FREE AND COFREE ACTS OF DCPO-MONOIDS ON DIRECTED COMPLETE POSETS

MOJGAN MAHMOUDI AND HALIME MOGHBELI

Dedicated to Professor M. Mehdi Ebrahimi on the occasion of his 65th Birthday

Abstract. In this paper, we study the existence of the free and cofree objects in the categories **Dcpo-**S (and **Cpo-**S) of all directed complete posets (with bottom element) equipped with a compatible right action of a dcpo-monoid (cpo-monoid) S, with (strict) continuous action-preserving maps between them. More precisely, we consider all forgetful functors between these categories and the categories **Dcpo** of dcpo's, (**CPo**) of cpo's, **Pos** of posets, and **Set** of sets, and study the existence of their left and right adjoints.

1. Introduction and Preliminaries

The category **Dcpo** of Directed Complete Partial Ordered sets plays an important role in Theoretical Computer Science, and specially in Domain Theory (see [1]). It has been proved that this category is complete and cocomplete (see [1, 7]). The free dcpo over a poset has been given in [4, 9, 12].

In this paper, we consider the free and cofree objects in the category **Dcpo-**S of all S-dcpo's; dcpo's equipped with a compatible right action of a dcpo-monoid S, with continuous action-preserving maps between them. We take the forgetful functors from this category to the categories of dcpo's, posets, and sets, and study the existence of their left and right adjoints. In fact, we consider the following three

2010 *Mathematics Subject Classification.* 06F05, 18A40, 20M30, 20M50.

Key words and phrases. Directed complete partially ordered set, free, cofree, monoid actions.

squares of forgetful functors

$$
\begin{array}{ccc}
\mathbf{Cpo}\text{-}S & \xrightarrow{U_1} & \mathbf{Cpo} \\
\downarrow{\scriptstyle U_3} & & \downarrow{\scriptstyle U_2} \\
\mathbf{Dcpo}\text{-}S & \xrightarrow{U_4} & \mathbf{Dcpo} \\
\downarrow{\scriptstyle U_6} & & \downarrow{\scriptstyle U_5} \\
\mathbf{Pos}\text{-}S & \xrightarrow{U_7} & \mathbf{Pos} \\
\downarrow{\scriptstyle U_9} & & \downarrow{\scriptstyle U_8} \\
\mathbf{Act}\text{-}S & \xrightarrow{U_{10}} & \mathbf{Set}
\end{array}
$$

and study the existence of the left and the right adjoints for these functors (that is, U_i-free and U_i-cofree objects). We recall that the bottom square has been considered in [3], where it has been shown that the horizontal forgetful functors, U_7 and U_{10}, have both left and right adjoints, while the vertical forgetful functors, U_8 and U_9, have just left adjoints (here we give a correction to the definition of the U_9-free functor given in [3]). Also, the left adjoint to the right vertical forgetful functor, U_5, in the middle square has been found in [4, 9, 12].

Here, we show that, although all the forgetful functors in these squares have left adjoints (for the existence of U_1-free, U_3-free, and U_6-free, we had to add a condition on S), none of the vertical forgetful functors in all the above three squares has a right adjoint. In finding the left adjoints (free objects), we observe that the definition of the left hand side vertical free functors is the same as the right hand side ones, and just we need to define a proper S-action. Also we prove that all the horizontal functors, except U_1, have right adjoints (cofree objects). In finding these right adjoints, we observe that the cofree horizontal functors (when existing) are in some sense the restrictions of the bottom horizontal cofree functors. The same is true for the free horizontal functors.

In the following, we give some preliminaries needed in the sequel. For more information about dcpo's we refer to [8], about S-sets see [6, 10], about S-posets see [3], and for S-dcpo's refer to [11].

Let **Pos** denote the category of all partially ordered sets (posets) with order-preserving (monotone) maps between them. A nonempty subset D of a partially ordered set is called *directed*, denoted by $D \subseteq^d P$, if for every $a, b \in D$ there exists $c \in D$ such that $a, b \leq c$; and P is called *directed complete*, or briefly a *dcpo*, if for every $D \subseteq^d P$, the directed join $\bigvee^d D$ exists in P. A dcpo which has a bottom element $\perp$ is said to be a *cpo*.

A *dcpo map* or a *continuous map* $f : P \to Q$ between dcpo's is a map with the property that for every $D \subseteq^d P$, $f(D)$ is a directed subset of Q and $f(\bigvee^d D) = \bigvee^d f(D)$. A dcpo map $f : P \to Q$ between cpo's is called *strict* if $f(\perp) = \perp$. Thus we have the categories **Dcpo** (and **Cpo**) of all dcpo's (cpo's) with (strict) continuous maps between them.

We repeatedly apply the following lemmas in this paper.

Lemma 1.1. [1, 5] *Let $\{A_i : i \in I\}$ be a family of dcpo's. Then the directed join of a directed subset $D \subseteq^d \prod_{i\in I} A_i$ is calculated as $\bigvee^d D = (\bigvee^d D_i)_{i\in I}$ where*

$$D_i = \{a \in A_i : \exists d = (d_k)_{k\in I} \in D, a = d_i\}$$

for all $i \in I$.

Lemma 1.2. [8] *Let P, Q, and R be dcpo's, and $f : P \times Q \to R$ be a function of two variables. Then f is continuous if and only if f is continuous in each variable; which means that for all $a \in P$, $b \in Q$, $f_a : Q \to R$ $(b \mapsto f(a, b))$ and $f_b : P \to R$ $(a \mapsto f(a, b))$ are continuous.*

Remark 1.3. Recall that for a poset P, a nonempty subset I is called an ideal if I is an (up-)directed down subset of P, and the collection of all ideals of P is usually denoted by $\mathrm{Id}(P)$. It has been stated in [4] that for a poset P, $\mathrm{Id}(P)$ is the free dcpo over P. Notice that $\mathrm{Id}(P)$ is a dcpo in which the supremum of every directed subset is given by union, and the down map $\downarrow: P \to \mathrm{Id}(P)$ is the universal monotone map of the free object.

We consider the cofree dcpo on a poset in Section 2.

Recall that a *po-monoid* is a monoid with a partial order $\leq$ which is compatible with the monoid operation: for $s, t, s', t' \in S$, $s \leq t$, $s' \leq t'$ imply $ss' \leq tt'$. Similarly, a *dcpo* (*cpo*)*-monoid* is a monoid which is also a dcpo (cpo) whose binary operation is a (strict) continuous map.

Recall that a (*right*) *S-act* or *S-set* for a monoid S is a set A equipped with an *action* $A \times S \to A$, $(a, s) \mapsto as$, such that $a1 = a$ and $a(st) = (as)t$, for all $a \in A$ and $s, t \in S$. Let **Act**-S denote the category of all S-acts with action-preserving maps (maps $f : A \to B$ with $f(as) = f(a)s$, for all $a \in A$, $s \in S$).

Also, for a po-monoid S, a (*right*) *S-poset* is a poset A which is also an S-act whose action $\lambda : A \times S \to A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order. The category of all S-posets with action-preserving monotone maps between them is denoted by **Pos**-S.

Finally, recall that for a dcpo (cpo)-monoid S, a (right) *S-dcpo* (*S-cpo*) is a dcpo (cpo) A which is also an S-act whose action $\lambda : A \times S \to A$ is a (strict) continuous map.

Also, by an *S-dcpo map* (*S-cpo map*) between S-dcpo's (S-cpo's), we mean a map $f : A \to B$ which is both (strict) continuous and action-preserving. We denote the categories of all S-dcpo's (S-cpo's) and S-dcpo (S-cpo) maps between them by **Dcpo-**S and **Cpo-**S, respectively.

Furthermore, notice that in the definition of an S-cpo, the monotonicity of the action implies that it is also strict (this is because, for an S-cpo A, $\bot_A \leq \bot_A \bot_S \leq \bot_A 1_S = \bot_A$). Also, by Lemma 1.2, the action $\lambda : A \times S \to A$ is continuous if and only if the maps $\lambda_a : S \to A$ and $\lambda_s : A \to A$, for all $a \in A$ and $s \in S$, are continuous.

2. Adjoint relations for Dcpo-S

In this section, we consider the middle square of the forgetful functors. We show that both U_4 and U_5 have left adjoints, while only U_4 has a right adjoint. Also, it is proved that if S satisfies a condition which we call "good", then U_6 has a left adjoint, but U_6 does not have a right adjoint.

Free S-dcpo over a dcpo. By a *free S-dcpo on a dcpo P* we mean an S-dcpo F_4 together with a continuous map $\tau : P \to F_4$ with the universal property that given any S-dcpo A and a continuous map $f : P \to A$ there exists a unique S-dcpo map $\overline{f} : F_4 \to A$ such that $\overline{f} \circ \tau = f$.

Theorem 2.1. *For a given dcpo P, the free S-dcpo on P is $F_4 = P \times S$, with componentwise order and the action given by $(x, s)t = (x, st)$, for $x \in P$, $s, t \in S$.*

Proof. Recall that $P \times S$ is an S-poset, and is a dcpo (see [1]). Now, we show that the action defined above on $P \times S$ is a continuous map. Applying Lemma 1.2, let $D \subseteq^d P \times S$ and $s \in S$. We show that

$$(\bigvee^d D)s = \bigvee^d_{(p,t) \in D} (p, ts).$$

By Lemma 1.1, $\bigvee^d D = (\bigvee^d D_1, \bigvee^d D_2)$ where $D_1 = DomD$ and $D_2 = ImD$ are directed subsets of P and S, respectively. Now,

$$(\bigvee^d D)s = (\bigvee^d D_1, (\bigvee^d D_2)s) = (\bigvee^d D_1, \bigvee^d_{s' \in D_2} s's) = \bigvee^d_{(p,t) \in D} (p, ts).$$

Notice that the first equality is true by the definition of the action on $P\times S$, also the last equality is proved straightforward. Now, let $T\subseteq^d S$ and $(p,s)\in P\times S$. Then

$$(p,s)(\bigvee^d T)=(p,s(\bigvee^d T))=(p,\bigvee^d_{t\in T} st)=\bigvee^d_{t\in T}(p,st)$$

where the last equality follows by applying the definition of the least upper bound.

Again, recalling that $P\times S$ is the free S-poset on the poset P, with the universal map $\tau : P\to P\times S$, given by $x\mapsto (x,1)$ (see [3]), we show that τ is continuous. Let $D\subseteq^d P$. Then

$$\tau(\bigvee^d D)=(\bigvee^d D,1)=\bigvee^d_{x\in D}(x,1)=\bigvee^d_{d\in D}\tau(d)$$

where the second equality is because of the definition of the upper bound.

Finally, to prove the universal property of $\tau : P\to P\times S$ for S-dcpo's, take a continuous map $f : P\to B$ to an S-dcpo B. Then the map $\overline{f} : P\times S\to B$ defined by $\overline{f}(p,s)=f(p)s$, which is the unique S-poset map with $\overline{f}\circ\tau=f$ (see [3]), is continuous. Applying Lemma 1.2, let first $D\subseteq^d P$ and $s\in S$. Then

$$\overline{f}(\bigvee^d D,s)=f(\bigvee^d D)s=(\bigvee^d_{x\in D} f(x))s=\bigvee^d_{x\in D} f(x)s=$$

$$=\bigvee^d_{x\in D}\overline{f}(x,s)$$

where the the third equality is because B is an S-dcpo. Secondly, assume that $T\subseteq^d S$ and $p\in P$, then

$$\overline{f}(p,\bigvee^d T)=f(p)\bigvee^d T=\bigvee^d_{t\in T} f(p)t=\bigvee^d_{t\in T}\overline{f}(p,t)$$

where the second equality is because B is an S-dcpo. □

Corollary 2.2. *The forgetful functor* $U_4 : \mathbf{Dcpo}\text{-}S\to\mathbf{Dcpo}$ *has a left adjoint.*

Cofree S-dcpo over a dcpo. By a *cofree S-dcpo on a dcpo P* we mean an S-dcpo K_4 together with a continuous map $\sigma : K_4\to P$ with the universal property that given any S-dcpo A and a continuous map $g : A\to P$ there exists a unique S-dcpo map $\overline{g} : A\to K_4$ such that $\sigma\circ\overline{g}=g$.

Theorem 2.3. *For a given dcpo P and dcpo-monoid S, the cofree S-dcpo on P is the set $K_4 = P^{(S)}$, of all dcpo maps from S to P, with pointwise order and the action given by $(fs)(t) = f(st)$, for $s, t \in S$ and $f \in P^{(S)}$.*

Proof. First we show that $P^{(S)}$ is an S-dcpo. Recall that $P^{(S)}$ is a dcpo, and the supremum in $P^{(S)}$ is calculated pointwise (see [8]). Also, the action defined above is a continuous map. It is well-defined, since for $f \in P^{(S)}$ and $s \in S$, fs is continuous. This is because for $T \subseteq^d S$,

$$(fs)(\bigvee^d T) = f(s(\bigvee^d T)) = f(\bigvee^d_{t\in T} st) = \bigvee^d_{t\in T} f(st) = \bigvee^d_{t\in T} (fs)(t)$$

where the second equality is because S is a dcpo-monoid, and the third equality is because f is continuous.

To prove the continuity of the action, we apply Lemma 1.2. Let first $F \subseteq^d P^{(S)}$ and $s \in S$. Then

$$((\bigvee^d F)s)(t) = (\bigvee^d F)(st) = \bigvee^d_{f\in F} f(st) = \bigvee^d_{f\in F} (fs)(t) = (\bigvee^d_{f\in F} fs)(t)$$

where the second and the last equality are because supremum in $P^{(S)}$ is calculated pointwise. Therefore, $(\bigvee^d F)s = \bigvee^d (Fs)$. Now assume that $T \subseteq^d S$ and $f \in P^{(S)}$. Then

$$(f(\bigvee^d T))(s) = f((\bigvee^d T)s) = f(\bigvee^d_{t\in T} ts) = \bigvee^d_{t\in T} f(ts) =$$

$$= \bigvee^d_{t\in T} (ft)(s) = (\bigvee^d_{t\in T} ft)(s)$$

as required. Consequently $P^{(S)}$ is an S-dcpo. Now, take the cofree map $\sigma : P^{(S)} \to P$ defined by $\sigma(f) = f(1)$. First, we show that it is continuous. Let $F \subseteq^d P^{(S)}$. Then

$$\sigma(\bigvee^d_{f\in F} f) = (\bigvee^d_{f\in F} f)(1) = \bigvee^d_{f\in F} f(1) = \bigvee^d_{f\in F} \sigma(f).$$

Further, given a continuous map $\alpha : A \to P$ from an S-dcpo A, the map $\overline{\alpha} : A \to P^{(S)}$, given by $\overline{\alpha}(a)(s) = \alpha(as)$, is an S-dcpo map and satisfies $\sigma \circ \overline{\alpha} = \alpha$. First, we show that $\overline{\alpha}$ is continuous. Let $D \subseteq^d A$ and $s \in S$, then

$$\overline{\alpha}(\bigvee^d D)(s) = \alpha((\bigvee^d D)s) = \alpha(\bigvee^d_{x\in D} xs) = \bigvee^d_{x\in D} \alpha(xs) =$$

$$= \bigvee_{x \in D}^{d} \overline{\alpha}(x)(s) = (\bigvee_{x \in D}^{d} \overline{\alpha}(x))(s).$$

Secondly, $\overline{\alpha}$ is action-preserving, since for all $s, t \in S$ and $a \in A$ we have

$$\overline{\alpha}(as)(t) = \alpha((as)t) = \alpha(a(st)) = \overline{\alpha}(a)(st) = (\overline{\alpha}(a)s)(t).$$

To establish the uniqueness of $\overline{\alpha}$, suppose that $h : A \to P^{(S)}$ is also an S-dcpo map such that $\sigma \circ h = \alpha$. Then for all $a \in A$ and $s \in S$,

$$\begin{aligned} h(a)(s) &= h(a)(s1) = (h(a)s)(1) = \sigma(h(a)s) \\ &= \sigma(h(as)) = \alpha(as) = \overline{\alpha}(a)(s). \end{aligned}$$

□

Corollary 2.4. *The forgetful functor* U_4 : **Dcpo**-S $\to$ **Dcpo** *has a right adjoint.*

Now, we consider the adjoints of U_5. Recall that for a poset P, the set $\mathrm{Id}(P)$ of ideals of P is the free dcpo over P (see Remark 1.3).

Corollary 2.5. *The forgetful functor* U_5 : **Dcpo** $\to$ **Pos** *has a left adjoint.*

In the following, we see that the right adjoint of U_5 does not necessarily exist.

Lemma 2.6. *If P is a nontrivial poset with non identity order, which is also a dcpo, then the cofree dcpo over P does not exist.*

Proof. Let P be a non trivial dcpo in which the order is not identity, and let $K(P)$ be the cofree dcpo over P as a poset. Take $k : K(P) \to P$ to be the cofree monotone map.

First we see that k is one-one. This is because, otherwise there exist $x \neq y \in K(P)$ such that $k(x) = k(y) = p_0$. Then, considering the monotone map $f : \{\Theta\} \to P$ from the singleton dcpo $\{\Theta\}$, defined by $f(\Theta) = p_0$, we see that there exist two dcpo maps $f_1, f_2 : \{\Theta\} \to K(P)$, given by $f_1(\Theta) = x$ and $f_2(\Theta) = y$, such that $k \circ f_1 = f$ and $k \circ f_2 = f$. This contradicts the universal property of the cofree map k.

Moreover, we see that k is a retraction, since for the monotone map $id_P : P \to P$, by the universal property of cofree maps, there exists a dcpo map $f : P \to K(P)$ with $k \circ f = id_P$. Therefore, k is a poset isomorphism.

Now, since the order on P is not identity, there exist $x, y \in P$ with $x < y$. Define the poset map $f : \mathcal{P}(\mathbb{N}) \to P$ by

$$f(M) = \begin{cases} x & \text{if M is finite} \\ y & \text{otherwise} \end{cases}$$

Then, by the universal property of cofree maps, there exists a unique dcpo map $\overline{f} : \mathcal{P}(\mathbb{N}) \to K(P)$ with $k \circ \overline{f} = f$. Now, f being a composition of two dcpo maps, is a dcpo map. But this is a contradiction, because taking the directed subset D of $\mathcal{P}(\mathbb{N})$ consisting of all finite subsets of $\mathbb{N}$, we have $f(\bigvee^d D) = f(\bigcup D) = f(\mathbb{N}) = y$ but $\bigvee^d f(D) = \bigvee^d \{x\} = x$. □

Corollary 2.7. *The forgetful functor* $U_5 : \mathbf{Dcpo} \to \mathbf{Pos}$ *does not have a right adjoint.*

Now, we consider U_6. First, using the above corollary, we have:

Remark 2.8. The forgetful functor U_6 from **Dcpo**-S to **Pos**-S does not have a right adjoint for a general dcpo-monoid S. This is implied by taking $S = \{1\}$, and applying Corollary 2.7.

Now, we give a condition on S under which U_6 has a left adjoint.

Definition 2.9. We say that a dcpo P is *good* if for every directed subset D of P, $\bigvee^d D \in D$.

Remark 2.10. A dcpo P is good if and only if each directed subset of P has a top element. This condition is also equivalent to the fact that each element of P is compact. Recall that the element x of a dcpo P is called *compact* if for every directed subset D of P, $x \leq \bigvee^d D$ implies $x \leq d$, for some $d \in D$.

Finite posets and Noetherian posets (satisfying ACC on chains of elements) are examples of good dcpo's. Also, for any poset P with discrete order, the posets $P \oplus \top$ and $\bot \oplus P$ are good dcpo's, where $P \oplus \top$ and $\bot \oplus P$ are obtained by adding a top element and a bottom element to P, respectively.

Theorem 2.11. *Let S be a good dcpo-monoid. For a given S-poset A, the free S-dcpo on A is the dcpo* $\mathrm{Id}(A)$ *with the action* $\lambda : \mathrm{Id}(A) \times S \to \mathrm{Id}(A)$, *given by* $(I, s) \mapsto I.s =: \downarrow(Is)$, *where* $Is = \{as : a \in I\}$ *for* $I \in \mathrm{Id}(A)$ *and* $s \in S$.

Proof. First we show that $\mathrm{Id}(A)$ is an S-dcpo. Notice that, by Remark 1.3, $\mathrm{Id}(A)$ is a dcpo in which the supremum of a directed subset D of $\mathrm{Id}(A)$ is $\bigcup D$. Also, it is clear that the given action is well-defined. Further, for all $I \in \mathrm{Id}(A)$ and $s, t \in S$, we have

(1) $I.1 = \downarrow(I1) = \downarrow I = I$,

(2) $I.(st) = \downarrow(I(st)) = \downarrow((Is)t) = \downarrow(\downarrow(Is))t = (I.s).t$,

where equalities are true by a straightforward computation using definitions. Now, we show that the action is also continuous. Applying

Lemma 1.2, let $\{I_\alpha : \alpha \in \Lambda\}$ be a directed subset of $\mathrm{Id}(A)$ and $s \in S$. We have

$$(\bigvee_{\alpha\in\Lambda}^{d} I_\alpha).s = (\bigcup_{\alpha\in\Lambda} I_\alpha).s = \downarrow((\bigcup_{\alpha\in\Lambda} I_\alpha)s) = \downarrow(\bigcup_{\alpha\in\Lambda}(I_\alpha s)) = \bigcup_{\alpha\in\Lambda}(\downarrow(I_\alpha s) =$$

$$= \bigvee_{\alpha\in\Lambda}^{d}(\downarrow(I_\alpha s)) = \bigvee_{\alpha\in\Lambda}^{d} I_\alpha.s$$

where the equalities are true by straightforward calculations.

Now, assume that $T \subseteq^d S$ and $I \in \mathrm{Id}(A)$. Then

$$I.(\bigvee^{d} T) = \downarrow(I(\bigvee^{d} T)) = \bigvee_{t\in T}^{d}\downarrow(It) = \bigvee_{t\in T}^{d} I.t,$$

where the second equality follows from the hypothesis that $\bigvee^d T \in T$, which gives that $\downarrow(I(\bigvee^d T)$ is the maximum element of the set $\{\downarrow It : \ t \in T\}$. Therefore,

$$\downarrow(I(\bigvee^{d} T)) = \bigvee_{t\in T}^{d}\downarrow(It),$$

and so $\mathrm{Id}(A)$ is an S-dcpo. Now, we show that $\downarrow: A \to \mathrm{Id}(A)$, $a \mapsto \downarrow a$ is an S-poset map. It is clear that $\downarrow$ is order-preserving. It is also action-preserving, since $(\downarrow a).s = \downarrow((\downarrow a)s) = \downarrow(as)$. Finally, we show that $\downarrow: A \to \mathrm{Id}(A)$ is a universal map. Let $f : A \to B$ be an S-poset map to an S-dcpo B. Then the map $\overline{f} : \mathrm{Id}(A) \to B$ given by $\overline{f}(I) = \bigvee^d f(I)$ is the unique S-dcpo map with $\overline{f} \circ \downarrow = f$. To see this, first we show that $\overline{f}$ is continuous. Let $\{I_\alpha : \alpha \in \Lambda\}$ be a directed subset of $\mathrm{Id}(A)$. Then

$$\overline{f}(\bigvee_{\alpha\in\Lambda}^{d} I_\alpha) = \overline{f}(\bigcup_{\alpha\in\Lambda} I_\alpha) = \bigvee^{d} f(\bigcup_{\alpha\in\Lambda} I_\alpha) = \bigvee_{\alpha\in\Lambda}^{d}(\bigvee^{d} f(I_\alpha)) = \bigvee_{\alpha\in\Lambda}^{d}\overline{f}(I_\alpha)$$

where the third equality follows by the definition of supremum. In fact, since $f(I_\alpha) \subseteq f(\bigcup_{\alpha\in\Lambda} I_\alpha)$, we get $\bigvee^d f(I_\alpha) \leq \bigvee^d f(\bigcup_{\alpha\in\Lambda} I_\alpha)$, for all $\alpha \in \Lambda$. Also, if $b \in B$ is an upper bound of the set $\{\bigvee^d f(I_\alpha) : \alpha \in \Lambda\}$, then for $x \in f(\bigcup_{\alpha\in\Lambda} I_\alpha)$, we have $x \in f(I_{\alpha_0})$, for some $\alpha_0 \in \Lambda$, and so $x \leq \bigvee^d f(I_{\alpha_0}) \leq b$, which gives $\bigvee^d f(\bigcup_{\alpha\in\Lambda} I_\alpha) \leq b$.

Also, $\overline{f}$ is action-preserving, since for $I \in \mathrm{Id}(A)$ and $s \in S$, we have

$$\overline{f}(I.s) = \overline{f}(\downarrow(Is)) = \bigvee^{d} f(\downarrow(Is)) = \bigvee^{d} f(Is) = \bigvee^{d} f(I)s =$$

$$= (\bigvee^{d} f(I))s = \overline{f}(I)s$$

where the third equality is because an element c is an upper bound of $f(\downarrow(Is))$ if and only if it is an upper bound of $f(Is)$. The fourth equality is because f is action-preserving. Also, the fifth equality is because B is an S-dcpo. Furthermore, we have $\overline{f}(\downarrow a) = \bigvee^d f(\downarrow a) = f(a)$. To show the uniqueness of $\overline{f}$, suppose that $h : \mathrm{Id}(A) \to B$ is also an S-dcpo map with $h \circ \downarrow = f$. Then for every $I \in \mathrm{Id}(A)$,

$$\overline{f}(I) = \bigvee^{d} f(I) = \bigvee^{d}_{a\in I} f(a) = \bigvee^{d}_{a\in I} h(\downarrow a) = h(\bigcup_{a\in I} \downarrow a) = h(I).$$

□

Corollary 2.12. *If S is a good dcpo-monoid, then the forgetful functor U_6 : **Dcpo**-$S \to$ **Pos**-S has a left adjoint.*

3. Adjoint relations for **Cpo-**S

In this section, we consider the top square of the forgetful functors. We show that U_2 has a left adjoint, and if we assume that S is a cpo-monoid whose identity is the bottom element, then U_1 has a left adjoint; also if S is a cpo-monoid whose identity is the top element, then U_3 has a left adjoint. But, none of U_1, U_2, and U_3 has a right adjoint.

Free S-cpo on a cpo P. By a *free S-cpo on a cpo P* we mean an S-cpo F_1 together with a strict continuous map $\tau : P \to F_1$ with the universal property that given any S-cpo A and a strict continuous map $f : P \to A$ there exists a unique S-cpo map $\overline{f} : F_1 \to A$ such that $\overline{f} \circ \tau = f$.

Theorem 3.1. *Let S be a cpo-monoid whose identity is the bottom element. Then for a given cpo P and cpo-monoid S, the free S-cpo on P is $F_1 = P \times S$, with componentwise order and the action given by $(x, s)t = (x, st)$, for $x \in P$, $s, t \in S$.*

Proof. First recall that $P \times S$ with the above action and order is the free S-dcpo on the dcpo P (see Theorem 2.1). Also, we know that $P \times S$ with the componentwise order is a cpo (see [1]). Now, we show that $\tau : P \to P \times S$ given by $x \mapsto (x, 1)$ is a universal strict continuous map. Since the identity element of S is the bottom element, we have

$$\tau(\bot_P) = (\bot_P, 1) = (\bot_P, \bot_S)$$

which means that τ is strict. The continuity of τ was proved in Theorem 2.1. To prove the universal property, let $f : P \to B$ be any strict continuous map to an S-cpo B. Then the map $\overline{f} : P \times S \to B$ defined by $\overline{f}(p, s) = f(p)s$ is the unique S-dcpo map with $\overline{f} \circ \tau = f$ (see Theorem 2.1). Now, we show that $\overline{f}$ is also strict. Since f is strict and B is an S-cpo,

$$\overline{f}(\bot_P, \bot_S) = f(\bot_P)\bot_S = \bot_B \bot_S = \bot_B.$$

□

Corollary 3.2. *If S is a cpo-monoid whose identity is the bottom element, then the forgetful functor U_1 : **Cpo**-$S \to$ **Cpo** has a left adjoint.*

Remark 3.3. The forgetful functor U_1 : **Cpo**-$S \to$ **Cpo** does not have a right adjoint. One can see this by noting that it does not necessarily preserve the initial object. For example, let S be the 2-element chain $\{1, a\}$ with $1 < a$, and $aa = a$, $1a = a = a1$. Then S is an S-cpo and, it is the initial object of **Cpo**-S (see [11]), whereas the initial object in the category **Cpo** is the singleton cpo.

Now, we consider U_2.

Theorem 3.4. *The forgetful functor U_2 : **Cpo** $\to$ **Dcpo** has a left adjoint.*

Proof. For a dcpo P, $P_\bot = \bot \oplus P$ is the free cpo on P. □

Remark 3.5. The forgetful functor U_2 : **Cpo** $\to$ **Dcpo** does not have a right adjoint. This is because, U_2 does not preserve the initial object. Notice that the initial object in **Cpo** is the singleton poset $\{\Theta\}$, while the initial object in **Dcpo** is the empty poset.

Finally, we study U_3.

Free S-cpo over an S-dcpo. By a *free S-cpo on an S-dcpo A* we mean an S-cpo F_6 together with a S-dcpo map $\tau : A \to F_6$ with the universal property that given any S-cpo B and a strict continuous map $f : A \to B$ there exists a unique S-cpo map $\overline{f} : F_6 \to A$ such that $\overline{f} \circ \tau = f$.

Theorem 3.6. *Let S be a cpo-monoid in which the identity element is the top element. Then the free S-cpo on an S-dcpo A is $A_\bot = \bot \oplus A$ with the action defined by:*

$$a.s = \begin{cases} as & \text{if } a \in A \\ \bot & \text{if } a = \bot \end{cases}$$

for all $a \in A_\bot$ and $s \in S$.

Proof. We show that this action is continuous. Applying Lemma 1.2, let $D \subseteq^d A_\perp$ and $s \in S$. First note that $D \subseteq A_\perp$ is directed if and only if $D \subseteq A$ is directed or $D = D' \cup \{\perp\}$ where $D' = \emptyset$ or D' is a directed subset of A. Therefore, two cases may occur:

Case (1): $D \subseteq^d A$. In this case,

$$(\bigvee^d D).s = (\bigvee^d D)s = \bigvee^d_{x \in D} xs = \bigvee^d_{x \in D} x.s$$

since the action on A is continuous.

Case (2): $D = D' \cup \{\perp\}$, where $D' \subseteq^d A$ or $D' = \emptyset$. If $D' = \emptyset$, then the result is clear. Let $D' \subseteq^d A$. Then,

$$(\bigvee^d D).s = (\bigvee^d D').s = (\bigvee^d D')s = \bigvee^d_{x \in D'} xs =$$

$$(\bigvee^d_{x \in D'} x.s) \vee (\perp.s) = \bigvee^d_{x \in D} x.s$$

Now assume that $T \subseteq^d S$ and $a \in A_\perp$. If $a = \perp$, then $\perp.(\bigvee^d T) = \perp = \bigvee^d (\perp.s)$. If $a \in A$, then

$$a.(\bigvee^d T) = a(\bigvee^d T) = \bigvee^d_{t \in T}(at) = \bigvee^d_{t \in T}(a.t)$$

where the second equality is because A is an S-dcpo. Therefore, $A_\perp$ is an S-cpo. Now, we show that the inclusion map $\iota : A \to A_\perp$ is the universal free map. Let $f : A \to B$ be any S-dcpo map to an S-cpo B. Then, the map $\overline{f} : A_\perp \to B$ defined by

$$\overline{f}(a) = \begin{cases} f(a) & \text{if} \quad a \in A \\ \perp_B & \text{if} \quad a = \perp \end{cases}$$

is the unique cpo map with $\overline{f} \circ \iota = f$. Now, we show that $\overline{f} : A_\perp \to B$ is action-preserving, and so it is an S-cpo map. Since the identity element of S is the top element, the bottom element of every S-cpo is a zero element ($s \leq 1$ implies $\perp_A s \leq \perp_A 1 = \perp_A$, and so $\perp_A s = \perp_A$), and hence $\overline{f}(\perp.s) = \overline{f}(\perp) = \perp_B = \perp_B s = \overline{f}(\perp)s$, for all $s \in S$. Also, for $a \neq \perp$ and $s \in S$, $\overline{f}(a.s) = \overline{f}(as) = f(as) = f(a)s = \overline{f}(a)s$. □

Corollary 3.7. *If S is a cpo-monoid whose identity is the top element, then the forgetful functor U_3 : **Cpo**-S → **Dcpo**-S has a left adjoint.*

Remark 3.8. The forgetful functor U_3 : **Cpo**-S → **Dcpo**-S does not have a right adjoint. Take $S = \{1\}$ and apply Remark 3.5. Another way to see this, is by showing that U_3 does not preserve the initial object. Consider the example S given in Remark 3.3. Then S is the initial object in the category **Cpo**-S, but the initial object in **Dcpo**-S is the empty poset.

4. Erratum to Adjoint relations for Pos-S

In this section, we consider the bottom square of forgetful functors. Recall that the adjoint situations related to the category of S-posets have been stated in [3]. The free functor from S-acts to S-posets is described in Theorem 17 of [3]. There is an error in that description which makes it true if and only if the pomonoid S has a trivial order. In fact, it is stated there that for a given S-act A, the free S-poset is (A, Δ), where Δ is the discrete (equality) order. But, if there are $s, t \in S$ with $s < t$, then we may have a in A such that $as \neq at$ and so $(as, at) \notin \Delta$. That is, (A, Δ) is not necessarily an S-poset.

In the following, we correct this, and find the free adjunction to the forgetful functor U_9 : **Pos**-$S \to$ **Act**-S.

Let A be an S-act. Consider the relation $R = \{(as, at) : a \in A, s \leq t\}$ on A. Recall the order Δ_R (see [2], and $\leq_R$ in [3]) given by

$$a\Delta_R b \quad \text{if and only if} \quad \begin{array}{l} \text{there exist } a_1, a_1', \ldots, a_n, a_n' \in A; \\ a = a_1 R a_1' = \cdots = a_n R a_n' = b \end{array}$$

which explicitly means that $a\Delta_R b$ if and only if there exist $a_1, a_2, ..., a_n \in A$, $s_1, ..., s_n \in S$, $t_1, ..., t_n \in S$ with $s_i \leq t_i$, for all $i = 1, ..., n$, and such that

$$\begin{array}{llllll} a = a_1 s_1 & & a_2 t_2 = a_3 s_3 & \cdots & & a_n t_n = b \\ & a_1 t_1 = a_2 s_2 & & a_3 t_3 = a_4 s_4 & \cdots \quad a_{n-1} t_{n-1} = a_n s_n & \end{array}$$

Then the relation θ given by

$$a\theta b \Leftrightarrow a\Delta_R b \Delta_R a$$

is an S-act congruence, and the quotient S-act A/θ turns into an S-poset with the order given by

$$[a]_\theta \leq [b]_\theta \Leftrightarrow a\Delta_R b.$$

Theorem 4.1. *For a given S-act A, the quotient S-act A/θ given above is the free S-poset on A.*

Proof. First note that the S-act A/θ with the above order is an S-poset. To see this, we show that the order is a well-defined partial order, and also the action is monotone. Let $[a]_\theta = [c]_\theta$, $[b]_\theta = [d]_\theta$, and $[a]_\theta \leq [b]_\theta$. Then $a\Delta_R c\Delta_R a$, $b\Delta_R d\Delta_R b$, and $a\Delta_R b$. This gives that $c\Delta_R a\Delta_R b\Delta_R d$, and so $c\Delta_R d$, since Δ_R is transitive. Thus, the order is well-defined. It is also a partial order, since Δ_R, and so $\leq$ is a preorder. Further, it is anti-symmetric. For, if $[a]_\theta \leq [b]_\theta \leq [a]_\theta$, then $a\Delta_R b\Delta_R a$. Therefore, by the definition of θ, $[a]_\theta = [b]_\theta$.

To see that the action is monotone, let $[a] \leq [b]$ and $s \leq t$. Then there exist $a_1, a'_1, ..., a_n, a'_n \in A$ such that $a = a_1Ra'_1 = a_2Ra'_2 = \cdots = a_nRa'_n = b$, and by the definition of R, using the fact that S is a pomonoid, this gives

$$as = a_1sRa'_1s = a_2sRa'_2s = \cdots = a_nsRa'_nsRa'_nt = bt$$

which means $as\Delta_R bt$.

Now, we show that the natural map $\pi : A \to A/\theta$, $a \mapsto [a]$, is a universal S-act map. Let $f : A \to B$ be any S-act map to an S-poset B. Then, the map $\overline{f} : A/\theta \to B$, defined by $\overline{f}([a]) = f(a)$, is the unique S-poset map with $\overline{f} \circ \pi = f$. To see this, notice that $\theta \subseteq Kerf$. For, if $a\Delta_R b$ then $f(a) \leq f(b)$, by the definition of Δ_R, the hypothesis that f is an S-act map, and that B is an S-poset. Therefore, by the Decomposition Theorem (Fundamental Homomorphism Theorem) of S-acts, there exists the unique S-act map $\overline{f}$ as above. We further see that $\overline{f}$ is an order-preserving map, since $[a] \leq [b]$ means $a\Delta_R b$, and so $f(a) \leq f(b)$. □

5. Conclusion

In this final section, applying the investigations done in the above sections, we consider the composition of forgetful functors given in all the three squares, and consider the questions whether they have a left or a right adjoint or do not have.

Remark 5.1. Applying the compositions of some of the free functors in the above sections, we get:

(1) If S is a cpo-monoid in which the identity element is the bottom element, then the free S-cpo over a set X is $X_\perp \times S$.

(2) If S is a cpo-monoid in which the identity element is the bottom element, then the free S-cpo over a poset P is $(\mathrm{Id}(P) \cup \{\emptyset\}) \times S$.

(3) If S is a good cpo-monoid in which the identity element is the top element, the free S-cpo over an S-act A is $\mathrm{Id}(A/\theta) \cup \{\emptyset\}$, where A/θ is the S-poset given in Theorem 4.1.

(4) If S is a good cpo-monoid in which the identity element is the top element, the free S-cpo over an S-poset A is $\mathrm{Id}(A) \cup \{\emptyset\}$.

(5) If S is an cpo-monoid in which the identity element is the bottom element, then the free S-cpo over a dcpo A is $(\perp \oplus A) \times S$. While if the identity element is the top element, then the free S-cpo over a dcpo A is $(A \times S)_\perp$.

(6) The free S-dcpo over a set X is $X \times S$, where X is considered as a dcpo with the identity order.

(7) The free S-dcpo over a poset P is $\mathrm{Id}(P) \times S$.

(8) If S is a good S-dcpo, then the free S-dcpo over an S-act A is $\mathrm{Id}(A/\theta)$, where A/θ is the S-poset given in Theorem 4.1.

(9) The free cpo over a poset P is $\mathrm{Id}(P) \cup \{\emptyset\}$.

(10) The free cpo over a set X is $\perp \oplus X$.

(11) The free dcpo over a set X is $(X, =)$.

Remark 5.2. About the cofree functors, we have:

(1) The cofree functor **Set** $\to$ **Cpo**-S does not necessarily exist. This is because, considering $S = \{1\}$, the forgetful functor U : **Cpo** $\to$ **Set** does not preserve coproducts. Let **2** denote the two elements chain $\{0, 1\}$ with $0 < 1$, and let **3** denote the three elements chain $\{0, a, 1\}$ with $0 < a < 1$. The coproduct of **2** and **3** in **Cpo** is their coalesced sum $\mathbf{2} \uplus \mathbf{3} = \perp \oplus ((\mathbf{2} \setminus \{0\}) \dot{\cup} (\mathbf{3} \setminus \{0\}))$ which has four elements, whereas the coproduct of $U_8(\mathbf{2})$ and $U_8(\mathbf{3})$ in **Set** is their disjoint union which has five elements.

Also, similar to Remark 3.8, it is seen that U does not preserve the initial object. And, similarly, the cofree S-cpo over a poset, a dcpo, an S-poset, and an S-act, do not necessarily exist.

(2) The cofree S-dcpo over **Act**-S does not necessarily exist. Take $S = \{1\}$, and see Theorem 5.3. Similarly, the same is true for the cofree S-dcpos over a set and over a poset.

(3) The forgetful functors from **Cpo** to **Pos** and to **Set** do not have right adjoints. This is because they do not preserve initial objects. Notice that the initial object in **Cpo** is the singleton poset $\{\Theta\}$, while the initial objects in **Pos** and **Set** are both the empty set.

In the following, we see that the cofree dcpo over a set does not necessarily exist.

Theorem 5.3. *The cofree dcpo over set X exists if and only if $|X| = 1$.*

Proof. If $|X| = 1$, then the cofree dcpo over X is $(X, =)$. If $|X| \geq 2$, we show that the cofree dcpo over the set X does not exist. Assume the contrary and let $K(X)$ be the cofree dcpo over X with the cofree map $k : K(X) \to X$. First we see that k is one-one. This is because, otherwise there exist $x \neq y \in K(X)$ such that $k(x) = k(y) = x_0$. Then, considering the map $f : \{\Theta\} \to X$ defined by $f(\Theta) = x_0$, we see that there exist two dcpo maps $f_1, f_2 : \{\Theta\} \to K(X)$, given by $f_1(\Theta) = x$, and $f_2(\Theta) = y$, such that $k \circ f_1 = f$ and $k \circ f_2 = f$. This contradicts the universal property of k, and therefore k is one-one.

Now, take $x \neq y \in X$ and define the map f from the three-element chain $\{0, a, 1\}$ with $0 < a < 1$, to X by $f(0) = x = f(1)$, $f(a) = y$. By the universal property of k, there exists a unique dcpo map $\overline{f} : \mathbf{3} \to K(X)$ with $k \circ \overline{f} = f$. Now, $k(\overline{f}(0)) = f(0) = x$ and $k(\overline{f}(1)) = f(1) = x$. Since k is one-one, we get $\overline{f}(0) = \overline{f}(1)$. By monotonicity of $\overline{f}$, $\overline{f}(0) \leq \overline{f}(a) \leq \overline{f}(1)$ holds. Whence $\overline{f}(a) = \overline{f}(0)$, and so $x = k(\overline{f}(0)) = k(\overline{f}(a)) = y$, which is a contradiction. □

Open problems.

(a) Are the conditions given on S for the existence of the free objects F_1, F_3, and F_6, necessary?

(b) Can we give a condition on S, under which the cofree vertical functors on the side of the diagram exist?

Acknowledgment. The authors gratefully acknowledge Professor M.Mehdi Ebrahimi's comments and conversations during this work, specially we thank his suggestion to draw the diagram of forgetful functors which helped us to improve the results and have a clear picture of what we were working on. Also our sincere thanks goes to the referees for their careful reading and helpful suggestions.

References

[1] Abramsky, S. and A. Jung, "Domain theory", Handbook of Computer Science and logic, Vol. 3, Clarendon Press, Oxford, 1995.

[2] Blyth, T.S. and M.F. Janowitz. "Residuation Theory", Pergamon Press, Oxford, 1972.

[3] Bulman-Fleming, S. and M. Mahmoudi, *The category of S-posets*, Semigroup Forum 71(3), 443-461 (2005).

[4] Crole, Roy. L., "Categories for types", Cambridge University Press, Cambridge, 1994.

[5] Davey, B.A. and H.A. Priestly, "Introduction to Lattices and Order", Cambridge University Press, Cambridge, 1990.

[6] Ebrahimi, M.M., and M. Mahmoudi, *The category of M-Sets*, Ital. J. Pure Appl. Math. 9, 123-132 (2001).

[7] Fiech, A., *Colimits in the category Dcpo*, Math. Structures Comput. Sci. 6, 455-468 (1996).

[8] Jung, A., "Cartesian closed categories of Domains", Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 110 pp (1989).

[9] Jung, A., Moshier, M.Andrew, and S. Vickers, *Presenting dcpos and dcpo algebras*, Electronic Notes in Theoretical Computer Science 219, 209-229 (2008).

[10] Kilp, M., U. Knauer, and A. Mikhalev, "Monoids, Acts and Categories", Walter de Gruyter, Berlin, New York, 2000.

[11] Mahmoudi, M. and H. Moghbeli, *The Categories of actions of a dcpo-monoid on directed complete posets*, submitted.

[12] Vickers, S. and C. Townsend, *A universal characterization of the double powerlocale*, Theoret. Comput. Sci. 316, 297-321 (2004).

Department of Mathematics, Shahid Beheshti University, G.C., Tehran 19839, Iran.

E-mail address: m-mahmoudi@sbu.ac.ir

E-mail address: h_moghbeli@sbu.ac.ir