Inverse limits with generalized Markov interval functions

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Abstract

In 2002 Markov interval maps were introduced by S. Holte. It was shown that any two inverse limits with Markov interval bonding maps with the same pattern were homeomorphic.

In this article we introduce generalized Markov interval functions, which are a generalization of Markov interval maps to set-valued functions, and show that any two generalized inverse limits with generalized Markov interval bonding functions with the same pattern are homeomorphic.

1 Introduction

In [7] Markov interval maps are defined as follows. Interval self-maps on $I = [a_0, a_m]$ are Markov with respect to $A = \{a_0, a_1, \ldots, a_m\}$, if

1. $a_0 < a_1 < \ldots < a_m$,
2. $f(A) \subseteq A$,
3. f is injective on every component of $I \backslash A$.

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Two interval self-maps, f and g, are Markov with the same pattern if f is Markov with respect to $A = \{a_0, a_1, \ldots, a_m\}$, g is Markov with respect to $B = \{b_0, b_1, \ldots, b_m\}$, and $f(a_j) = a_k$ if and only if $g(b_j) = b_k$.

The main theorem in [7] says that any two Markov interval maps with the same pattern have homeomorphic inverse limits:

Theorem 1.1. Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of surjective maps from $I = [a_0, a_m]$ to I, which are all Markov interval maps with respect to $A = \{a_0, a_1, \ldots, a_m\}$ and let $\{g_n\}_{n=0}^{\infty}$ be a sequence of surjective maps from $J = [b_0, b_m]$ to J, which are all Markov interval maps with respect to $B = \{b_0, b_1, \ldots, b_m\}$. If for each n, f_n and g_n are Markov interval maps with the same pattern, then (I, f_n) is homeomorphic to (J, g_n).

In this paper we introduce generalized Markov interval functions, which generalize Markov interval maps from [7] (in such a way that every Markov interval map is naturally interpreted as a generalized Markov interval function). In this generalization we allow a generalized Markov interval function to be non single-valued only on points in A, and include a condition that provides the injectivity of f on every component of $I \backslash A$. The definition of two generalized Markov interval functions with the same pattern will generalize the definition of two Markov interval maps with the same pattern (as it is defined in [7]). We prove the following theorem, which is a generalization of Theorem 1.1, as the main result of the paper:

Theorem 1.2. Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of u.s.c. functions from $I = [a_0, a_m]$ to 2^I with surjective graphs, which are all generalized Markov interval functions with respect to $A = \{a_0, a_1, \ldots, a_m\}$ and let $\{g_n\}_{n=0}^{\infty}$ be a sequence of u.s.c. functions from $J = [b_0, b_m]$ to 2^J with surjective graphs, which are all generalized Markov interval functions with respect to $B = \{b_0, b_1, \ldots, b_m\}$. If for each n, f_n and g_n are generalized Markov interval functions with the same pattern, then (I, f_n) is homeomorphic to (J, g_n).

Since techniques we used in the proof of Theorem 1.2 are quite different from the ones used in [7], our proof can serve as an alternative proof of Holte's result.

2 Definitions and notation

A *map* is a continuous function. In the case when $X = Y = \mathbb{R}$, $a \in \mathbb{R}$, and $f : X \to Y$ is a map, we use $\lim\limits_{x \downarrow a} f(x)$ to denote the *right-hand limit* and

$\lim\limits_{x \uparrow a} f(x)$ to denote the *left-hand limit* of a function f at the point $a \in \mathbb{R}$. A detailed introduction of such limits can be found in [17, p. 83–95].

Let X be a compact metric space, then 2^X denotes the set of all nonempty closed subsets of X.

If $f : X \to 2^Y$ is a function, then the *graph* of f, $\Gamma(f)$, is defined as $\Gamma(f) = \{(x, y) \in X \times Y | \; y \in f(x)\}$.

A function $f : X \to 2^Y$ *has a surjective graph*, if for each $y \in Y$ there is an $x \in X$, such that $y \in f(x)$.

Let $f : X \to 2^Y$ be a function. If for each open set $V \subseteq Y$, the set $\{x \in X \mid f(x) \subseteq V\}$ is open in X, then f is an *upper semicontinuous* function (abbreviated u.s.c.) from X to 2^Y.

The following is a well-known characterization of u.s.c. functions between metric compacta (for example, see [9, p. 120, Theorem 2.1]).

Theorem 2.1. Let X and Y be compact metric spaces and $f : X \to 2^Y$ a function. Then f is u.s.c. if and only if its graph $\Gamma(f)$ is closed in $X \times Y$.

Note that for any continuous function $f : X \to Y$, where X and Y are compact metric spaces, the graph of f is a closed subset of $X \times Y$. Therefore the function $F : X \to 2^Y$, defined by $F(x) = \{f(x)\}$, is an u.s.c. function, since $\Gamma(F) = \Gamma(f)$. Also if $F : X \to 2^Y$ is an u.s.c. function such that $F(x) = \{y_x\}$ for each $x \in X$, then the function $f : X \to Y$, defined by $f(x) = y_x$, is continuous. Such functions F will be addressed as single-valued functions. In the paper we frequently deal with such u.s.c. functions. Understanding them as mappings will simplify the notation and make the proof more reader-friendly. That is why in this case we write $y = F(x)$ instead of $y \in F(x)$. In addition, we say that F is *injective* if f is injective.

Let A be a subset of X and let $f : X \to 2^Y$ be a function. The *restriction* of f on the set A, $f|_A$, is the function from A to 2^Y such that $f|_A(x) = f(x)$ for every $x \in A$.

Let $f : [a, b] \to 2^{[c,d]}$ be a function. Then we say that f is *single-valued at some point* $x \in [a, b]$ if $f(x)$ consists of a single point. We also say that f is *single-valued on some interval* $I \subseteq [a, b]$ if the above holds for each $x \in I$.

A sequence $\{X_k, f_k\}_{k=0}^{\infty}$ of compact metric spaces X_k and u.s.c. functions $f_k : X_{k+1} \to 2^{X_k}$, is an *inverse sequence with u.s.c. bonding functions*.

The *inverse limit* of an inverse sequence $\{X_k, f_k\}_{k=0}^{\infty}$ with u.s.c. bonding functions is defined as the subspace of $\prod_{k=0}^{\infty} X_k$ of all points $(x_0, x_1, x_2, \ldots)$, such that $x_k \in f_k(x_{k+1})$ for each k. The inverse limit of an inverse sequence $\{X_k, f_k\}_{k=0}^{\infty}$ is denoted by $\varprojlim (X_k, f_k)$.

In this paper we deal only with the case when for each k, X_k is a closed interval I and $f_k : I \to 2^I$. So, we denote the inverse limit simply by (I, f_k).

The notion of inverse limits of inverse sequences with upper semicontinuous bonding functions (also known as generalized inverse limits) was introduced by Mahavier in [12] and later by Ingram and Mahavier in [9]. Since then, inverse limits have appeared in many papers, such as [1, 2, 3, 4, 5, 6, 8, 10, 11, 13, 14, 15, 16, 18].

3 Proof of Theorem 1.2

In this section we introduce the notion of generalized Markov interval functions and prove Theorem 1.2.

Definition 3.1. Let $a, b \in \mathbb{R}$, $a < b$, and m a positive integer. We say that an u.s.c. function f from $I = [a, b]$ to 2^I is a *generalized Markov interval function with respect to* A, where $A = \{a_0, a_1, \ldots, a_m\}$ is a subset of I, if

1. $a = a_0 < a_1 < \ldots < a_m = b$,
2. the restriction of f on every component of $I \backslash A$ is an injective single-valued function,
3. for each $j = 0, 1, \ldots, m$, the image $f(a_j)$ is an interval (possibly degenerate) $\big[a_{r_1(j)}, a_{r_2(j)}\big]$, where $a_{r_1(j)}, a_{r_2(j)} \in A$ $(a_{r_1(j)} \le a_{r_2(j)})$,
4. for each $j = 0, 1, \ldots, m-1$: $\lim\limits_{x \uparrow a_{j+1}} f(x)$, $\lim\limits_{x \downarrow a_j} f(x) \in A$.

Obviously, f can be single-valued at some points a_j in A. In this case $r_1(j) = r_2(j)$ for some $0 \le j \le m$ and $f(a_j) = \{a_{r_1(j)}\}$. Additionally, taking into account property 4. above, we see that:

1. if $0 < j < m$, then $\lim\limits_{x \uparrow a_j} f(x) = \lim\limits_{x \downarrow a_j} f(x) = a_{r_1(j)} = a_{r_2(j)}$,
2. if $j = 0$, $\lim\limits_{x \downarrow a_j} f(x) = a_{r_1(j)} = a_{r_2(j)}$,
3. if $j = m$, $\lim\limits_{x \uparrow a_j} f(x) = a_{r_1(j)} = a_{r_2(j)}$.

An example of a generalized Markov interval function can be seen in Figure 1. We point out that many set-valued functions and their inverse limits have already been studied and many of these functions are examples of generalized Markov interval functions (for example, see [3]).

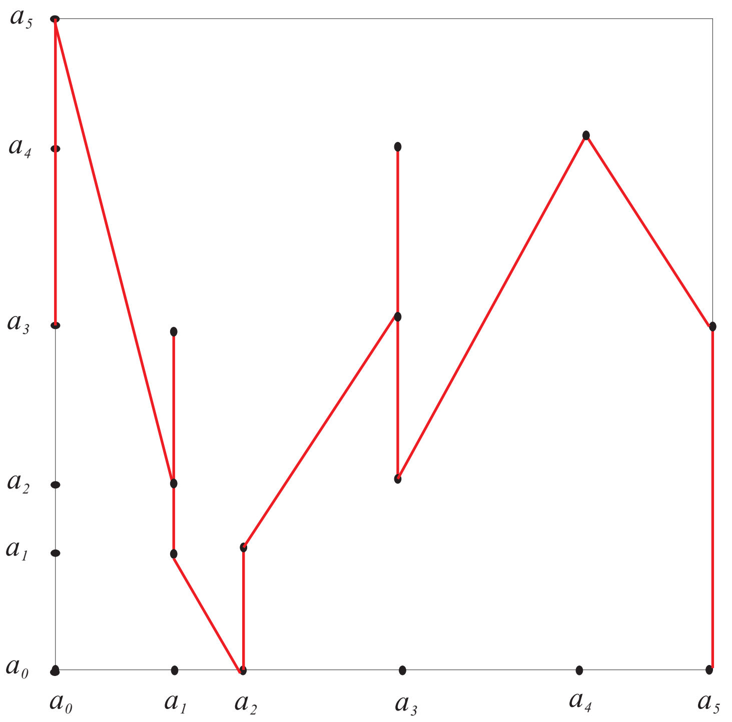

Figure 1: A generalized Markov interval function with respect to $A = \{a_0, a_1, a_2, a_3, a_4, a_5\}$.

Definition 3.2. Let $A = \{a_0, a_1, \ldots, a_m\}$ and $B = \{b_0, b_1, \ldots, b_m\}$, where $a_0 < a_1 < \ldots < a_m$ and $b_0 < b_1 < \ldots < b_m$. Then we say that $(a, b) \in A \times B$ is *a pair of similar points (with respect to A and B)*, if $a = a_i$ and $b = b_i$ for some $i = 0, 1, \ldots, m$.

In the following definition we define what it means for two generalized Markov interval functions to follow the same pattern.

Definition 3.3. Let $f : I = [a_0, a_m] \to 2^I$ be a generalized Markov interval function with respect to $A = \{a_0, a_1, \ldots, a_m\}$ and let $g : J = [b_0, b_m] \to 2^J$ be a generalized Markov interval function with respect to $B = \{b_0, b_1, \ldots, b_m\}$.

We say that f and g are *generalized Markov interval function with the same pattern* if $i)$ and $ii)$ hold true:

$i)$ for every $j = 0, 1, \ldots, m$: $f(a_j) = \left[a_{r_1(j)}, a_{r_2(j)}\right]$ if and only if $g(b_j) = \left[b_{r_1(j)}, b_{r_2(j)}\right]$,

$ii)$ for every $j = 0, 1, \ldots, m-1$: $(\lim\limits_{x \uparrow a_{j+1}} f(x), \lim\limits_{y \uparrow b_{j+1}} g(y))$ and $(\lim\limits_{x \downarrow a_j} f(x), \lim\limits_{y \downarrow b_j} g(y))$ are pairs of similar points.

Finally we prove Theorem 1.2.

Proof. Since we have different functions f_k, g_k, we introduce functions $r_1^k, r_2^k : \{0,1,\dots,m\} \to \{0,1,\dots,m\}$ serving as r_1, r_2 from Definition 3.1, i.e. such that $f_k(a_j) = [a_{r_1^k(j)}, a_{r_2^k(j)}]$ for each $j = 0,1,\dots,m$ and each $k = 0,1,2,\dots$. According to Definition 3.3 the same functions r_1^k, r_2^k are also used for g_k, i.e. $g_k(b_j) = [b_{r_1^k(j)}, b_{r_2^k(j)}]$.

For each $j = 0,1,\dots,m-1$ we define the subinterval $I_j = [a_j, a_{j+1}] \subseteq I = [a_0, a_m]$, and the subinterval $J_j = [b_j, b_{j+1}] \subseteq J = [b_0, b_m]$. We also define a piecewise linear mapping $h : I \to J$ such that $h(a_j) = b_j$ for all $j = 0,1,\dots,m$ by

$$h(x) = \begin{cases} ((b_1 - b_0)/(a_1 - a_0))(x - a_0) + b_0; & \text{if } x \in I_0, \\ ((b_2 - b_1)/(a_2 - a_1))(x - a_1) + b_1; & \text{if } x \in I_1, \\ \vdots & \\ ((b_m - b_{m-1})/(a_m - a_{m-1}))(x - a_{m-1}) + b_{m-1}; & \text{if } x \in I_{m-1}. \end{cases}$$

The mapping $h : I \to J$ is obviously continuous, monotone and surjective, therefore it is a homeomorphism.

Let $\boldsymbol{x} = (x_0, x_1, x_2, \dots)$ be any element of (I, f_n). We show first that there is a uniquely determined point $\boldsymbol{y} = (y_0, y_1, y_2, \dots)$ in (J, g_n), where $y_0 = h(x_0)$, and for all $i = 0,1,2,\dots$, $I(i)$ and $II(i)$ hold true. Here for each i, $I(i)$ and $II(i)$ are defined as the following statements:

$I(i)$ ... $x_i \in \mathrm{Int}(I_j)$ if and only if $y_i \in \mathrm{Int}(J_j)$, for each $j = 0,1,\dots,m-1$,

$II(i)$ ... $x_i = a_j$ if and only if $y_i = b_j$, for each $j = 0,1,\dots,m$.

To determine the point $\boldsymbol{y}$ we construct inductively the coordinates y_i of $\boldsymbol{y}$ as follows.

First we construct y_0 as $y_0 = h(x_0)$. It follows from the definition of h that $I(0)$ and $II(0)$ hold true.

Suppose we have already constructed $y_0, y_1, y_2, \dots, y_k$ such that $I(i)$ and $II(i)$ hold true for each $i = 0,1,\dots,k$, and $y_{i-1} \in g_{i-1}(y_i)$ holds true for each $i = 1,2,\dots,k$.

Now we construct y_{k+1} such that $I(k+1)$, $II(k+1)$, and $y_k \in g_k(y_{k+1})$. We consider the following two possible cases.

1. $x_{k+1} = a_j$ for some $j = 0,1,\dots,m$. In this case we define $y_{k+1} = b_j$. Obviously, $I(k+1)$ and $II(k+1)$ hold true. Next we show that $y_k \in g_k(y_{k+1})$. Since $x_k \in f_k(x_{k+1}) = f_k(a_j) = [a_{r_1^k(j)}, a_{r_2^k(j)}]$ for some

$a_{r_1^k(j)}, a_{r_2^k(j)} \in A$, and since g_k and f_k have the same pattern, it follows that $g_k(y_{k+1}) = g_k(b_j) = [b_{r_1^k(j)}, b_{r_2^k(j)}]$.

If $a_{r_1^k(j)} \neq a_{r_2^k(j)}$, then fix an integer ℓ_0 such that $x_k \in I_{\ell_0} \subseteq [a_{r_1^k(j)}, a_{r_2^k(j)}]$. Then $y_k \in J_{\ell_0} \subseteq [b_{r_1^k(j)}, b_{r_2^k(j)}] = g_k(y_{k+1})$. If $a_{r_1^k(j)} = a_{r_2^k(j)}$, then $x_k = a_{r_1^k(j)}$. It follows from the induction assumption $II(k)$ that $y_k = b_{r_1^k(j)}$ and therefore $y_k \in [b_{r_1^k(j)}, b_{r_2^k(j)}] = g_k(y_{k+1})$.

2. $x_{k+1} \in \operatorname{Int}(I_j)$ for some $j = 0, 1, \ldots, m-1$. In this case, since $f_k|_{\operatorname{Int}(I_j)}$ is single-valued,

$$x_k = f_k(x_{k+1}) = f_k|_{\operatorname{Int}(I_j)}(x_{k+1}) \in f_k(\operatorname{Int}(I_j)) = (a_{\ell_1}, a_{\ell_2}),$$

for some $a_{\ell_1}, a_{\ell_2} \in A$ (where $\{a_{\ell_1}, a_{\ell_2}\} = \{\lim\limits_{x \downarrow a_j} f(x), \lim\limits_{x \uparrow a_{j+1}} f(x)\}$). Therefore $y_k \in (b_{\ell_1}, b_{\ell_2}) = g_k(\operatorname{Int}(J_j))$ since f_k and g_k follow the same pattern. We choose $y_{k+1} \in \operatorname{Int}(J_j)$ such that $y_k = g_k|_{\operatorname{Int}(J_j)}(y_{k+1})$. Such a point y_{k+1} exists and is uniquely determined since $g_k|_{\operatorname{Int}(J_j)} : \operatorname{Int}(J_j) \to (b_{\ell_1}, b_{\ell_2})$ is bijective.

Next we show, that if we fix $y_0 = h(x_0)$, there is exactly one point $\boldsymbol{y}= (y_0, y_1, y_2, \ldots)$ in (J, g_n), such that for each nonnegative integer i, $I(i)$ and $II(i)$ hold true. Suppose that $\boldsymbol{y}= (y_0, y_1, y_2, \ldots)$ and $\boldsymbol{y'}= (y_0, y'_1, y'_2, \ldots) \in (J, g_n)$ are two such points. We show using induction on i that $y_i = y'_i$ for any i, hence it follows that $\boldsymbol{y} = \boldsymbol{y'}$. Suppose that for each $k = 0, 1, 2, \ldots, i-1$, $y_k = y'_k$. We prove that $y_i = y'_i$. We examine the following two cases.

1. For some $j = 0, 1, 2, \ldots, m-1$, $x_i \in \operatorname{Int}(I_j)$. Then y_i, y'_i are both in $\operatorname{Int}(J_j)$ by $I(i)$.

   Since $y_{i-1} = y'_{i-1}$, it follows that $g_{i-1}|_{\operatorname{Int}(J_j)}(y_i) = g_{i-1}(y_i) = y_{i-1} = y'_{i-1} = g_{i-1}(y'_i) = g_{i-1}|_{\operatorname{Int}(J_j)}(y'_i)$. Since $g_{i-1}|_{\operatorname{Int}(J_j)}$ is injective, it follows that $y_i = y'_i$.

2. For each $j = 0, 1, 2, \ldots, m-1$, $x_i \notin \operatorname{Int}(I_j)$. This means that $x_i = a_j$ for some $j = 0, 1, \ldots, m$. In this case $y_i = b_j = y'_i$ by $II(i)$.

Next we define a function $H : (I, f_n) \to (J, g_n)$ and prove that it is a homeomorphism.

For each $\boldsymbol{x}= (x_0, x_1, x_2, \ldots) \in (I, f_n)$ we define $H(\boldsymbol{x})$ to be the unique point $\boldsymbol{y}= (y_0, y_1, y_2, \ldots)$ in (J, g_n) such that $y_0 = h(x_0)$ and for each $i = 0, 1, 2, \ldots$, $I(i)$ and $II(i)$ hold true.

We have already seen that H is well defined. Next we show that H is continuous, by proving that for any sequence $\{\boldsymbol{x}^i\}_{i=0}^{\infty}$ in (I, f_n) converging to

$\boldsymbol{x} \in (I, f_n)$, the sequence $\{\boldsymbol{y}^i\}_{i=0}^{\infty}$, where $\boldsymbol{y}^i = H(\boldsymbol{x}^i)$ for each i, is convergent and its limit equals $H(\boldsymbol{x})$.

(I, f_n) and (J, g_n) are both compact metric spaces since I and J are compact (for details see [9]).

Let $\{\boldsymbol{x}^i\}_{i=0}^{\infty}$ be a convergent sequence of elements in (I, f_n), where $\boldsymbol{x}^i = (x_0^i, x_1^i, x_2^i, \ldots)$ for all $i = 0, 1, \ldots$. Let $\boldsymbol{x} = (x_0, x_1, x_2, \ldots)$ be the limit of this sequence. This means that x_j is the limit of the sequence $\{x_j^i\}_{i=0}^{\infty}$ for each j.

Let $\boldsymbol{s} = (s_0, s_1, s_2, \ldots) \in (J, g_n)$ be any accumulation point of the sequence $\{\boldsymbol{y}^i\}_{i=0}^{\infty}$. Let k_i be a strictly increasing sequence of nonnegative integers such that $\lim\limits_{i\to\infty} \boldsymbol{y}^{k_i} = \boldsymbol{s}$, see Figure 2.

$$(I, f_n) \ni (x_0^{k_0}, x_1^{k_0}, x_2^{k_0}, \ldots) \stackrel{H}{\longmapsto} (y_0^{k_0}, y_1^{k_0}, y_2^{k_0}, \ldots) \in (J, g_n)$$

$$(I, f_n) \ni (x_0^{k_1}, x_1^{k_1}, x_2^{k_1}, \ldots) \stackrel{H}{\longmapsto} (y_0^{k_1}, y_1^{k_1}, y_2^{k_1}, \ldots) \in (J, g_n)$$

$$(I, f_n) \ni (x_0^{k_2}, x_1^{k_2}, x_2^{k_2}, \ldots) \stackrel{H}{\longmapsto} (y_0^{k_2}, y_1^{k_2}, y_2^{k_2}, \ldots) \in (J, g_n)$$

$$\big\downarrow{\scriptstyle i\to\infty} \qquad\qquad\qquad\qquad \big\downarrow{\scriptstyle i\to\infty}$$

$$(s_0, s_1, s_2, \ldots) \in (J, g_n)$$

$$(I, f_n) \ni (x_0, x_1, x_2, \ldots) \stackrel{H}{\longmapsto} (y_0, y_1, y_2, \ldots) \in (J, g_n)$$

Figure 2: The diagram.

Let $\boldsymbol{y} = (y_0, y_1, \ldots) = H(\boldsymbol{x})$. We prove that $\boldsymbol{s} = \boldsymbol{y}$.

One can easily see, that

$$s_0 = \lim_{i\to\infty} y_0^{k_i} = \lim_{i\to\infty} h(x_0^{k_i}) = h(\lim_{i\to\infty} x_0^{k_i}) = h(x_0) = y_0.$$

Suppose we have already shown that $y_k = s_k$ for each $k = 0, 1, 2, \ldots, \ell - 1$. We show that $y_\ell = s_\ell$ by distinguishing the following cases.

1. $s_\ell \in \mathrm{Int}(J_j)$ for some $0 \le j \le m - 1$. The point s_ℓ is the limit of the sequence $\{y_\ell^{k_i}\}_{i=0}^{\infty}$. This means that there exists a nonnegative integer i_0 such that $y_\ell^{k_i} \in \mathrm{Int}(J_j)$, for all $i \ge i_0$. Therefore $x_\ell^{k_i} \in \mathrm{Int}(I_j)$, for all

$i \geq i_0$ by $I(\ell)$. Since $x_\ell = \lim_{i\to\infty} x_\ell^{k_i}$, it follows that $x_\ell \in I_j$. We consider the following two subcases.

(a) If $x_\ell \in \mathrm{Int}(I_j)$, then $y_\ell \in \mathrm{Int}(J_j)$ by $I(\ell)$. Then

$$g_{\ell-1}|_{\mathrm{Int}(J_j)}(y_\ell) = y_{\ell-1} = s_{\ell-1} = g_{\ell-1}|_{\mathrm{Int}(J_j)}(s_\ell)$$

and since $g_{\ell-1}|_{\mathrm{Int}(J_j)}$ is single-valued and injective, $y_\ell = s_\ell$ follows.

(b) If $x_\ell \in A$, then $x_\ell = a_j$ or $x_\ell = a_{j+1}$ (recall that $x_\ell \in I_j$). Without loss of generality, assume that $x_\ell = a_j$. It follows from

i. $\lim_{i\to\infty} x_\ell^{k_i} = a_j$, and a_j is the left-hand endpoint of I_j, and

ii. for all $i \geq i_0$, $x_\ell^{k_i} \in \mathrm{Int}(I_j)$ and $f_{\ell-1}$ is single-valued on $\mathrm{Int}(I_j)$,

that

$$x_{\ell-1} = \lim_{i\to\infty} x_{\ell-1}^{k_i} \overset{\text{ii.}}{=} \lim_{i\to\infty} f_{\ell-1}(x_\ell^{k_i}) \overset{\text{i.}}{=} \lim_{t\downarrow a_j} f_{\ell-1}(t) = a_r$$

where $a_r \in [a_{r_1^{\ell-1}(j)}, a_{r_2^{\ell-1}(j)}] = f_{\ell-1}(a_j)$ (recall that $f_{\ell-1}$ is a generalized Markov interval function with respect to A). Therefore, $x_{\ell-1} = a_r$ and by the definition of H, it follows that $y_{\ell-1} = b_r$. We also know that $(\lim_{t\downarrow a_j} f_{\ell-1}(t), \lim_{t\downarrow b_j} g_{\ell-1}(t))$ is a pair of similar points since $f_{\ell-1}$ and $g_{\ell-1}$ follow the same pattern and therefore $\lim_{t\downarrow b_j} g_{\ell-1}(t) = b_r$.

Using the fact that $g_{\ell-1}$ is injective on $\mathrm{Int}(J_j)$ and that $s_\ell \in \mathrm{Int}(J_j)$, we conclude that

$$s_{\ell-1} = g_{\ell-1}(s_\ell) \neq \lim_{t\downarrow b_j} g_{\ell-1}(t) = b_r = y_{\ell-1}.$$

Therefore $y_{\ell-1} \neq s_{\ell-1}$ which contradicts the inductive assumption.

2. $s_\ell = b_j$ for some $0 \leq j \leq m$.
If there exists a nonnegative integer i_1 such that $y_\ell^{k_i} = b_j$, for all $i \geq i_1$, then by $II(\ell)$, $x_\ell^{k_i} = a_j$ holds true for all $i \geq i_1$. This means that $x_\ell = a_j$, since it is the limit of the sequence $\{x_\ell^{k_i}\}_{i=0}^\infty$. Therefore, $y_\ell = b_j = s_\ell$.
If such an integer i_1 does not exist, then we consider the following two possible cases:

(a) $0 < j < m$. We chose a strictly increasing sequence of positive integers n_i, such that

- $\{y_\ell^{k_{n_i}}\}_{i=0}^{\infty}$ is a subsequence of $\{y_\ell^{k_i}\}_{i=0}^{\infty}$,
- $y_\ell^{k_{n_i}} \neq b_j$ for all i,
- $y_\ell^{k_{n_i}} \in \mathrm{Int}(J_{j-1})$ for all i or $y_\ell^{k_{n_i}} \in \mathrm{Int}(J_j)$ for all i.

Assume without loss of generality that $y_\ell^{k_{n_i}} \in \mathrm{Int}(J_j)$ for all i. Recall that $\lim\limits_{i\to\infty} y_\ell^{k_{n_i}} = s_\ell = b_j$ and that by $I(\ell)$, $x_\ell^{k_{n_i}} \in \mathrm{Int}(I_j)$ for all i. This means that $x_\ell \in I_j$ and we distinguish the following possibilities:

i. If $x_\ell \in A$, then either $x_\ell = a_j$ or $x_\ell = a_{j+1}$. One can see, using similar arguments as in 1.(b), that

$$s_{\ell-1} = \lim_{i\to\infty} y_{\ell-1}^{k_{n_i}} = \lim_{i\to\infty} g_{\ell-1}(y_\ell^{k_{n_i}}) = \lim_{t\downarrow b_j} g_{\ell-1}(t) = b_r$$

where $b_r \in [b_{r_1^{\ell-1}(j)}, b_{r_2^{\ell-1}(j)}] = g_{\ell-1}(b_j)$. By inductive assumption $y_{\ell-1} = s_{\ell-1} = b_r$. By the definition of H, it follows that $x_{\ell-1} = a_r$. Since $f_{\ell-1}$ and $g_{\ell-1}$ follow the same pattern and $f_{\ell-1}$ is injective on $Int(I_j)$ it follows that $x_{\ell-1} = a_r = \lim\limits_{t\downarrow a_j} f_{\ell-1}(t) \neq \lim\limits_{t\uparrow a_{j+1}} f_{\ell-1}(t)$. Therefore, x_ℓ cannot equal a_{j+1}, hence $x_\ell = a_j$. By the definition of H, it follows that $y_\ell = b_j = s_\ell$.

ii. If $x_\ell \in \mathrm{Int}(I_j)$, then

$$x_{\ell-1} = \lim_{i\to\infty} x_{\ell-1}^{k_{n_i}} = \lim_{i\to\infty} f_{\ell-1}(x_\ell^{k_{n_i}}) \neq \lim_{t\downarrow a_j} f_{\ell-1}(t)$$

since $f_{\ell-1}$ is single-valued and injective on $\mathrm{Int}(I_j)$. Using the same arguments as in i. we conclude that also in this case $s_{\ell-1} = b_r$ where $b_r \in [b_{r_1^{\ell-1}(j)}, b_{r_2^{\ell-1}(j)}] = g_{\ell-1}(b_j)$. Since $f_{\ell-1}$ and $g_{\ell-1}$ follow the same pattern, $\lim\limits_{t\downarrow a_j} f_{\ell-1}(t) = a_r$ follows from $\lim\limits_{t\downarrow b_j} g_{\ell-1}(t) = b_r$. Therefore $x_{\ell-1} \neq a_r$. It follows that $y_{\ell-1} \neq b_r = s_{\ell-1}$, which contradicts our inductive assumption.

(b) $j = 0$ or $j = m$. Assume without loss of generality that $j = 0$. We chose a strictly increasing sequence of positive integers n_i, such that

- $\{y_\ell^{k_{n_i}}\}_{i=0}^{\infty}$ is a subsequence of $\{y_\ell^{k_i}\}_{i=0}^{\infty}$,
- $y_\ell^{k_{n_i}} \neq b_j$ for all i,
- $y_\ell^{k_{n_i}} \in \mathrm{Int}(J_0)$ for all i.

The rest of the proof is similar to the proof of (a), replacing j with 0.

We can define $H^{-1} : (J, g_n) \to (I, f_n)$ in the same fashion as we did with H. Every element $\boldsymbol{y}= (y_0, y_1, \ldots)$ of (J, g_n) has the unique image $\boldsymbol{x}= (x_0, x_1, \ldots)$ in (I, f_n), such that $x_0 = h^{-1}(y_0)$ and for each $i = 0, 1, 2, \ldots$, $I(i)$ and $II(i)$ hold true. Therefore H is a homeomorphism. □

We conclude the paper with the following corollary that easily follows from Theorem 1.2.

Corollary 3.4. Let $f : I = [a_0, a_m] \to 2^I$ be a generalized Markov interval function with respect to $A = \{a_0, a_1, \ldots, a_m\}$ with a surjective graph and $g : J = [b_0, b_m] \to 2^J$ be a generalized Markov interval function with respect to $B = \{b_0, b_1, \ldots, b_m\}$ with a surjective graph. If f and g are generalized Markov interval functions with the same pattern, then (I, f) is homeomorphic to (J, g). □

Remark 3.5. Theorem 1.1 is a corollary of Theorem 1.2.

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References

[1] I. Banič, M. Črepnjak, M. Merhar and U. Milutinović, *Limits of inverse limits*, Topology Appl. 157 (2010), 439–450.

[2] I. Banič, M. Črepnjak, M. Merhar and U. Milutinović, *Paths through inverse limits*, Topology Appl. 158 (2011), 1099–1112.

[3] I. Banič, M. Črepnjak, M. Merhar and U. Milutinović, *Towards the complete classification of generalized tent maps inverse limits*, Topology Appl. 160 (2013), 63–73.

[4] W. J. Charatonik and R. P. Roe, *Inverse limits of continua having trivial shape*, Houston J. Math. 38 (2012), 1307–1312.

[5] A. N. Cornelius, *Weak crossovers and inverse limits of set-valued functions*, preprint (2009),

[6] S. Greenwood and J. A. Kennedy, *Generic generalized inverse limits*, Houston J. Math. 38 (2012) 1369–1384.

[7] S. Holte, *Inverse limits of Markov interval maps*, Topology Appl. 123, 2002, 421–427.

[8] A. Illanes, *A circle is not the generalized inverse limit of a subset of* $[0,1]^2$, Proc. Amer. Math. Soc. 139 (2011), 2987–2993.

[9] W. T. Ingram, W. S. Mahavier, *Inverse limits of upper semicontinuous set valued functions*, Houston J. Math. 32 (2006), 119–130.

[10] W. T. Ingram, *Inverse limits of upper semicontinuous functions that are unions of mappings*, Topology Proc. 34 (2009), 17–26.

[11] W. T. Ingram, *Inverse limits with upper semicontinuous bonding functions: problems and some partial solutions*, Topology Proc. 36 (2010), 353–373.

[12] W. S. Mahavier, *Inverse limits with subsets of* $[0,1]\times[0,1]$, Topology Appl. 141 (2004), 225–231.

[13] V. Nall, *Inverse limits with set valued functions*, Houston J. Math. 37 (2011), 1323–1332.

[14] V. Nall, *Connected inverse limits with set valued functions*, Topology Proc. 40 (2012), 167–177.

[15] V. Nall, *Finite graphs that are inverse limits with a set valued function on* $[0,1]$, Topology Appl. 158 (2011), 1226–1233.

[16] A. Palaez, *Generalized inverse limits*, Houston J. Math. 32 (2006), 1107–1119.

[17] W. Rudin, *Principles of Mathematical Analysis* , McGraw Hill Book Co., 1976.

[18] S. Varagona, *Inverse limits with upper semi-continuous bonding functions and indecomposability*, Houston J. Math. 37 (2011), 1017–1034.