# Properties of Chip-firing Games on Complete Graphs*

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**Abstract:** Björner, Lovász and Shor introduced a chip-firing game on a finite graph $G$ as follows. We put some chips on each vertex of $G$, we say that a vertex is ready if it has at least as many chips as its degree, in which case we can fire it and the result is that it distributes one chip to each of its neighbors, this may cause other vertices to be ready, and so on. This game continues until no vertex can be fired. In this paper, we study chip-firing games on complete graphs. We obtain a sufficient and necessary condition for chip-firing games on complete graphs to be finite.

**Keywords**: Chip-firing games; Complete graphs

## 1 Introduction

All graphs considered in this paper are simple, finite and undirected. For notations and terminologies not defined here, we follow West [15].

In 1986, Spencer [13] introduced a "balancing game" in an infinite undirected path, which was generalized by Björner, Lovász and Shor [4] to a graph as follows.

We obtain an *initial configuration* by putting some chips on each vertex of a graph $G$, the sum of chips on all vertices is $N$. We say that a vertex is *ready* if it has at least as many chips as its degree, in which case the vertex distributes one chip to each of its neighbors, this process is called a *firing*. This may cause other vertices to be ready, and so on. This game terminates if each vertex has fewer chips than its degree. The game introduced above is called *chip-firing game*.

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Chip-firing game has been around for no more than 20 years, but it has rapidly become an important and interesting object of study in structural combinatorics. The reason for this is partly due to its relation with the Tutte polynomial and group theory, but also because of the contribution of people in theoretical physics who know it as the Abelian sandpile model.

For the partial studies on this topic, see for examples: In [4, 3], some important properties of chip-firing games on graphs and digraphs were studied. In [10], Jeffs and seager described infinite configurations on an $n$-cycle with $n$ chips. In [2], Biggs showed that the set of configurations that are stable and recurrent for a game can be given the structure of an Abelian group, and that the order of the group is equal to the tree number of the graph. In [11], López showed that the generating function of critical configurations of a version of a chip-firing game on a graph $G$ is an evaluation of the Tutte polynomial of $G$. And Heuvel proved that the number of steps needed to reach a critical configuration is polynomial in the number of edges of the graph and the number of chips in the initial configuration in [9]. In particular, chip-firing game is closely related to the Abelian sandpile model introduced in [7], and a detailed argument for sandpile model can be found in [6]. From [12], we can know that for a Abelian sandpile model, if the toppling matrix $\Delta_{ij}$ is symmetric and the loading of the system at site $i$ equals the dissipation at $i$, then the Abelian sandpile model coincides with the parallel chip-firing game on a graph, in which all ready vertices must fire simultaneously. For other related topics, we suggest readers to refer to [1, 5, 8].

In this paper, we study the finiteness of chip-firing games on complete graphs. We obtain a sufficient and necessary condition for chip-firing games to be finite.

# 2 Some properties of chip-firing games on complete graphs

Let $G$ be a connected graph with vertex set $\{v_1, \cdots, v_n\}$, and put $a_i$ chips on vertex $v_i$, $i = 1, 2, \cdots, n$. Denote the configuration of the chip-firing game at this moment by a vector $\alpha = (a_1, \cdots, a_n) \in \mathbb{N}^n, \sum_i a_i = N$. Assume a vertex $v_i$ is ready and we fire it, this means decreasing $a_i$ by the degree $d(v_i)$ of vertex $v_i$, and increasing $a_j$ by 1 for each neighbor $v_j$ of $v_i$. We call a sequence of configurations $(\alpha_0, \alpha_1, \cdots, \alpha_k, \cdots)$ a *fired sequence* if the configurations $\alpha_k$ is obtained from $\alpha_{k-1}$ by firing a ready vertex, says $v_{\alpha_{k-1}}$. A fired sequence corresponds to a *fired vertex sequence*. Clearly, a chip-firing game with an initial distribution may have many distinct fired sequences and each of them is called a *legal game*. Bjöner et al [4] reported the following results:

**Theorem 2.1** ([4]). *Given a connected graph and an initial distribution of chips, either every legal game can be continued infinitely, or every legal game terminates after the same number of moves with the same final configuration. The number of times a given vertex is fired is the same in every legal game.*

**Theorem 2.2** ([4]). *Let $G$ be a graph with $n$ vertices, $m$ edges, $N$ chips. We have:*

*(a) if $N > 2m - n$, then the game is infinite.*

*(b) if $m \leq N \leq 2m - n$, then there exists an initial configuration guaranteeing finite termination and also one guaranteeing infinite game.*

*(c) if $N < m$, then the game is finite.*

**Theorem 2.3** ([4]). *If a chip-firing game is infinite, then every vertex is fired infinitely often.*

Furthermore, Tardos [14] proved the following theorem.

**Theorem 2.4** ([14]). *If a chip-firing game terminates, then there is a vertex which is not fired at all.*

From Theorem 2.3 and Theorem 2.4, it is easy to see that:

**Corollary 2.1.** *A chip-firing game on a simple finite connected graph is finite if and only if there is a vertex which is not fired at all.*

By Theorem 2.1, if the initial configuration of a chip-firing game is determined, then the finiteness of the game is also determined. If a chip-firing game with initial configuration $\alpha$ is finite, we say that $\alpha$ is a *finite configuration* (or simply, $\alpha$ is *finite*), and *infinite configuration* (or *infinite*) otherwise. Furthermore, $c_\alpha(v)$ denotes the number of chips on vertex $v$ in configuration $\alpha$.

From now on, we assume the chip-firing games mentioned below are based on complete graph $K_n$. By Theorem 2.2, we know that chip-firing games on $K_n$ are finite when $N < \binom{n}{2}$, and are infinite when $N > 2\binom{n}{2} - n$. Thus, it is sufficient to consider the finiteness of chip-firing games when $\binom{n}{2} \leq N \leq 2\binom{n}{2} - n$. Assume that $N = \binom{n}{2}$. For the configuration $\alpha = (0, 1, 2, \cdots, n-1)$, we can see that there always is a vertex having $n-1$ chips after any number of firings, then the configuration $\alpha = (0, 1, 2, \cdots, n-1)$ is infinite. Two configurations $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ of $K_n$ are said to be *equivalent*, denoted by $(a_1, \cdots, a_n) \cong (b_1, \cdots, b_n)$, if $\{a_1, \cdots, a_n\} = \{b_1, \cdots, b_n\}$. Clearly, the equivalent configurations has the same finiteness. Furthermore, if a configuration $\alpha'$ can reach an equivalent configuration of $(0, 1, 2, \cdots, n-1)$ by firing a sequence of vertices, then $\alpha'$ is infinite.

**Theorem 2.5.** *Let $\alpha = (0, 1, 2, \cdots, n-1)$ be a configuration on $K_n$. If a configuration $\beta$ can reach an equivalent configuration $\alpha'$ of $\alpha$ by firing a sequence of vertices, then $\beta$ is an equivalent configuration of $\alpha$.*

*Proof.* Assume $\beta$ reaches $\alpha'$ along a fired sequence $(\beta, \alpha_1, \cdots, \alpha_{k-1}, \alpha_k, \alpha')$. Without loss of generality we assume that $\alpha_k$ reaches $\alpha'$ by firing a vertex $v_1$. Then each of other vertices of $K_n$ obtains a chip by firing $v_1$. Thus $c_{\alpha'}(v_i) > 0$ for $i = 2, 3, \cdots, n$. Since

$\alpha'$ is equivalent configuration of $\alpha$, we have $c_{\alpha'}(v_1) = 0$, and $c_{\alpha_k}(v_1) = n-1$. Note that $c_{\alpha_k}(v_i) = c_{\alpha'}(v_i) - 1$ for $i = 2, \cdots, n$. Thus $\alpha_k \cong (0, 1, 2, \cdots, n-1)$. Similarly, $\alpha_{k-1}, \cdots, \alpha_1, \beta$ are all equivalent configurations of $\alpha$. $\square$

**Theorem 2.6.** *Given a chip-firing game on $K_n$ with initial configuration $\alpha$ and $N = \binom{n}{2}$. Then $\alpha$ is infinite if and only if $\alpha \cong (0, 1, 2, \cdots, n-1)$.*

*Proof.* The sufficiency is true clearly. We next prove the necessity.

Let $\alpha_0 = (c_{\alpha_0}(v_1), \cdots, c_{\alpha_0}(v_n)) = (c_1, \cdots, c_n)$ be any infinite configuration on $K_n$ with $\sum\limits_{k=1}^{n} c_k = \binom{n}{2}$ such that the vertex $v_i$ has $c_i$ chips. Without loss of generality, we assume $c_1 \leq \cdots \leq c_n$. Since $\alpha_0$ is infinite and $v_n$ possesses the largest number of chips in $\alpha_0$, $c_n \geq n-1$. We play the chip-firing game according to the following rules:

$(O_1)$ We fire the vertex with the maximum number of chips in each configuration (this is allowed because of the Abelian property of the chip-firing game).

$(O_2)$ If more than one vertex have maximum number of chips in the same configuration, then we fire the one with the maximum subscript.

**Claim 1.** $c_1 = 0$.

Suppose $c_1 > 0$. Since $\alpha_0$ is infinite, there exists a fired sequence $(\alpha_0, \alpha_1, \cdots, \alpha_k, \cdots)$ and a fired vertex sequence $(v_{i_1}, v_{i_2}, \cdots, v_{i_k}, \cdots)$ such that $v_{i_k} = v_1$ and $v_{i_j} \neq v_1$ for $j < k$. Note that the rules $O_1$ and $O_2$, and $c_1 \leq c_2 \leq \cdots \leq c_n$. We have $k \geq n$ and each of the vertices $v_2, \cdots, v_n$ has been fired at least once when $v_1$ is fired. Now, we consider a new game with initial configuration $\alpha'_0$ obtained from $\alpha_0$ by removing a chip on $v_1$. Clearly, we can fire the vertices $v_{i_1}, \cdots, v_{i_{k-1}}$ along the sequence $(v_{i_1}, v_{i_2}, \cdots, v_{i_k}, \cdots)$, and let the corresponding fired sequence is $(\alpha'_0, \alpha'_1, \cdots, \alpha'_{k-1})$. Note that $k-1 \geq n-1$, $v_1$ has at least $n-1$ chips in $\alpha'_{k-1}$, and then $v_1$ is ready. Consequently, each vertex can be fired at least once in this new game. Thus $\alpha'_0$ is infinite by Theorem 2.4. But, the sum of chips in $\alpha'_0$ is $\binom{n}{2} - 1$. So $\alpha'_0$ is finite by Theorem 2.2, a contradiction. The proof of Claim 1 is completed.

According to $O_1$ and $O_2$, we have $v_{i_1} = v_n$. Since $\alpha_1$ is also infinite, $\min\{c_{\alpha_1}(v_i) | i = 1, 2, \cdots, n\} = 0$ by Claim 1. Note that in $\alpha_1$, each $v_h (1 \leq h \leq n-1)$ gets a chip from $v_n$ when $v_n$ is fired. Thus $c_{\alpha_1}(v_n) = 0$ and $c_{\alpha_0}(v_n) = n-1$. Similarly, we have $v_{i_2} = v_{n-1}$, $c_{\alpha_2}(v_{n-1}) = 0$, $c_{\alpha_0}(v_{n-1}) = n-2; \cdots; v_{i_{n-1}} = v_2$, $c_{\alpha_{n-1}}(v_2) = 0$, $c_{\alpha_0}(v_2) = 1$. Thus $\alpha_0 = (0, 1, \cdots, n-1)$. $\square$

**Theorem 2.7.** *Let $\alpha$ be an initial configuration of $K_n$ such that $\alpha \cong \beta = (c_\beta(v_1), c_\beta(v_2), \cdots, c_\beta(v_n)) = (c_1, c_2, \cdots, c_n)$ with $x = c_1 = c_2 = \cdots = c_k \leq c_{k+1} \leq \cdots \leq c_n$. Then $\alpha$ is finite if $N < \binom{n}{2} + \max\{\binom{k}{2}, \binom{x+1}{2}\}$.*

*Proof.* In the following discussion, we play each chip-firing game according to the rules $O_1$ and $O_2$.

If $0 \leq x \leq k-1$, then $N < \binom{n}{2} + \binom{k}{2}$. Suppose $\beta$ is infinite. Let $\beta_1$ be a configuration obtained by firing $v_n, \cdots, v_{k+1}$ continuously such that $\beta_1 = (n-1, \cdots, n-1, c'_{k+1}, \cdots, c'_n)$.

As $\beta_1$ is also infinite, each vertex can be fired in a chip-firing game with initial configuration $\beta_1$ by Theorem 2.3. Assume the fired vertex sequence of above game is $\mathscr{A}$. We consider a new configuration $\beta'_1 = (n-k, n-k+1, \cdots, n-1, c'_{k+1}, \cdots, c'_n)$ and play a new game with initial configuration $\beta'_1$ along the fired vertex sequence $\mathscr{A}$. Clearly, each of vertices $v_1, \cdots, v_n$ can be fired in the new game. Thus, $\beta'_1$ is infinite by Theorem 2.4. But, the sum of chips is $N - \binom{k}{2} < \binom{n}{2}$ in $\beta'_1$. Thus $\beta'_1$ is finite by Theorem 2.2, a contradiction.

If $k \leq x < n-1$, then $N < \binom{n}{2} + \binom{x+1}{2}$. Assume $\beta$ is infinite. Now we consider a new configuration $\beta' = (0, 1, 2, \cdots, x, c_{x+2}, \cdots, c_n)$. Let the sum of chips in $\beta'$ is $N'$. Then, $N - N' = \sum_{i=0}^{x}(c_{i+1} - i) \geq \sum_{i=0}^{x}(x - i) = \binom{x+1}{2}$. Note that $N < \binom{n}{2} + \binom{x+1}{2}$. Thus, $N' \leq N - \binom{x+1}{2} < \binom{n}{2} + \binom{x+1}{2} - \binom{x+1}{2} = \binom{n}{2}$. And so $\beta'$ is finite by Theorem 2.2. As $\beta$ is infinite and $c_1 = c_2 = \cdots = c_k \leq \cdots \leq c_{x+1} \leq \cdots \leq c_n$, there exists a fired vertex sequence $\mathscr{B} = (u_1, u_2, \cdots, u_{n-x-1})$, where $u_i \in \{v_{x+2}, v_{x+3}, \cdots, v_n\}$ for $i = 1, 2, \cdots, n-x-1$. Then a chip-firing game with initial configuration $\beta'$ can reach a configuration $\beta'' = (n-x-1, n-x, \cdots, n-1, c'_{x+2}, \cdots, c'_n)$ along the fired vertex sequence $\mathscr{B}$. Clearly, $v_{x+1}, \cdots, v_1$ can also be fired one by one in this game. In $\beta''$, for any vertex $u \in \{v_{x+2}, v_{x+3}, \cdots, v_n\}$ which has not been fired, it is easy to see that $c_{\beta''}(u) \geq n-1$. By the Theorem 2.4, $\beta'$ is infinite. A contradiction.

Therefore, $\beta$ and $\alpha$ are finite. $\square$

# 3 A sufficient and necessary condition

In this section, we will give a sufficient and necessary condition for chip-firing games on complete graphs to be finite.

**Lemma 3.1.** *Let $\alpha$ be an initial configuration of $K_n$ with $N$ chips, $\binom{n}{2} \leq N \leq 2\binom{n}{2} - n$. Then by firing some sequence of vertices starting at $\alpha$, we can reach a configuration $\beta$ such that $c_\beta(v) \leq 2n-3$ for each $v \in V(K_n)$.*

*Proof.* We consider a configuration $\alpha'$ on $K_n$: For each $v \in V(K_n)$, let $c'_\alpha(v) = k$ if $c_\alpha(v) = 2k$ or $2k+1, k \in \mathbb{N}$. Clearly, the sum of chips in $\alpha'$ is less than $\binom{n}{2}$. By Theorem 2.2, $\alpha'$ is finite. Assume $(v'_1, v'_2, \cdots, v'_t)$ is a fired vertex sequence such that the chip-firing game with initial configuration $\alpha'$ terminates. Now, we play a new chip-firing game with initial configuration $\alpha$ along the fired vertex sequence $(v'_1, v'_1, v'_2, v'_2, \cdots, v'_t, v'_t)$. Let the corresponding fired sequence be $(\alpha, \cdots, \beta)$. We have that $\beta$ satisfies the condition of lemma. $\square$

Take a configuration $\alpha = (c(v_1), \cdots, c(v_n)) = (c_1, \cdots, c_n)$ of $K_n$ such that $c_1 \geq \cdots \geq c_n$. If there are $v_k, v_{k+1}$ such that $c_k - c_{k+1} \geq 2$, we call $(v_k, v_{k+1})$ a *faultage.*

Take a chip-firing game on $K_n$ with $\binom{n}{2} \leq N \leq 2\binom{n}{2} - n$. It can always reach a configuration $\beta$ depicted in Lemma 3.1: $c_\beta(v) \leq 2n-3$ for each $v \in V(K_n)$. Without loss of generality, we assume that $c_\beta(v_1) \geq c_\beta(v_2) \geq \cdots \geq c_\beta(v_n)$ and $\beta$ contains $k-1$ faultages. We now divide $v_1, v_2, \cdots, v_n$ into $k$ ordered parts by above $k-1$ faultages. Denote the $h^{\text{th}}$ part by $\rho_h$. Then we have that the vertices in $\rho_h$ are ordered in decreasing number of chips. Denote the first vertex of $\rho_h$ (the one with the smallest index in $\rho_h$) by $v'_h$, $1 \leq h \leq k$. Clearly, $v'_h$ possesses the largest number of chips in $\rho_h$. We have that there is no faultage in each part. Let $c_\beta(v'_h) = c^*_h$. Denote the number of vertices of the $h^{\text{th}}$ part by $s_h$. Now, we have:

**Property 3.1.** *In a chip-firing game, if there exists a vertex $u$ which can be fired at least three times, then the game is infinite. In fact, let $u \in \rho_h$. Then, the first vertex $v'_h$ of $\rho_h$ can also be fired at least three times. Note that $c^*_k \leq c^*_h \leq 2n-3+c^*_k$. Furthermore, if $v'_h$ fires three times, it must have received at least $n - c^*_k$ chips before its third firing. Thus $v'_k$ receives at least $n - c^*_k$ chips as well, so it fires at least once. Thus all vertices will have been fired, so by Theorem 2.4, our claim holds.*

**Property 3.2.** *Let $u_1, u_2$ be two vertices with $c_\beta(u_1) - c_\beta(u_2) = p$. It is not difficult to see that if we have a fired sequence $(\beta, \beta_1, \cdots, \beta_t)$ such that $u_1, u_2$ were fired the same number of times, then $c_{\beta_t}(u_1) - c_{\beta_t}(u_2) = p$.*

We call *firing a part $\rho_i$* if the vertices of the part $\rho_i$ are fired in succession. In the following theorem, we will complete the characterization of the finiteness of chip-firing games on complete graphs. Combining Lemma 3.1 with the above argument, we only need to consider such chip-firing games: with initial configuration $\beta = (c(v_1), c(v_2), \cdots, c(v_n))$, $\binom{n}{2} \leq N \leq 2\binom{n}{2} - n$, $c_\beta(v_1) \geq \cdots \geq c_\beta(v_n)$ and $\beta$ contains $k-1$ faultages. For each $v \in V(K_n)$, $c_\beta(v) \leq 2n-3$. Suppose the number of vertices of the $h^{\text{th}}$ part is $s_h$, and the first vertex of the $h^{\text{th}}$ part is $v'_h$, $c_\beta(v'_h) = c^*_h, 1 \leq h \leq k$. We have:

**Theorem 3.1.** *A chip-firing game with initial configuration $\beta$ is finite if and only if there exists two integers $i, j \in \{1, 2, \cdots, k\}, j \leq i \leq k$ such that $c^*_i + \sum_{b=1}^{i-1} s_b + \sum_{c=1}^{j-1} s_c < n-1$, $c^*_j + \sum_{b=1}^{i-1} s_b + \sum_{c=1}^{j-1} s_c - n < n-1$.*

*Proof.* First, $\beta$ contains $k-1$ faultages, we denote the $h^{\text{th}}$ part by $\rho_h, 1 \leq h \leq k$. We play the chip-firing game according to the rules $O_1$ and $O_2$.

Now, we assume that the game is finite. By Property 3.1, each of the vertices on the game is fired at most twice. We define $H_1, H_2$ as follows: Let $H_1$ (resp. $H_2$) be the set of the parts whose vertices were fired at least once (resp. twice). Assume $H_1 = \{\rho_1, \rho_2, \cdots, \rho_{i-1}\}, H_2 = \{\rho_1, \rho_2, \cdots, \rho_{j-1}\}$. Obviously, $j \leq i$. As $\beta$ is finite, $i - 1 < k$ by Theorem 2.4. Note that the game terminates, then $v'_i$ can not satisfy the condition of firing and the vertices of $\rho_j$ can not be fired twice. That is, $c_i^* + \sum_{b=1}^{i-1} s_b + \sum_{c=1}^{j-1} s_c < n-1$ and $c_j^* + \sum_{b=1}^{i-1} s_b + \sum_{c=1}^{j-1} s_c - n < n-1$.

Conversely, let $i, j \in \{1, 2, \cdots, k\}$ be two minimum integers such that $j \le i \le k$, $c_i^* + \sum_{b=1}^{i-1} s_b + \sum_{c=1}^{j-1} s_c < n-1$, $c_j^* + \sum_{b=1}^{i-1} s_b + \sum_{c=1}^{j-1} s_c - n < n-1$. Then we have that $v_i'$ can not satisfy the condition of firing and $v_j'$ can not be fired twice. If $j = 1$, the chip-firing game clearly terminates and $\beta$ is finite. If not, since $c_i^* + \sum_{b=1}^{i-1} s_b + \sum_{c=1}^{j-1} s_c < n-1$, we have $c_k^* + \sum_{b=1}^{i-1} s_b + \sum_{c=1}^{j-1} s_c < n-1$. It follows from $0 \le c_1^* - c_k^* \le 2n-3$ that $c_1^* + \sum_{b=1}^{i-1} s_b + \sum_{c=1}^{j-1} s_c - 2n < n-1$ and then $v_1'$ can not satisfy the condition of firing at this moment. Note that $v_1', v_i', v_j'$ can not be fired at this moment, thus no vertex can be fired. Therefore, the chip-firing game terminates. That is, $\beta$ is finite. $\square$

# 4 Conclusions

For a chip-firing game on a graph, a natural question is to consider its finiteness. But it is difficult to find exactly the boundary between infinite and finite games, so we hope to make progress for some special graphs. In [10], Jeffs and seager accurately described infinite configurations on an $n$-cycle with $n$ chips. And in this paper, we provide a necessary and sufficient description of whether a chip-firing game is infinite on the complete graph $K_n$, and find two other results: one upper bound on the total number of chips for which the chip-firing game is finite, and another necessary and sufficient condition for which a game with $N = \binom{n}{2}$ chips is infinite. Although the two papers as above depend heavily on the symmetry of the cycle or complete graph, but which are new and significative attempt in this area. In the next step, we can try to do some work in other special graphs, such as wheel, complete bipartite graph, complete k-partite graph, and so on. We hope to find a broader characterization of what exactly makes a chip-firing game infinite.

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