Cohen-Macaulayness of bipartite graphs, revisited

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Abstract

Cohen-Macaulayness of bipartite graphs is investigated by several mathematicians and has been characterized combinatorially. In this paper, we give some different combinatorial conditions for a bipartite graph which are equivalent to Cohen-Macaulayness of the graph. We prove that a bipartite graph is Cohen-Macaulay if and only if it is well-covered and has a unique perfect matching. We also provide a fast algorithm to check Cohen-Macaulayness of a given bipartite graph.

Key words: Edge ideal, Cohen-Macaulay ring, bipartite graph, well-covered.
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1 Introduction an preliminaries

Characterization and classification of Cohen-Macaulay graphs, specially bipartite graphs, have been extensively studied in the last decades (e.g. see [2], [4], [6], [11] and [3]). A thorough background to the subject is provided in the above references and [9]. To make this note self-contained, we review some of the basic notions.

Throughout this paper, all graphs are finite and simple with no vertex of degree zero. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For two adjacent vertices v and w in G, we write $v \sim w$. The set of all vertices of G adjacent to a vertex v is denoted by $N(v)$. We say that a set $F \subseteq V(G)$ is an independent set in G if no two vertices of F are adjacent. A set $P \subseteq E(G)$ is called a perfect matching if there is no pair of distinct edges in P with a common vertex and any vertex in G belongs to one of the edges in P. A graph G is called bipartite if $V(G) = V \cup W$ such that $V \cap W = \varnothing$ and both V and W and independent sets in G. A bipartite graph is called complete bipartite if each vertex in V is adjacent to each vertex of W.

Let G be a graph on the vertex set $V(G) = \{v_1, \dots, v_n\}$. Let K be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring on n variables with coefficients in K. The edge ideal $I(G)$ of G is defined to be the ideal of S generated by all square-free monomials $x_i x_j$, provided that, $v_i \sim v_j$ in G. The quotient ring $R(G) = S/I(G)$ is called the edge ring of the graph G.

Let $[n] = \{1, 2, \dots, n\}$. A (finite) simplicial complex Δ on n vertices is a collection of subsets of $[n]$ such that the following conditions hold:

i) $\{i\} \in \Delta$ for each $i \in [n]$,
ii) if $E \in \Delta$ and $F \subseteq E$, then $F \in \Delta$.
An element of Δ is called a face and a maximal face with respect to inclusion order, is called a facet. The dimension of a face $F \in \Delta$ is defined to be $|F| - 1$ and the dimension of Δ is the maximum of the dimensions of its faces. Faces with dimension 0 are called vertices.

Let Δ be a simplicial complex on $[n]$ and $S = K[x_1, \ldots, x_n]$. Let I_Δ be the ideal of S generated by all square-free monomials $x_{i_1} \cdots x_{i_s}$, provided that, $\{i_1, \ldots, i_s\} \notin \Delta$. The quotient ring $K[\Delta] = S/I_\Delta$ is called the Stanley-Reisner ring of the simplicial complex Δ. Let v_i be a vertex in a graph G and x_i be the corresponding variable in the polynomial ring. In this paper, we usually use x_i instead of v_i. The same notation is used for vertices of a simplicial complex.

For a graph G, there is a simplicial complex which is called independence complex of G and is defined by

$$\Delta_G = \{F \subseteq V : F \text{ is an independent set in } G\}.$$

Let R be a commutative ring with an identity. The depth of R, denoted by $\operatorname{depth}(R)$, is the largest integer r such that there is a sequence $f_1, \ldots, f_r$ of elements of R such that f_i is not a zero-divisor in $S/(f_1, \ldots, f_{i-1})$ for all $1 \leq i \leq r$, and $(f_1, \ldots, f_r) \neq R$. Such a sequence is called a regular sequence. The depth is an important invariant of a ring. It is bounded by another important invariant, the Krull dimension of the ring denoted by $\dim(R)$; the length of the longest chain of prime ideals in the ring. The ring R is called Cohen-Macaulay if $\operatorname{depth}(R) = \dim(R)$. A graph G (a simplicial complex Δ, respectively) is called Cohen-Macaulay if the ring $R(G)$ (the ring $K[\Delta]$, respectively) is Cohen-Macaulay. It is well known that $\dim(K[\Delta]) = \dim(\Delta) + 1$ (see [5]).

A simplicial complex Δ is called pure if all its facets have the same cardinality. A graph G is called well covered or unmixed if all maximal independent sets of vertices of G have the same cardinality. It is clear that a graph G is unmixed if and only if the simplicial complex Δ_G is pure. It is well known that a Cohen-Macaulay simplicial complex is pure ([5]), but the converse is not true, i.e., there are pure simplicial complexes which are not Cohen-Macaulay. Also it is known that if G is a Cohen-Macaulay graph, then $\overline{G}$, the complement of G, is connected [5].

A pure simplicial complex Δ on vertex set $[n]$ is called completely balanced if there is a partition of $[n]$ as $C_1, \ldots, C_r$ such that each facet of Δ has exactly one vertex in common with each C_i. Here, a partition means that $C_1 \cup \cdots \cup C_r = [n]$, and for each $i \neq j$, $C_i \cap C_j = \varnothing$. Such simplicial complexes were studied by R. Stanley [8]. He proved that, in a completely balanced simplicial complex with partition $C_1, \ldots, C_r$, the elements $\theta_1, \ldots, \theta_r$ form a homogeneous system of parameters, where

$$\theta_i = \sum_{j \in C_i} \bar{x}_j,$$

where, $\bar{x}_i$ denotes the image of x_i in $K[\Delta]$. By a homogeneous system of parameters in a standard graded ring R, we mean a set of homogeneous elements $\theta_1, \ldots, \theta_r$ of nonzero degrees such that $\dim(R/(\theta_1, \ldots, \theta_r)) = 0$.

2 Main Results

M. Estrada and R. H. Villarreal in [2] have proved that, for a bipartite graph G, Cohen-Macaulayness and shellability are equivalent and, if G is Cohen-Macaulay, then, there is a vertex v in G such that $G \setminus \{v\}$ is again Cohen-Macaulay.

Villarreal has proved in [12] that a bipartite graph G with parts V and W is unmixed if and only if $|V| = |W|$ and there is an order on vertices of V and W as $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively, such that:
1) $x_i \sim y_i$ for $i = 1, \ldots, n$,
2) for each $1 \leq i < j < k \leq n$ if $x_i \sim y_j$ and $x_j \sim y_k$, then $x_i \sim y_k$.

J. Herzog and T. Hibi in [4] have proved that, a bipartite graph G is Cohen-Macaulay if and only if it is unmixed (has properties (1) and (2) above) and:
3) if $x_i \sim y_j$, then $i \leq j$.

In these results, one needs to find an appropriate order on the vertices of G for which the criterion holds. Therefore, checking the Cohen-Macaulayness of a given bipartite graph in practice is rather complicated. In this paper, we show that one can check Cohen-Macaulayness of a given bipartite graph in a quite easy way. We prove that, a graph G is Cohen-Macaulay if and only if it is unmixed and has a unique perfect matching. These kind of graphs have applications in computational biology. The unique maximum-weight perfect matching can be used to predict the folding structure of RNA molecules (see [10]). Our main theorem is the following.

Theorem 1 *Let G be a bipartite graph with parts V and W. Then, G is Cohen-Macaulay if and only if there is a perfect matching in G as $\{x_1, y_1\}, \ldots, \{x_n, y_n\}$, such that, $x_i \in V$ and $y_i \in W$ for $i = 1, \ldots, n$, and the following two conditions hold.*
1) The induced subgraph on $N(x_i) \cup N(y_i)$ is a complete bipartite graph, for $i = 1, \ldots, n$.
2) If $x_i \sim y_j$ for $i \neq j$, then, $x_j \not\sim y_i$.

Before proving the theorem, we prove some preliminary lemmas.

Lemma 2 *Let G be an unmixed bipartite graph with a perfect matching $\{x_1, y_1\}, \ldots, \{x_n, y_n\}$. Then, G is Cohen-Macaulay if and only if the sequence $\bar{x}_1 + \bar{y}_1, \ldots, \bar{x}_n + \bar{y}_n$ is a regular sequence in $R(G)$.*

Proof. The sets $\{x_1, y_1\}, \ldots, \{x_n, y_n\}$ form a partition of vertices of G and any maximal independent set intersects each of them in exactly one vertex. Thus, the simplicial complex Δ_G is completely balanced. By Corollary 4.2 and its remark in [8], the sequence $x_1 + y_1, \ldots, x_n + y_n$ is a system of parameters in $R(G)$. By Theorem 17.4 in [7], $R(G)$ is Cohen-Macaulay if and only if every system of parameters is a regular sequence in $R(G)$. □

Lemma 3 *Let I be an ideal of $S = K[x_1, \ldots, x_n]$, generated by some quadratic monomials. Let for some i, j, $1 \leq i < j \leq n$, $x_i^2 \notin I$ and $x_j^2 \notin I$. Then, $\bar{x}_i + \bar{x}_j$ is a zero-divisor in S/I if and only if one of the following conditions holds.*
i) There is x_k, $k \notin \{i, j\}$ such that $\bar{x}_k(\bar{x}_i + \bar{x}_j) = 0$ or,

ii) There are integers k, l, $1 \leq k < l \leq n$, both distinct from i and j, such that $x_k x_l \notin I$ and $\bar{x}_k \bar{x}_l(\bar{x}_i + \bar{x}_j) = 0$.
Here, $\bar{x}_i$ denotes the image of x_i in S/I.

Proof. Without loss of generality, we may assume that $i = 1$ and $j = 2$. It is well known that a polynomial f in S belongs to a monomial ideal I if and only if all monomials of f belong to I. Let $\prec$ be the lexicographic order on monomials of S induced by $x_1 \succ x_2 \succ \cdots \succ x_n$. Let $\bar{x}_1 + \bar{x}_2$ be a zero-divisor in S/I. Then, there is a polynomial h in S such that $\bar{h}$ is nonzero in S/I and $\bar{h}(\bar{x}_1 + \bar{x}_2) = 0$ or equivalently, $f = h(x_1 + x_2) \in I$. Let $h = h_1 + h_2 + \cdots + h_r$ such that h_i's are monomials and $h_1 \succ h_2 \succ \cdots \succ h_r$. We may assume that $h_1 \notin I$. Now, $h_1 x_1$ is the greatest monomial of f with respect to the order $\prec$ which can not be canceled by other monomials. Therefore, $h_1 x_1 \in I$ and there is a quadratic monomial in the monomial generating set of I which divides $h_1 x_1$ and does not divide h_1. This monomial must be of the form $x_1 x_k$ for some k, $1 < k \leq n$. On the other hand, $x_1 \nmid h_1$ because $x_k | h_1$ and $h_1 \notin I$. Since the order is lexicographic, x_1 does not divide any other monomials in h. Therefore, in the polynomial $hx_1 + hx_2$, there is no monomial of the summand hx_2, which is divisible by x_1. In this summand, $h_1 x_2$ is the greatest monomial with respect to $\prec$ and can not be canceled by other monomials. This implies that $h_1 x_2 \in I$. Similar to the above argument, there is a quadratic monomial in the generating set of I which divides $h_1 x_2$ but not h_1. This monomial must be of the form $x_2 x_l$ for some $2 < l \leq n$. We also have $x_2 \nmid h_1$ but $x_k | h_1$ and $x_l | h_1$. If $k = l$, then, $x_k(x_1 + x_2) \in I$ and if $k \neq l$, then, $x_k x_l(x_1 + x_2) \in I$. Note that, $x_k x_l \notin I$ because $x_k x_l | h_1$ and $h_1 \notin I$. This completes the proof in one direction. The converse is trivial. $\square$

Proof of the Theorem 1. We proceed the proof in 3 steps. First we prove that a bipartite graph G is unmixed if and only if there is a perfect matching in G satisfying condition 1. Then, in Step 2, we prove that for an unmixed bipartite graph, condition 2 is necessary for Cohen-Macaulayness. In Step 3 we prove that, condition 2 is also sufficient for Cohen-Macaulayness of such a graph.

Step 1. Let G be unmixed. There is no isolated vertex and any vertex in V is adjacent to some vertices in W. Therefore, there is no vertex in V independent to the set W. This means that W is a maximal independent set in G. Similarly, V is a maximal independent set. By unmixedness of G, $|V| = |W|$. Let $A \subseteq V$ be a nonempty set. Suppose $|N(A)| < |A|$. There is no edge between A and $W \setminus N(A)$. Therefore, $A \cup (W \setminus N(A))$ is an independent set and its size is strictly greater than $|W|$, which is a contradiction. Therefore, $|N(A)| \geq |A|$ for each nonempty subset A of V. Therefore, by the marriage theorem of Hall [1], there is a perfect matching between V and W.

Now, let $V = \{x_1, \ldots, x_n\}$, $W = \{y_1, \ldots, y_n\}$ and $\{x_1, y_1\}, \ldots, \{x_n, y_n\}$ be a perfect matching in G. The graph G is unmixed and any maximal independent set of vertices in G has cardinality n. Therefore, any maximal independent set intersects each edge of the perfect matching in exactly one vertex. Suppose for some j, $1 \leq j \leq n$, the induced subgraph on $N(x_j) \cup N(y_j)$ is not a complete bipartite graph. Then, there are $x \in N(y_j)$ and $y \in N(x_j)$ such that $x \not\sim y$. The set $\{x, y\}$ is independent and so there is a maximal independent set containing it. But, this maximal independent set does not meet the edge $\{x_j, y_j\}$ which is a contradiction. Therefore, condition 1 holds.

Conversely, assume that there is a perfect matching $\{x_1, y_1\}, \ldots, \{x_n, y_n\}$ in G which satisfies condition 1. Let A be a maximal independent set in G. Then A meets each edge in the perfect matching in at most one vertex. Suppose that for some j, $1 \leq j \leq n$, $A \cap \{x_j, y_j\} = \varnothing$. Then, neither x_j nor y_j is independent to A, and there are $x, y \in A$ such that $x \sim y_j$ and $y \sim x_j$. But, x and y are not adjacent and the induced subgraph on $N(x_j) \cup N(y_j)$ is not a complete bipartite graph, which is a contradiction. Therefore, A meets any edge in the perfect matching and has cardinality n. It means that G is unmixed.

Step 2. Let G be a bipartite graph with a perfect matching which satisfies condition 1 but condition 2 fails. That is, for some i and j, $1 \leq i < j \leq n$, we have $x_i \sim y_j$ and $x_j \sim y_i$. Then, in the quotient ring $R(G)/(x_i + y_i)$, the element $\bar{x}_i$ is nonzero and $\bar{x}_i(\bar{x}_j + \bar{y}_j) = 0$ because $\bar{x}_i = -\bar{y}_i$. Therefore, $\bar{x}_j + \bar{y}_j$ is a zero-divisor in $R(G)/(x_i + y_i)$. This means that the sequence $\bar{x}_1 + \bar{y}_1, \ldots, \bar{x}_n + \bar{y}_n$ is not a regular sequence in $R(G)$ and hence by Lemma 2, $R(G)$ is not Cohen-Macaulay.

Step 3. Let G be a bipartite graph with a perfect matching satisfying condition 1. In this case, $\dim(R(G)) = n$ and to prove that $R(G)$ is Cohen-Macaulay, it is enough to show that the sequence $\bar{x}_1 + \bar{y}_1, \ldots, \bar{x}_n + \bar{y}_n$ is a regular sequence in $R(G)$ (Lemma 2). For an integer i, $1 \leq i < n$, the ring $R(G)/(x_1 + y_1, \ldots, x_{i-1} + y_{i-1})$ can be viewed as the ring $R'(G)$ obtained by $R(G)$ identifying x_j with $-y_j$ for $j = 1, \ldots, i-1$. By the Lemma 3 and its proof, the only possibility for $\bar{x}_i + \bar{y}_i$ to be zero-divisor in $R'(G)$ is that there is j, $1 \leq j \leq i-1$, such that $\bar{x}_j(\bar{x}_i + \bar{y}_i) = 0$. Therefore, $\bar{x}_j\bar{y}_i = 0$ and $\bar{x}_j\bar{x}_i = 0$ or equivalently, $\bar{y}_j\bar{x}_i = 0$. Therefore, $x_j \sim y_i$ and $y_j \sim x_i$. But, in this case, condition 2 fails. This completes the proof. □

Proposition 4 *Condition 1 in Theorem 1 which is equivalent to unmixedness of a bipartite graph is also equivalent to say that non of the polynomials* $x_1+y_1, \ldots, x_n+y_n$ *is a zero-divisor in* $R(G)$.

Proof. The assertion is clear by the Lemma 3 and the Theorem 1. □

Remark 5 *Condition 2 in Theorem 1 is equivalent to say that, for each* i *and* j, $1 \leq i < j \leq n$, *the induced subgraph on vertices* $\{x_i, y_i, x_j, y_j\}$ *has connected complement.*

Corollary 6 *Let* G *be a bipartite Cohen-Macaulay graph and let* $\{x_i, y_i\}$ *be any edge in the perfect matching mentioned in Theorem 1. Then,* $G \setminus \{x_i, y_i\}$ *is again Cohen-Macaulay.*

Proof. Here, by $G \setminus \{x_i, y_i\}$ we mean the graph obtained by deleting vertices x_i and y_i and all edges passing through one of these vertices. It is clear that if condition 1 or 2 in Theorem 1 holds for G, then, it holds for $G \setminus \{x_i, y_i\}$ for each $i = 1, \ldots, n$. □

Proposition 7 *Let* G *be a bipartite Cohen-Macaulay graph with parts* V *and* W*. Then, there is at least one vertex of degree one in each part.*

Proof. Let y be a vertex in W such that for any other vertex $y' \in W$, we have $\deg(y') \leq \deg(y)$. Let $x \in V$ be the vertex such that $\{x, y\}$ is in a perfect matching in G. If $\deg(x) > 1$, then there is a vertex $y' \in W \setminus \{y\}$ such that $x \sim y'$. Let x' be a vertex in $V \setminus \{x\}$ such that $\{x', y'\}$ is in the perfect matching. Since G is Cohen-Macaulay, the induced subgraph on $N(x) \cup N(y)$ is a complete bipartite graph and $x' \notin N(y)$. Then, y' is adjacent to each vertex in $N(y) \cup \{x'\}$. This means that $\deg(y') > \deg(y)$, which is a contradiction. Therefore, $\deg(x) = 1$. □

Let G be a Cohen-Macaulay bipartite graph. There are some vertices in both parts with degree one. If we remove the vertex adjacent to a vertex of degree one, the edge consisting of these two vertices in a perfect matching will be removed and the remaining graph is also Cohen-Macaulay.

Corollary 8 *Let G be a Cohen-Macaulay bipartite graph. Then, there is a unique perfect matching in G.*

Proof. Let V and W be two parts of G. By the Theorem 1, there is a perfect matching. Let P be a perfect matching in G. By the above proposition, there is a vertex x_1 of degree one in V. Let $y_1 \in W$ be the unique vertex adjacent to x_1. Then $\{x_1, y_1\} \in P$. The graph $G \setminus \{x_1, y_1\}$ is again Cohen-Macaulay and $V \setminus \{x_1\}$ has a vertex of degree one as x_2. Let $y_2 \in W \setminus \{y_1\}$ be the unique vertex adjacent to x_2. Then, $\{x_2, y_2\} \in P$. Continuing this process, P will be uniquely determined. □

Corollary 9 *Let G be an unmixed bipartite graph. Then, the following conditions are equivalent.*

i) G is Cohen-Macaulay.

ii) There is a unique perfect matching in G.

iii) For each two edges e_1, e_2 in a perfect matching, the complement of the induced subgraph on vertices of e_1 and e_2 is connected.

iv) For a perfect matching P in G, there is an order on edges of P such that $P = \{\{x_1, y_1\}, \ldots, \{x_n, y_n\}\}$ and $x_i \sim y_j$ implies $i \leq j$.

Proof. (i→ii) This is proved in Corollary 8. Let G be unmixed but not Cohen-Macaulay. Then, there is a perfect matching and two edges in the perfect matching as $\{x_i, y_i\}$ and $\{x_j, y_j\}$ such that $x_i \sim y_j$ and $x_j \sim y_i$. Substituting $\{x_i, y_i\}$ and $\{x_j, y_j\}$ by $\{x_i, y_j\}$ and $\{x_j, y_i\}$, we get a different perfect matching. This proves (ii→i). The equivalence of (i) and (iii) is clear by Theorem 1 and Remark 5. To prove (i→iv), let $x_1, \ldots, x_n$ be vertices of V such that $\deg(x_i) \geq \deg(x_{i+1})$ for each $i = 1, \ldots, n-1$. Then, $\deg(y_1) = 1$. Remove $\{x_1, y_1\}$. In the remaining graph, $\deg(y_2) = 1$ and continuing this process shows that there is no any edge between x_i and y_j if $i > j$. Finally, condition (iv) clearly implies condition (iii). □

Now for a given bipartite graph G, we present a fast polynomial-time algorithm to check whether G is Cohen-Macaulay.

Algorithm 10 *Let G be a given bipartite graph with parts V and W, $|V| = |W| = n$.*

Step 1. *Take $i = 0$.*

Step 2. *If there is no vertex with degree 1 in V, go to Step 6.*

Step 3. *Set $i = i + 1$. Choose a vertex of degree one in V and name it x_i. Name the vertex in W adjacent to x_i to be y_i. Take $V = V \setminus \{x_i\}, W = W \setminus \{y_i\}$. If $i < n$, go to Step 2.*

Step 4. *If there is j, $2 \leq j \leq n$ such that, a vertex in $N(x_j)$ and a vertex in $N(y_j)$ are not adjacent, then, go to Step 6.*

Step 5. *Write "G is Cohen-Macaulay" and end the algorithm.*

Step 6. *Write "G is not Cohen-Macaulay" and end the algorithm.*

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