if and only if S is non-isomorphic to $\mathbb{Z}_p$, for every prime number p.*

(ii) *If S is non-abelian, then $G(\mathcal{U}(S))$ is connected.*

Proof. (i) Let S be abelian. Then the action of S over $\mathcal{U}(S)$ is trivial. So, in view of Remark 3.4(ii), $G(\mathcal{U}(S))$ is disconnected if and only if $|\mathcal{U}(S)| = 2$. Hence, a known fact of groups gives the result.

(ii) If S is non-abelian, then S contains a non-trivial subgroup. This gives that $\mathcal{U}(S)$ has at least two zero elements $\{\{1\}\}, \{S\}$ and $|\mathcal{U}(S)| \geq 3$. Thus $G(\mathcal{U}(S))$ is connected by Lemma 3.10. □

In light of Theorem 4.6 and Proposition 4.7, the following result is obtained.

Corollary 4.8. *Let S be a non-trivial group. If S is non-isomorphic to $\mathbb{Z}_p$ for every prime number p, then girth$(G(\mathcal{U}(S))) = 3$ and diam$(G(\mathcal{U}(S))) = 2$.*

References

[1] J. Bosák, *Theory of Graphs and Application*, Academic Press, New York, 1964.

[2] I. Chakrabarty, S. Ghosh, T. K. Mukherjee and M. K. Sen, Intersection graphs of ideals of rings, *Discrete Math.* **309** (2009), 5381–5392.
[3] B. Csákány and G. Pollák, The graph of subgroups of a finite group, *Czech. Math. J.* **19** (1969), 241–247.
[4] S. Eilenberg, *Automata, Languages, and Machines*, Vol. B, Academic Press, New York, San Francisco, London, 1976.
[5] M. Kilp, U. Knauer and A. Mikhalev, *Monoids, Acts and Categories*, de Gruyter, Berlin, 2000.
[6] U. Knauer, Projectivity of acts and Morita equivalence of monoids, *Semigroup Forum* **3** (1972), 359–370.
[7] G. Lallement, *Semigroups and Combinatorial Applications*, John Wiley, New York, 1979.
[8] S. Lie, Die Grundlagen für die Theorie der unendlichen kontinuierlichen Transformationsgruppen I und II, *Leipzig. Berichte.* **3** (1891), 316–352, 353–393.
[9] B. Pondělíček, On the intersection graph of a commutative distributive groupoid, *Math. Slovaca*, **29** (1979), no. 1, 57–62.
[10] R. Shen, Intersection graphs of subgroups of finite groups, *Czech. Math. J.* **60** (2010), no. 4, 945–950.
[11] D. B. West, *Introduction to Graph Theory*, Prentice-Hall of India Pvt. Ltd, India, 2003.
[12] B. Zelinka, Intersection graphs of finite abelian groups, *Czech. Math. J.* **25** (1975), no. 2, 171–174.
[13] B. Zelinka, Intersection graphs of lattices, *Čas. Mat.* **23** (1973), no. 3, 216–222.
[14] B. Zelinka, Intersection graphs of semilattices, *Čas. Mat.* **25** (1975), no. 4, 345–350.

H. Rasouli, Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
E-mail address: hrasouli@srbiau.ac.ir

A. Tehranian, Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
E-mail address: tehranian@srbiau.ac.ir