CONVERGENCE TO A COMMON FIXED POINT OF A FINITE FAMILY OF GENERALIZED ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. We introduce an iterative process which converges strongly to a common fixed point of a finite family of uniformly continuous generalized asymptotically nonexpansive mappings in Hilbert spaces. As a consequence, results on convergence to a common fixed point of a finite family of uniformly continuous asymptotically nonexpansive in the intermediate sense and asymptotically nonexpansive mappings are proved.

1. Introduction

Let C be a nonempty subset of a real Hilbert space H; a mapping $T : C \to C$ is called a *L-Lipschitzian* if and only if there exists $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \forall x, y \in C. \tag{1.1}$$

If in (1.1) we have that $L \leq 1$ then T is called *nonexpansive*. T is said to be *asymptotically nonexpansive* [10] if and only if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$, as $n \to \infty$, such that for all $x,\ y \in C$ the following inequality holds:

$$\|T^n x - T^n y\| \leq k_n\|x - y\|, \ \ \forall\, n \geq 1. \tag{1.2}$$

T is said to be *asymptotically nonexpansive in the intermediate sense* [3] if and only if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0. \tag{1.3}$$

Put

$$\nu_n := \max\{0, \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|)\}.$$

It follows that $\nu_n \to 0$, as $n \to \infty$. In fact, we see that (1.3) is equivalent to

$$\|T^n x - T^n y\| \leq \|x - y\| + \nu_n, \ \forall n \geq 1, \ x, y \in C, \tag{1.4}$$

where $\nu_n \in [0, \infty)$ with $\nu_n \to 0$, as $n \to \infty$.

It is worth mentioning that, when C is bounded, the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings (see e.g., [15]).

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Let C be a closed subset of a Hilbert space H and T be a self-mapping nonexpansive mapping. The classical *Mann iteration method* [17] given by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \tag{1.5}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers in $[0, 1]$, has extensively been investigated in literature (see, e.g., [4, 8, 23, 34, 35] and references therein). If the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by (1.5) *converges weakly* to a fixed point of T (this is indeed true in a uniformly convex Banach space with a Frechét differentiable norm [23]). Related works can also be found in [2, 6, 12, 19, 21, 22, 28, 29, 33]. However, this convergence is in general not strong (see the counter example in [9]; see also [11]). Attempts to modify the Mann iteration method (1.5) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [18] proposed the following modification of the Mann iteration method (1.5):

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : ||y_n - z|| \le ||x_n - z||\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}(x_0), n \ge 0. \end{cases} \tag{1.6}$$

They proved that the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by (1.6) *converges strongly* to the fixed point of nonexpansive self-mapping T.

It is worth mentioning that Scheme (1.6) involves computation of intersection of closed convex subsets C_n and Q_n of C for each $n \ge 1$ and hence is not easy to compute.

In [26], Schu introduced a Mann type process given by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \ge 1, \tag{1.7}$$

to approximate fixed point of asymptotically nonexpansive self-mapping. He proved that, if C is a nonempty, closed and bounded and T is completely continuous asymptotically nonexpansive self-mapping with sequence $\{k_n\} \subset [1, \infty)$, for all $n \ge 1$, and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ then the sequence $\{x_n\}$ given by (1.7) converges strongly to some fixed point of T.

Rhoades [24] and Chidume *et al.* [5] extended the results of Schu [26] to uniformly convex Banach spaces which are more general than Hilbert spaces using a modified Ishikawa iteration method [13] under different settings. In [20], Osilike and Aniagbosor proved that the theorems of Schu and Rhoades remain true without the boundedness condition imposed on C, provided that $F(T) \ne \emptyset$.

Recently, Chidume *et al.* [7] proved that, if T is completely continuous an asymptotically nonexpansive mapping in the intermediate sense with a sequence $\{\nu_n\}$ such that $\sum \nu_n < \infty$ with $F(T) \ne \emptyset$, then, for arbitrary $x_0 \in C$, the sequence defined by:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \ge 1, \tag{1.8}$$

where $\{\alpha_n\}$ is a sequence in $[\epsilon, 1 - \epsilon]$, for some $\epsilon > 0$, converges strongly to some fixed point of T. They also proved *weak convergence of the scheme* without the assumption that T be completely continuous.

But it is worth mentioning that in all the above results, either *compactness* assumption or completely continuous, is imposed on the map T or the convergence is *weak convergence*. Our concern now is the following:

Let C be a nonempty, closed and convex subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be *generalized asymptotically nonexpansive* if and only if there exist $\{\mu_n\}, \{\nu_n\} \subset [0, \infty)$ such that $\mu_n, \nu_n \to 0$, as $n \to \infty$, satisfying the following inequality:

$$||T^n x - T^n y|| \leq ||x - y|| + \mu_n ||x - y|| + \nu_n, \forall x, y \in C. \tag{1.9}$$

An example of *generalized asymptotically nonexpansive* mapping is the following:

Example 1.1. Let $C := (-\infty, 1]$ and for $k \in (0, 1)$ define $T : C \to C$ by

$$T(x) = \begin{cases} x, & x \in (-\infty, 0), \\ kx, & x \in [0, \frac{1}{2}], \\ 0, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Then, clearly $F(T) = (-\infty, 0]$. Let $C_1 := (-\infty, 0)$, $C_2 := [0, \frac{1}{2}]$ and $C_3 := (\frac{1}{2}, 1]$. Then, if $x, y \in C_1$, $x, y \in C_2$ or $x, y \in C_3$ we get that $|T^n x - T^n y| = |x - y| + k^n |x - y|$. If $x \in C_2$ and $y \in C_1$ then we obtain that

$$\begin{aligned} |T^n x - T^n y| &= |k^n x - y| = |k^n x - k^n y + k^n y - y| \leq k^n |x - y| + k^n |y| + |y| \\ &\leq |x - y| + k^n |x - y| + k^n |x - y| = |x - y| + 2k^n |x - y| \end{aligned} \tag{1.10}$$

If $x \in C_1$ and $y \in C_3$ then we get that $|T^n x - T^n y| = |x| \leq |x - y|$. If $x \in C_2$ and $y \in C_3$ then we have that

$$\begin{aligned} |T^n x - T^n y| = |k^n x| &\leq |k^n x - k^n y + k^n y| \leq k^n |x - y| + k^n |y| \\ &\leq |x - y| + k^n |x - y| + k^n |y| \end{aligned} \tag{1.11}$$

Therefore, from the above discussions we have that T is generalized asymptotically nonexpansive mapping with $\mu_n = 2k^n$ and $\nu_n = k^n$.

The class of generalized asymptotically nonexpansive mappings was introduced by Shahzad and Zegeye [27]. It is clear from the definition that a class of generalized asymptotically nonexpansive mappings includes the classes of asymptotically nonexpansive in the intermediate sense, asymptotically nonexpansive and nonexpansive mappings. In [32], Shahzad and Zegeye proved that if T_i, for $i \in \{1, 2, ..., N\}$, are continuous generalized asymptotically nonexpansive self mappings of C subset of a real Hilbert space H with $F := \cap_{i=1}^N F(T_i) \neq \emptyset$, then Mann's type scheme given by:

$$x_0 \in C, x_{n+1} = \alpha_0 x_n + \alpha_1 T_1^n x_n + ... + \alpha_N T_N^n x_n, \forall n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying certain conditions, converges strongly to a common fixed point of mappings $\{T_i : i = 1, 2, ..., N\}$ provided that the *interior of F is nonempty*. In fact the assumption that *interior of F is nonempty* is a severe restriction.

It is our purpose, in this paper to construct an explicit iterative scheme which converges strongly to a common fixed point of a finite family of uniformly continuous generalized asymptotically nonexpansive mappings in Hilbert spaces. As a consequence, results on convergence to a common fixed point of a finite family of uniformly continuous asymptotically nonexpansive in the intermediate sense and asymptotically nonexpansive mappings are proved. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear mappings.

2. Preliminaries

In what follows, we shall make use of the following lemmas.

Lemma 2.1. *Let H be a real Hilbert space. Then the following inequality holds:*

(1) $||x+y||^2 \leq ||x||^2 + 2\langle y, x+y\rangle$, *for all* $x, y \in H$.
(2) *If* $\{x_n\}$ *is a sequence in* H *weakly convergent to* z, *then* $\limsup_{n\to\infty} ||x_n - y||^2 = \limsup_{n\to\infty} ||x_n - z||^2 + ||z-y||^2, \forall y \in H$.

Lemma 2.2. [31] *Let H be a real Hilbert space and $B_R(0)$ be a closed ball of H. Then, for any given subset $\{x_0, x_1, ..., x_N\} \subset B_r(0)$ and for any positive numbers $\alpha_0, \alpha_1, ..., \alpha_N$ with $\sum_{i=0}^{N} \alpha_i = 1$, we have that*

$$||\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_N x_N||^2 \quad \leq \quad \sum_{i=0}^{N} \alpha_i ||x_i||^2 - \alpha_i\alpha_j ||x_i - x_j||^2,$$

for any $i, j \in \{0, 1, 2, ..., N\}$ with $i < j$,

Lemma 2.3. [16] *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, ..., k\}$ such that the condition $a_n \leq a_{n+1}$ holds.

Lemma 2.4. [1] *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $x \in H$. Then $x_0 = P_C x$ if and only if*

$$\langle z - x_0, x - x_0\rangle \leq 0, \forall z \in C.$$

Lemma 2.5. *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n + \gamma_n, n \geq n_0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset R$ satisfying the following conditions: $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n\to\infty} \delta_n \leq 0$ and $\frac{\gamma_n}{\alpha_n} \to 0$, as $n \to \infty$. Then, $\lim_{n\to\infty} a_n = 0$.

Proof. Let $\theta_n = \frac{\gamma_n}{\alpha_n}$ then by the hypothesis we have that $\theta_n \to 0$, as $n \to \infty$. Thus, we obtain that $\gamma_n = \alpha_n \theta_n$ and that $\limsup_{n\to\infty}(\delta_n + \theta_n) \leq 0$ and hence the conclusion follows from Lemma 2.1 of Xu [30]. □

Lemma 2.6. *Let C be a nonempty, closed and convex subset of a Hilbert space H and $T : C \to C$ be a continuous generalized asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \bar{x} \in C$ and $\limsup_{m\to\infty} \limsup_{n\to\infty} ||x_n - T^m x_n|| = 0$, then $(I - T)\bar{x} = 0$.*

Proof. Since $\{x_n\}$ is bounded, we can define a function on H by $f(x) = \limsup_{n\to\infty} ||x_n - x||^2, x \in H$. By Lemma 2.1, the weak convergence $x_n \rightharpoonup \bar{x}$ implies that $f(x) = f(\bar{x}) + ||x - \bar{x}||^2$, $x \in H$. In particular, for each $m \geq 1$, we have that

$$f(T^m\bar{x}) = f(\bar{x}) + ||T^m\bar{x} - \bar{x}||^2. \tag{2.1}$$

On the other hand, since T is generalized asymptotically nonexpansive we get that

$$\begin{aligned}
f(T^m\bar{x}) &= \limsup_{n\to\infty} ||x_n - T^m\bar{x}||^2 \\
&= \limsup_{n\to\infty} ||(x_n - T^m x_n) + (T^m x_n - T^m\bar{x})||^2 \\
&\leq \limsup_{n\to\infty} \big[||x_n - T^m x_n|| + ||T^m x_n - T^m\bar{x}||\big]^2 \\
&= \limsup_{n\to\infty} \big[||x_n - T^m x_n||^2 + ||T^m x_n - T^m\bar{x}||^2 + 2M||x_n - T^m x_n||\big] \\
&\leq \limsup_{n\to\infty} ||x_n - T^m x_n||^2 + \limsup_{n\to\infty} ||T^m x_n - T^m\bar{x}||^2 \\
&\quad + \limsup_{n\to\infty} 2M||x_n - T^m x_n||\big] \\
&\leq \limsup_{n\to\infty} ||x_n - T^m x_n||^2 + \limsup_{n\to\infty} \big[(1 + \mu_m)||x_n - \bar{x}|| + \nu_m\big]^2 \\
&\quad + \limsup_{n\to\infty} 2M||x_n - T^m x_n||\big],
\end{aligned}$$

for some $M > 0$. Now, taking $\limsup_{m\to\infty}$ both sides and using the facts that $\limsup_{m\to\infty} \mu_m = 0$, $\limsup_{m\to\infty} \nu_m = 0$ and $\limsup_{m\to\infty} \limsup_{n\to\infty} ||x_n - T^m x_n|| = 0$, we obtain that

$$\limsup_{m\to\infty} f(T^m\bar{x}) \leq \limsup_{n\to\infty} ||x_n - \bar{x}||^2. \tag{2.2}$$

Now, from (2.1) and (2.2) we get that $\limsup_{m\to\infty} ||\bar{x} - T^m\bar{x}|| = 0$, that is, $T^m\bar{x} \to \bar{x}$ and hence by continuity of T we obtain that $T\bar{x} = \bar{x}$. □

3. Main Result

We now state and prove our main theorem.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : C \to C$ be uniformly continuous generalized asymptotically nonexpansive mappings with sequences $\{\mu_{n,i}\}$ and $\{\nu_{n,i}\}$, for $i = 1, 2, ..., N$. Assume that $\mathcal{F} := \cap_{i=1}^N F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = w \in C, \text{ chosen arbitrarily,} \\ y_n = \alpha_n w + (1 - \alpha_n)x_n, \\ x_{n+1} = \beta_{n,0}x_n + \sum_{i=1}^N \beta_{n,i}T_i^n y_n, n \geq 1, \end{cases} \tag{3.1}$$

where $\alpha_n \in (0,1)$ *such that* $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\frac{\mu_{n,i}}{\alpha_n} \to 0$, $\frac{\nu_{n,i}}{\alpha_n} \to 0$, *as* $n \to \infty$, $\{\beta_{n,i}\} \subset [a,b] \subset (0,1)$, *for* $i = 1,2,...,N$, *satisfying* $\beta_{n,0}+\beta_{n,1}+...+\beta_{n,N} = 1$, *for each* $n \geq 1$. *Then* $\{x_n\}$ *converges strongly to an element of* $\mathcal{F}$.

Proof. Let $x^* \in \mathcal{F}$. Let $\mu_n := \max\{\mu_{n,i} : i = 1,2,...,N\}$ and $\nu_n := \max\{\nu_{n,i} : i = 1,2,...,N\}$. Then, from (3.1) and generalized asymptotically nonexpansiveness of T_i, for each $i \in \{1,2,...,N\}$, we have that

$$\begin{aligned}
||x_{n+1} - x^*|| &= ||\beta_{n,0}x_n + \sum_{i=1}^{N}\beta_{n,i}T_i^n y_n - x^*|| \\
&\leq \beta_{n,0}||x_n - x^*|| + ||\sum_{i=1}^{N}\beta_{n,i}(T_i^n y_n - x^*)|| \\
&\leq \beta_{n,0}||x_n - x^*|| + (1-\beta_{n,0})(1+\mu_n)||y_n - x^*|| + (1-\beta_{n,0})\nu_n \\
&\leq \beta_{n,0}||x_n - x^*|| + (1-\beta_{n,0})(1+\mu_n)\big[\alpha_n||w - x^*|| \\
&\quad +(1-\alpha_n)||x_n - x^*||\big] + (1-\beta_{n,0})\nu_n \\
&\leq \big[\beta_{n,0} + (1-\beta_{n,0})(1+\mu_n)(1-\alpha_n)\big]||x_n - x^*|| \\
&\quad +\big[(1-\beta_{n,0})(1+\mu_n)\alpha_n\big]||w - x^*|| + (1-\beta_{n,0})\nu_n \\
&\leq \delta_n\big[||w - x^*|| + 1\big] + \big[1 - (1-\epsilon)\delta_n\big]||x_n - x^*|| \\
&= (1-\epsilon)\delta_n\big[(1-\epsilon)^{-1}(||w - x^*|| + 1)\big] + \big(1 - (1-\epsilon)\delta_n\big)||x_n - x^*||, \qquad (3.2)
\end{aligned}$$

where $\delta_n = (1-\beta_{n,0})(1+\mu_n)\alpha_n$, since there exists $N_0 > 0$ such that $\frac{\mu_n}{\alpha_n} \leq \epsilon(1+\mu_n)$ and $\frac{\nu_n}{\alpha_n} \leq (1+\mu_n)$, for all $n \geq N_0$ and for some $\epsilon > 0$ satisfying $0 \leq (1-\epsilon)\delta_n \leq 1$. Thus, by induction,

$$||x_{n+1} - x^*|| \leq \max\{||x_{N_0} - x^*||, (1-\epsilon)^{-1}(||w - x^*|| + 1)\}, \forall n \geq N_0,$$

which implies that $\{x_n\}$, and hence $\{y_n\}$, is bounded. Moreover, from (3.1) and Lemma 2.1 we obtain that

$$\begin{aligned}
||y_n - x^*||^2 &= ||\alpha_n(w - x^*) + (1-\alpha_n)(x_n - x^*)||^2 \\
&\leq (1-\alpha_n)||x_n - x^*||^2 + 2\alpha_n\langle w - x^*, y_n - x^*\rangle. \qquad (3.3)
\end{aligned}$$

Furthermore, from (3.1), Lemma 2.2, generalized asymptotically nonexpansiveness of T_i, for each $i = 1,2,...,N$, and (3.3) we have that

$$\begin{aligned}
||x_{n+1} - x^*||^2 &= ||\beta_{n,0}x_n + \sum_{i=1}^{N}\beta_{n,i}T_i^n y_n - x^*||^2 \\
&\leq \beta_{n,0}||x_n - x^*||^2 + \sum_{i=1}^{N}\beta_{n,i}||T_i^n y_n - x^*||^2 \\
&\quad -\beta_{n,0}\beta_{n,i}||x_n - T_i^n y_n||^2 \\
&\leq \beta_{n,0}||x_n - x^*||^2 + (1-\beta_{n,0})(1+\mu_n M')||y_n - x^*||^2 \\
&\quad +(1-\beta_{n,0})M'\nu_n - \beta_{n,0}\beta_{n,i}||x_n - T_i^n y_n||^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
||x_{n+1}-x^*||^2 &\leq \beta_{n,0}||x_n-x^*||^2+(1-\beta_{n,0})\Big[(1-\alpha_n)||x_n-x^*||^2 \\
&\quad +2\alpha_n\langle w-x^*, y_n-x^*\rangle\Big]+(1-\beta_{n,0})\mu_n M'||y_n-x^*||^2 \\
&\quad +(1-\beta_{n,0})M'\nu_n-\beta_{n,0}\beta_{n,i}||x_n-T_i^n y_n||^2 \\
&\leq (1-\theta_n)||x_n-x^*||^2+2\theta_n\langle w-x^*, y_n-x^*\rangle+(\mu_n+\nu_n)M \\
&\quad -\beta_{n,0}\beta_{n,i}||x_n-T_i^n y_n||^2 \qquad (3.4)\\
&\leq (1-\theta_n)||x_n-x^*||^2+2\theta_n\langle w-x^*, y_n-x^*\rangle \\
&\quad +(\mu_n+\nu_n)M, \qquad (3.5)
\end{aligned}$$

for some $M', M>0$, where $\theta_n:=\alpha_n(1-\beta_{n,0})$.

Now, the rest of the proof is divided into two parts:

Case 1. Suppose that there exists $n_0\in\mathbb{N}$ such that $\{||x_n-x^*||\}$ is decreasing for all $n\geq n_0$. In this situation, $\{||x_n-x^*||)\}$ is convergent. Then from (3.4) we have that

$$x_n-T_i^n y_n\to 0, \text{ as } n\to\infty, \qquad (3.6)$$

for each $i\in\{1,2,...,N\}$. Moreover, from (3.1) and (3.6) and the fact that $\alpha_n\to 0$, we get that

$$||x_{n+1}-x_n||=\beta_{n,1}||T_1^n y_n-x_n||+...+\beta_{n,N}||T_N^n y_n-x_n||\to 0, \qquad (3.7)$$

and

$$y_n-x_n=\alpha_n(w-x_n)\to 0, \qquad (3.8)$$

as $n\to\infty$ and hence

$$||y_{n+1}-y_n||\leq||y_{n+1}-x_{n+1}||+||x_{n+1}-x_n||+||x_n-y_n||\to 0, \qquad (3.9)$$

as $n\to\infty$. Furthermore, from (3.6) and (3.8) we get that

$$||y_n-T_i^n y_n||\leq||y_n-x_n||+||x_n-T_i^n y_n||\to 0, \text{ as } n\to\infty. \qquad (3.10)$$

Therefore, since

$$\begin{aligned}
||y_n-T_i y_n|| &\leq ||y_n-y_{n+1}||+||y_{n+1}-T_i^{n+1}y_{n+1}||+||T_i^{n+1}y_{n+1}-T_i^{n+1}y_n|| \\
&\quad +||T_i^{n+1}y_n-T_i y_n||, \\
&\leq ||y_n-y_{n+1}||+||y_{n+1}-T_i^{n+1}y_{n+1}||+(1+\mu_{n+1})||y_{n+1}-y_n|| \\
&\quad +\nu_{n+1}+||T_i(T_i^n y_n)-T_i y_n||, \qquad (3.11)
\end{aligned}$$

we have from (3.9), (3.10), (3.11) and uniform continuity of T_i that

$$||y_n-T_i y_n||\to 0, \text{ as } n\to\infty, \text{ for each } i=1,2,...,N\ , \qquad (3.12)$$

and hence again uniform continuity of T_i for each $i\in\{1,2,...,N\}$ implies that

$$\limsup_{m\to\infty}\limsup_{n\to\infty}||y_n-T_i^m y_n||=0. \qquad (3.13)$$

Moreover, since $\{y_n\}$ is bounded and H is reflexive, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i}\rightharpoonup z$ and $\limsup_{n\to\infty}\langle w-v, y_n-v\rangle=\lim_{i\to\infty}\langle w-v, y_{n_i}-v\rangle$, where $v=P_{\mathcal{F}}(w)$. Then, from (3.13) and Lemma 2.6 we have that $z\in F(T_i)$, for each

$i = 1, 2, ..., N$. Hence, $z \in \cap_{i=1}^{N} F(T_i)$. Therefore, by Lemma 2.4, we immediately obtain that

$$\begin{aligned} \limsup_{n\to\infty}\langle w - v, y_n - v\rangle &= \lim_{i\to\infty}\langle w - v, y_{n_i} - v\rangle \\ &= \langle w - v, z - v\rangle \leq 0. \end{aligned} \tag{3.14}$$

Thus, now putting $x^* := v$ in inequality (3.5) we get that, for $n \geq n_0$,

$$\begin{aligned} ||x_{n+1} - v||^2 &\leq (1 - \theta_n)||x_n - v||^2 + 2\theta_n\langle w - v, y_n - v\rangle \\ &\quad + M(\mu_n + \nu_n) \end{aligned} \tag{3.15}$$

for some $M > 0$. Note that θ_n satisfies $\lim_n \theta_n = 0$ and $\sum_{n=1}^{\infty} \theta_n = \infty$. Thus, it follows from (3.15) and Lemma 2.5 that $||x_n - v|| \to 0$, as $n \to \infty$. Consequently, $x_n \to v$.

Case 2. Suppose that for each $n_0 \in \mathbb{N}$, $\{||x_n - x^*||\}_{n\geq n_0}$ is not decreasing. Then there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$||x_{n_i} - x^*|| < ||x_{n_i+1} - x^*||$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.3, there exists an increasing sequence $\{m_k\}_{k\geq n_0}$ such that $m_k \to \infty$, $||x_{m_k} - x^*|| \leq ||x_{m_k+1} - x^*||$ and $||x_k - x^*|| \leq ||x_{m_k+1} - x^*||$ for all $k \geq n_0$. Then from (3.4) and the fact that $\theta_n \to 0$ we have

$$\begin{aligned} &\beta_{m_k,0}\beta_{m_k,i}||x_{m_k} - T_i^{m_k} y_{m_k}||^2 \\ \leq\ &||x_{m_k} - x^*||^2 - ||x_{m_k+1} - x^*||^2 + \theta_{m_k}||x_{m_k} - x^*||^2 \\ &+ 2\theta_{m_k}\langle w - x^*, y_{m_k} - x^*\rangle + M(\mu_{m_k} + \nu_{m_k}) \to 0, \text{ as } k \to \infty. \end{aligned}$$

This implies that $x_{m_k} - T_i^{m_k} y_{m_k} \to 0$, as $k \to \infty$. Thus, as in Case 1, we obtain that $x_{m_k} - y_{m_k}, y_{m_k} - T_i y_{m_k} \to 0$, as $k \to \infty$, and $\limsup_{m\to\infty}\limsup_{k\to\infty} ||y_{n_k} - T_i^m y_{n_k}|| = 0$, for each $i = 1, 2, ..., N$ and hence we get that

$$\limsup_{k\to\infty}\langle w - v, y_{m_k} - v\rangle \leq 0. \tag{3.16}$$

where $v = P_{\mathcal{F}}(w)$. Now, putting $x^* := v$ in (3.5) we obtain that, for some $M > 0$,

$$\begin{aligned} ||x_{m_k+1} - v||^2 &\leq (1 - \theta_{m_k})||x_{m_k} - v||^2 + 2\theta_{m_k}\langle w - v, y_{m_k} - v\rangle \\ &\quad + M(\mu_{m_k} + \nu_{m_k}). \end{aligned} \tag{3.17}$$

Since $||x_{m_k} - v||^2 \leq ||x_{m_k+1} - v||^2$, (3.17) implies that

$$\begin{aligned} \theta_{m_k}||x_{m_k} - v||^2 &\leq ||x_{m_k} - v^*||^2 - ||x_{m_k+1} - v||^2 + 2\theta_{m_k}\langle w - v, y_{m_k} - v\rangle \\ &\quad + M(\mu_{m_k} + \nu_{m_k}). \\ &\leq 2\theta_{m_k}\langle w - v, y_{m_k} - v\rangle + M(\mu_{m_k} + \nu_{m_k}). \end{aligned}$$

In particular, since $\theta_{m_k} > 0$, we get

$$||x_{m_k} - v||^2 \leq 2\langle w - v, y_{m_k} - v\rangle + \frac{\mu_{m_k}}{\theta_{m_k}}M + \frac{\nu_{m_k}}{\theta_{m_k}}M.$$

Then, from (3.16) and the fact that $\frac{\mu_{m_k}}{\theta_{m_k}}, \frac{\nu_{m_k}}{\theta_{m_k}} \to 0$, we obtain that $||x_{m_k} - v|| \to 0$, as $k \to \infty$. This together with (3.17) give $||x_{m_k+1} - v|| \to 0$, as $k \to \infty$. But $||x_k - v|| \leq ||x_{m_k+1} - v||$, for all $k \geq n_0$, thus we obtain that $x_k \to v$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to an element of $\mathcal{F}$ and the proof is complete. □

The following example is a uniformly continuous generalized asymptotically nonexpansive mapping which is not Lipshitzian.

Example 3.2. Let $C := [\frac{-1}{\pi}, \frac{1}{\pi}]$ and $T : C \to C$ be given by

$$T(x) = \begin{cases} \frac{x}{2}\sin(\frac{1}{x}), & x \in (0, \frac{1}{\pi}] \\ x, & x \in [\frac{-1}{\pi}, 0] \end{cases}$$

Then it is shown in [14, 32] that T is uniformly continuous generalized asymptotically nonexpansive mapping which is not Lipschitzian.

If in Theorem 3.1, we assume that each T_i is asymptotically nonexpansive in the intermediate sense we get the following corollary.

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : C \to C$ be uniformly continuous mappings which are asymptotically nonexpansive in the intermediate sense with sequences $\{\nu_{n,i}\}$, for $i = 1, 2, ..., N$. Assume that $\mathcal{F} := \cap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = w \in C, \text{ chosen arbitrarily}, \\ y_n = \alpha_n w + (1 - \alpha_n)x_n, \\ x_{n+1} = \beta_{n,0}x_n + \sum_{i=1}^{N} \beta_{n,i}T_i^n y_n, \; n \geq 1, \end{cases} \tag{3.18}$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\frac{\nu_{n,i}}{\alpha_n} \to 0$, as $n \to \infty$, $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$, for $i = 1, 2, ..., N$ satisfying $\beta_{n,0} + \beta_{n,1} + ... + \beta_{n,N} = 1$, for each $n \geq 1$. Then $\{x_n\}$ converges strongly to an element of $\mathcal{F}$.

Proof. Since every asymptotically nonexpansive mapping in the intermediate sense is generalized asymptotically nonexpansive mapping with $\mu_{n,i} \equiv 0$, for all $n \geq 0$ and for $i = 1, 2, ..., N$, the conclusion follows from Theorem 3.1. □

If in Theorem 3.1, we assume that each T_i is asymptotically nonexpansive we have that T_i is uniformly continuous generalized asymptotically nonexpansive with $\nu_{n,i} \equiv 0$, for all $n \geq 1$ and for $i = 1, 2, ..., N$, and hence we get the following corollary.

Corollary 3.4. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : C \to C$ be asymptotically nonexpansive mappings with sequence $\{k_{n,i} := (1 + \mu_{n,i})\}$, for $i = 1, 2, ..., N$. Assume that $\mathcal{F} := \cap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = w \in C, \text{ chosen arbitrarily}, \\ y_n = \alpha_n w + (1 - \alpha_n)x_n, \\ x_{n+1} = \beta_{n,0}x_n + \sum_{i=1}^{N} \beta_{n,i}T_i^n y_n, n \geq 1, \end{cases} \tag{3.19}$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\frac{\mu_{n,i}}{\alpha_n} \to 0$, as $n \to \infty$, $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$, for $i = 1, 2, ..., N$ satisfying $\beta_{n,0} + \beta_{n,1} + ... + \beta_{n,N} = 1$, for each $n \geq 1$. Then $\{x_n\}$ converges strongly to an element of $\mathcal{F}$.

If in Theorem 3.1, we assume that each T_i is nonexpansive we get that T_i is generalized asymptotically nonexpansive with $\mu_{n,i} \equiv 0 \equiv \nu_{n,i}$, for all $n \geq 1$ and $i = 1, 2, ..., N$ and hence the following corollary holds.

Corollary 3.5. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : C \to C$ be nonexpansive mappings for $i = 1, 2, ..., N$. Assume that $\mathcal{F} := \cap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = w \in C, \text{ chosen arbitrarily,} \\ y_n = \alpha_n w + (1 - \alpha_n)x_n, \\ x_{n+1} = \beta_{n,0}x_n + \sum_{i=1}^{N} \beta_{n,i}T_i^n y_n, \ n \geq 1, \end{cases} \tag{3.20}$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$ for $i = 1, 2, ..., N$ satisfying $\beta_{n,0} + \beta_{n,1} + ... + \beta_{n,N} = 1$, for each $n \geq 1$. Then $\{x_n\}$ converges strongly to an element of $\mathcal{F}$.

Remark 3.6. *Theorem 3.1 provides convergent sequence to a common fixed point of a finite family of uniformly continuous generalized asymptotically nonexpansive mappings in Hilbert spaces. No compactness assumption is imposed either on T or on C and our scheme does not involve computation of C_{n+1} from sets C_n and Q_n for each $n \geq 1$. Moreover, the assumption that interior of $\mathcal{F} := \cap_{i=1}^{N} F(T_i)$ is nonempty is not required.*

Remark 3.7. *Our results extend and unify most of the results that have been proved for this important class of nonlinear mappings. In particular, Theorem 3.1 extends Theorem 3.4 of Nakajo and Takahashi* [18] *and Theorem 1 of Kim and Kim* [15] *to a more general class of generalized asymptotically nonexpansive mappings without the compactness assumption on T or our scheme does not involve computation of C_{n+1} from sets C_n and Q_n for each $n \geq 1$. Corollary 3.4 extends Theorem 2.1 and 2.2 of Schu* [25] *and Theorem 3.3 of Zhou et.al.* [36] *to a more general class of generalized asymptotically nonexpansive mappings without any compactness assumption on T or C.*

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