Graph convergence for the $H(\cdot,\cdot)$-co-accretive mapping with an application

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Abstract

In this paper, we introduce a concept of graph convergence for the $H(\cdot,\cdot)$-co-accretive mapping in Banach spaces and prove an equivalence theorem between graph convergence and resolvent operator convergence for the $H(\cdot,\cdot)$-co-accretive mapping. Further, we consider a system of generalized variational inclusions involving $H(\cdot,\cdot)$-co-accretive mapping in real q-uniformly smooth Banach spaces. Using resolvent operator technique, we prove the existence and uniqueness of solution and suggest an iterative algorithm for the system of generalized variational inclusions under some suitable conditions. Further, we discuss the convergence of iterative algorithm using the concept of graph convergence. Our results can be viewed as a refinement and generalization of some known results in the literature.

Keywords: Graph convergence, $H(\cdot,\cdot)$-co-accretive mapping, Iterative algorithm, System of generalized variational inclusions, Convergence criteria.

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1 Introduction

Variational inclusion is an important and useful generalization of the variational inequality. One of the most interesting and important problems in the theory of variational inclusion is the development of an efficient and implementable iterative algorithm. Variational inclusions include variational, quasi-variational, variational-like inequalities as special cases. For the application of variational inclusions, see for example, [5]. Various kinds of iterative methods have been studied to find the approximate solutions for variational inclusions, among these methods, the proximal point mapping method for solving variational inclusions have been widely used by many authors for details, we refer to see, [7, 8, 11-17, 22-28, 31-33, 35-37, 40, 42, 43] and the references therein.

Recently, many authors have studied the perturbed algorithms for variational inequalities involving maximal monotone mappings in Hilbert spaces. Using the concept of graph convergence for maximal

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monotone mappings, the equivalence between graph convergence and resolvent operator convergence considered by Attouch [4], they constructed some perturbed algorithms for variational inequality and proved the convergence of sequences generated by perturbed algorithm under some suitable conditions, see for examples, [1, 9, 18-20, 30].

In 2001, Huang and Fang [22] introduce the generalize m-accretive mapping and gave the definition of proximal point mapping for the generalized m-accretive mapping in the Banach spaces. Since then many researchers investigated several class of generalized m-accretive mappings such as H-accretive, (H,η)-accretive and (A,η)-accretive mappings, see for examples, [7, 10-12, 14-17, 22, 23, 26-28, 37, 40]. Zhang et al. [41] introduced a new system of nonlinear variational inclusions with (A_i,η_i)-accretive operators in Banach spaces. By using the resolvent operator associated with (A_i,η_i)-accretive operator, they constructed a Mann iterative algorithm with errors for finding the approximate solutions of the system of nonlinear variational inclusions in Banach spaces. Sun et al. [36] introduced a new class of M-monotone mapping in Hilbert spaces. Cai and Bu [6] introduced a new iterative algorithm for finding a common element of the set of solutions of a general variational inequality problem for finite inverse-strongly accretive mappings and the set of common fixed points of a countable family of strict pseudo-contractive mappings in a Banach space. Onjai and Kumam [34] introduced a new iterative scheme for finding a common element of the set of fixed points of strict pseudo-contractions, the set of common solutions of a generalized mixed equilibrium problem and the set of common solutions of the quasi-variational inclusion in Banach spaces.

Recently, Zou and Huang [42, 43] and Kazmi et al. [28] introduced and studied a class of $H(\cdot,\cdot)$-accretive mappings in Banach spaces. Very recently, Luo and Huang [31] introduced and studied a new class of B-monotone mappings in Banach spaces, an extension of H-monotone mappings [14]. They showed some properties of the proximal point mapping associated with B-monotone mapping and obtained some applications for solving variational inclusions in Banach spaces. Very recently, Ahmad et al. [2, 3] introduced and studied the concept of $H(\cdot,\cdot)$-cocoercive and $H(\cdot,\cdot)$-η-cocoercive operators. Very recently, Li and Huang [29] studied the concept of graph convergence for the $H(\cdot,\cdot)$-accretive mapping in Banach spaces and show an application for solving a variational inclusion.

Motivated and inspired by the work above, in this paper, we introduce the concept of graph convergence for the $H(\cdot,\cdot)$-co-accretive mapping in q-uniformly smooth Banach spaces. We show an equivalence theorem between graph convergence and resolvent operator convergence for the $H(\cdot,\cdot)$-co-accretive mapping. Further, we prove the existence and uniqueness of solution and suggest an iterative algorithm for the system of generalized variational inclusions under some mild conditions. Furthermore, we discuss the convergence of iterative algorithm using the concept of graph convergence.

2 Preliminaries

Let E be a real Banach space with its norm $\|\cdot\|$, E^* be the topological dual of E, d is the metric induced by the norm $\|\cdot\|$. Let $\langle\cdot,\cdot\rangle$ be the dual pair between E and E^* and $CB(E)$(respectively 2^E) be the family of all nonempty closed and bounded subsets(respectively, all non empty subsets) of E and $\mathcal{D}(\cdot,\cdot)$ be the Häusdorff metric on $CB(E)$ defined by

$$\mathcal{D}(A,B) = \max\{ \sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\},$$

where $A, B \in CB(E)$, $d(x,B) = \inf_{y\in B} d(x,y)$ and $d(A,y) = \inf_{x\in A} d(x,y)$.

The generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \forall x \in E,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is well known that $J_q(x) = \|x\|^{q-1}J_2(x), \forall x(\neq 0) \in E$. In the sequel, we assume that E is a real Banach space such that J_q is single-valued. If E is a real Hilbert space, then J_2 becomes the identity mapping on E.

The modulus of smoothness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(t) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| \leq 1, \|y\| \leq t\right\}.$$

A Banach space E is called uniformly smooth, if

$$\lim_{t\to 0}\frac{\rho_E(t)}{t} = 0;$$

E is called q-uniformly smooth, if there exists a constant $C > 0$ such that

$$\rho_E(t) \leq Ct^q, \quad q > 1.$$

Note that J_q is single-valued, if E is uniformly smooth. In the study of characterstic inequalities in q-uniformly smooth Banach spaces, Xu [39] proved the following lemma.

Lemma 2.1. Let $q > 1$ be a real number and let E be a real smooth Banach space. Then E is q-uniformly smooth if and only if there exists a constant $C_q > 0$ such that for every $x, y \in E$,

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + C_q\|y\|^q.$$

3 $H(\cdot,\cdot)$-co-accretive mapping

Throughout the paper unless otherwise specified, we take E to be q-uniformly smooth Banach space. First, we recall the following definitions.

Definition 3.1. A mapping $A : E \to E$ is said to be

(i) accretive, if

$$\langle Ax - Ay, J_q(x-y) \rangle \geq 0, \ \forall \ x, y \in E;$$

(ii) strictly accretive, if

$$\langle Ax - Ay, J_q(x-y) \rangle > 0, \ \forall \ x, y \in E,$$

and the equality holds if and only if $x = y$;

(iii) δ-strongly accretive, if there exists a constant $\delta > 0$ such that

$$\langle Ax - Ay, J_q(x-y) \rangle \geq \delta \|x - y\|^q, \ \forall \ x, y \in E;$$

(iv) β-relaxed accretive, if there exists a constant $\beta > 0$ such that

$$\langle Ax - Ay, J_q(x-y) \rangle \geq (-\beta) \|x - y\|^q, \ \forall \ x, y \in E;$$

(iv) μ-cocoercive, if there exists a constant $\mu > 0$ such that

$$\langle Ax - Ay, J_q(x-y) \rangle \geq \mu \|Ax - Ay\|^q, \ \forall \ x, y \in E;$$

(v) γ-relaxed cocoercive, if there exists a constant $\gamma > 0$ such that

$$\langle Ax - Ay, J_q(x-y) \rangle \geq (-\gamma) \|Ax - Ay\|^q, \ \forall \ x, y \in E;$$

(vi) σ-Lipschitz continuous, if there exists a constant $\sigma > 0$ such that

$$\|Ax - Ay\| \leq \sigma \|x - y\|, \ \forall \ x, y \in E;$$

(vi) η-expansive, if there exists a constant $\eta > 0$ such that

$$\|Ax - Ay\| \geq \eta \|x - y\|, \ \forall \ x, y \in E;$$

if $\eta = 1$, then it is expansive.

Definition 3.2. A multi-valued mapping $T : E \to CB(E)$ is said to be $\mathcal{D}$-Lipschitz continuous, if there exists a constant $\lambda_{D_T} > 0$ such that

$$\mathcal{D}(Tx, Ty) \leq \lambda_{D_T} \|x - y\|, \quad \forall \ x, y \in E.$$

Definition 3.3. Let $H : E \times E \to E$ and $A, B : E \to E$ be three single-valued mappings. Then

(i) $H(A, \cdot)$ is said to be μ_1-cocoercive with respect to A, if there exists a constant $\mu_1 > 0$ such that
$$\langle H(Ax, u) - H(Ay, u), J_q(x-y) \rangle \geq \mu_1 \|Ax - Ay\|^q, \ \forall \ x, y, u \in E;$$

(ii) $H(\cdot, B)$ is said to be γ_1-relaxed cocoercive with respect to B, if there exists a constant $\gamma_1 > 0$ such that

$$\langle H(u, Bx) - H(u, By), J_q(x - y)\rangle \geq (-\gamma_1)\|Bx - By\|^q, \ \forall\, x, y, u \in E;$$

(iii) $H(A, B)$ is said to be symmetric cocoercive with respect to A and B, if $H(A, \cdot)$ is cocoercive with respect to A and $H(\cdot, B)$ is relaxed cocoercive with respect to B;

(iv) $H(A, \cdot)$ is said to be α_1-strongly accretive with respect to A, if there exists a constant $\alpha_1 > 0$ such that

$$\langle H(Ax, u) - H(Ay, u), J_q(x - y)\rangle \geq \alpha_1\|x - y\|^q, \ \forall\, x, y, u \in E;$$

(v) $H(\cdot, B)$ is said to be β_1-relaxed accretive with respect to B, if there exists a constant $\beta_1 > 0$ such that

$$\langle H(u, Bx) - H(u, By), J_q(x - y)\rangle \geq (-\beta_1)\|x - y\|^q, \ \forall\, x, y, u \in E;$$

(vi) $H(A, B)$ is said to be symmetric accretive with respect to A and B, if $H(A, \cdot)$ is strongly accretive with respect to A and $H(\cdot, B)$ is relaxed accretive with respect to B;

(vii) $H(A, \cdot)$ is said to be ξ_1-Lipschitz continuous with respect to A, if there exists a constant $\xi_1 > 0$ such that

$$\|H(Ax, u) - H(Ay, u)\| \leq \xi_1\|x - y\|, \ \forall\, x, y, u \in E;$$

(viii) $H(\cdot, B)$ is said to be ξ_2-Lipschitz continuous with respect to B, if there exists a constant $\xi_2 > 0$ such that

$$\|H(u, Bx) - H(u, By)\| \leq \xi_2\|x - y\|, \ \forall\, x, y, u \in E.$$

Definition 3.4. Let $f, g : E \to E$ be two single-valued mappings and $M : E \times E \rightrightarrows 2^E$ be a multi-valued mapping. Then

(i) $M(f, \cdot)$ is said to be α-strongly accretive with respect to f, if there exists a constant $\alpha > 0$ such that

$\langle u - v, J_q(x - y)\rangle \geq \alpha\|x - y\|^q, \ \forall\, x, y, w \in E$ and $\forall u \in M(f(x), w)$,
$v \in M(f(y), w)$;

(ii) $M(\cdot, g)$ is said to be β-relaxed accretive with respect to g, if there exists a constant $\beta > 0$ such that
$\langle u - v, J_q(x - y)\rangle \geq (-\beta)\|x - y\|^q, \ \forall\, x, y, w \in E$ and $\forall u \in M(w, g(x))$,
$v \in M(w, g(y))$;

(iii) $M(f, g)$ is said to be symmetric accretive with respect to f and g, if $M(f, \cdot)$ is strongly accretive with respect to f and $M(\cdot, g)$ is relaxed accretive with respect to g.

Now we define following $H(\cdot,\cdot)$-co-accretive mapping.

Definition 3.5. Let $A, B, f, g : E \to E$ and $H : E \times E \to E$ be single-valued mappings. Let $M : E \times E \rightrightarrows 2^E$ be a multi-valued mapping. The mapping M is said to be $H(\cdot,\cdot)$-co-accretive with respect to A, B, f and g, if $H(A, B)$ is symmetric cocoercive with respect to A and B, $M(f, g)$ is symmetric accretive with respect to f and g and $(H(A, B) + \lambda M(f, g))(E) = E$, for every $\lambda > 0$.

Example 3.1. Let $E = \mathbb{R}^2$, we define inner product by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 + x_2 y_2.$$

Let $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ be the mappings defined by

$$A(x_1, x_2) = (\frac{x_1}{2}, \frac{x_2}{3}),\ \forall\ (x_1, x_2) \in \mathbb{R}^2,$$

$$B(x_1, x_2) = (-x_1, -\frac{3}{2}x_2),\ \forall\ (x_1, x_2) \in \mathbb{R}^2.$$

Let $H : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping defined by

$$H(A(x), B(x)) = A(x) + B(x),\ \forall\ x \in \mathbb{R}^2.$$

Then for any $u \in \mathbb{R}^2$, we have

$$\begin{aligned}
\langle H(A(x), u) - H(A(y), u), x - y \rangle &= \langle A(x) - A(y), x - y \rangle \\
&= \langle (\frac{1}{2}(x_1 - y_1), \frac{1}{3}(x_2 - y_2)), (x_1 - y_1, x_2 - y_2) \rangle \\
&= \frac{1}{2}(x_1 - y_1)^2 + \frac{1}{3}(x_2 - y_2)^2,
\end{aligned}$$

and

$$\begin{aligned}
\|A(x) - A(y)\|^2 &= \langle A(x) - A(y), A(x) - A(y) \rangle \\
&= \langle (\frac{1}{2}(x_1 - y_1), \frac{1}{3}(x_2 - y_2)), (\frac{1}{2}(x_1 - y_1), \frac{1}{3}(x_2 - y_2)) \rangle \\
&= \frac{1}{4}(x_1 - y_1)^2 + \frac{1}{9}(x_2 - y_2)^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
\langle H(A(x), u) - H(A(y), u), x - y \rangle &= \frac{1}{2}(x_1 - y_1)^2 + \frac{1}{3}(x_2 - y_2)^2 \\
&= \frac{18(x_1 - y_1)^2 + 12(x_2 - y_2)^2}{36} \\
&= 2[\frac{9(x_1 - y_1)^2 + 6(x_2 - y_2)^2}{36}] \\
&\geq 2[\frac{9(x_1 - y_1)^2 + 4(x_2 - y_2)^2}{36}] \\
&= 2\|A(x) - A(y)\|^2
\end{aligned}$$

That is,

$$\langle H(A(x),u) - H(A(y),u), x-y\rangle \geq 2\|A(x)-A(y)\|^2.$$

Hence, $H(A,B)$ is 2-cocoercive with respect to A and

$$\begin{aligned}\langle H(u,B(x)) - H(u,B(y)), x-y\rangle &= \langle B(x)-B(y), x-y\rangle \\ &= \langle(-(x_1-y_1), -\frac{3}{2}(x_2-y_2)), (x_1-y_1, x_2-y_2)\rangle \\ &= -[(x_1-y_1)^2 + \frac{3}{2}(x_2-y_2)^2],\end{aligned}$$

and

$$\begin{aligned}\|B(x)-B(y)\|^2 &= \langle B(x)-B(y), B(x)-B(y)\rangle \\ &= \langle(-(x_1-y_1), -\frac{3}{2}(x_2-y_2)), (-(x_1-y_1), -\frac{3}{2}(x_2-y_2))\rangle \\ &= (x_1-y_1)^2 + \frac{9}{4}(x_2-y_2)^2,\end{aligned}$$

which implies that

$$\begin{aligned}\langle H(u,B(x)) - H(u,B(y)), x-y\rangle &= -[(x_1-y_1)^2 + \frac{3}{2}(x_2-y_2)^2] \\ &= -[\frac{4(x_1-y_1)^2 + 6(x_2-y_2)^2}{4} \\ &\geq -[\frac{4(x_1-y_1)^2 + 9(x_2-y_2)^2}{4}] \\ &= (-1)\|B(x)-B(y)\|^2.\end{aligned}$$

That is,

$$\langle H(u,B(x)) - H(u,B(y)), x-y\rangle \geq (-1)\|B(x)-B(y)\|^2.$$

Hence, $H(A,B)$ is 1-relaxed cocoercive with respect to B. Thus $H(A,B)$ is symmetric cocoercive with respect to A and B.

Now, we show symmetric accretivity of $M(f,g)$.

Let $f,g:\mathbb{R}^2 \to \mathbb{R}^2$ be the single-valued mappings such that

$$f(x_1,x_2) = (\frac{1}{3}x_1 - x_2, x_1 + \frac{1}{4}x_2),\ \forall\ (x_1,x_2)\in\mathbb{R}^2,$$
$$g(x_1,x_2) = (\frac{1}{2}x_1 + \frac{1}{2}x_2, -\frac{1}{2}x_1 + \frac{1}{3}x_2),\ \forall\ (x_1,x_2)\in\mathbb{R}^2.$$

Let $M:\mathbb{R}^2\times\mathbb{R}^2\times \to \mathbb{R}^2$ be a mapping defined by

$$M(f(x),g(x)) = f(x) - g(x).$$

Now for any $w\in\mathbb{R}^2$

$$\begin{aligned}\langle M(f(x),w) - M(f(y),w), x-y\rangle &= \langle f(x)-f(y), x-y\rangle \\ &= \langle(\frac{1}{3}(x_1-y_1) - (x_2-y_2), (x_1-y_1) + \frac{1}{4}(x_2-y_2)), \\ &\quad (x_1-y_1, x_2-y_2)\rangle \\ &= [\frac{1}{3}(x_1-y_1)^2 + \frac{1}{4}(x_2-y_2)^2],\end{aligned}$$

and

$\|x-y\|^2 = \langle (x_1-y_1, x_2-y_2), (x_1-y_1, x_2-y_2)\rangle = (x_1-y_1)^2 + (x_2-y_2)^2,$

which implies that

$$\begin{aligned}\langle M(f(x),w) - M(f(y),w), x-y\rangle &= [\frac{1}{3}(x_1-y_1)^2 + \frac{1}{4}(x_2-y_2)^2] \\ &\geq \frac{1}{4}[(x_1-y_1)^2 + (x_2-y_2)^2] \\ &= \frac{1}{4}\|x-y\|^2.\end{aligned}$$

That is,

$$\langle u-v, x-y\rangle \geq \frac{1}{4}\|x-y\|^2,\ \forall\ x,\ y \in\ \mathbb{R}^2, u \in M(f(x),w)\ \text{ and } v \in M(f(y),w).$$

Hence, $M(f,g)$ is $\frac{1}{4}$-strongly accretive with respect to f and

$$\begin{aligned}\langle M(w,g(x)) - M(w,g(y)), x-y\rangle &= -\langle g(x)-g(y), x-y\rangle \\ &= -\langle (\frac{1}{2}(x_1-y_1) + \frac{1}{2}(x_2-y_2), -\frac{1}{2}(x_1-y_1) \\ &\quad + \frac{1}{3}(x_2-y_2)), (x_1-y_1, x_2-y_2)\rangle \\ &= -[\frac{1}{2}(x_1-y_1)^2 + \frac{1}{3}(x_2-y_2)^2],\end{aligned}$$

and

$\|x-y\|^2 = \langle (x_1-y_1, x_2-y_2), (x_1-y_1, x_2-y_2)\rangle = (x_1-y_1)^2 + (x_2-y_2)^2,$

which implies that

$$\begin{aligned}\langle M(w,g(x)) - M(w,g(y)), x-y\rangle &= -[\frac{1}{2}(x_1-y_1)^2 + \frac{1}{3}(x_2-y_2)^2] \\ &\geq -\frac{1}{2}[(x_1-y_1)^2 + (x_2-y_2)^2] \\ &= -\frac{1}{2}\|x-y\|^2.\end{aligned}$$

That is,

$$\langle u-v, x-y\rangle \geq -\frac{1}{2}\|x-y\|^2,\ \forall\ x,\ y \in\ \mathbb{R}^2, u \in M(w,g(x))\ \text{ and } v \in M(w,g(y)).$$

Hence, $M(f,g)$ is $\frac{1}{2}$-relaxed accretive with respect to g. Thus $M(f,g)$ is symmetric accretive with respect to f and g. Also for any $x \in \mathbb{R}^2$, we have

$$\begin{aligned}[H(A,B) + \lambda M(f,g)](x) &= H(A(x),B(x)) + \lambda M(f(x),g(x)) \\ &= (A(x)+B(x)) + \lambda(f(x)-g(x)) \\ &= (-\frac{x_1}{2}, -\frac{7}{6}x_2) + \lambda(-\frac{1}{6}x_1 - \frac{3}{2}x_2, \frac{3}{2}x_1 - \frac{1}{12}x_2) \\ &= \big[-(\frac{1}{2}+\frac{\lambda}{6})x_1 - (\frac{3\lambda}{2})x_2,\ (\frac{3\lambda}{2})x_1 - (\frac{7}{6}+\frac{\lambda}{12})x_2\big],\end{aligned}$$

it can be easily verify that the vector on right hand side generate the whole $\mathbb{R}^2$,

$$\text{i.e.,}\ \ [H(A,B)+\lambda M(f,g)](\mathbb{R}^2)=\mathbb{R}^2,\ \ \forall\ \lambda>0.$$

Hence, M is $H(\cdot,\cdot)$-co-accretive with respect to A, B, f and g.

Example 3.2. Let $E = C[0,1]$ be the space of all real valued continuous functions defined on $[0,1]$ with the norm

$$\|x\| = \max_{t\in[0,1]} |x(t)|.$$

Let $A, B, f, g : E \to E$ be mappings defined, respectively by

$$A(x)=e^x, B(x)=e^{-x}, f(x)=x+1 \text{ and } g(x)=\sin x, \forall\ x\in E.$$

Let $H : E\times E \to E$ be a single-valued mapping defined by

$$H(A(x),B(x)) = A(x)+B(x),\ \forall\ x\in E,$$

and let $M : E\times E \to 2^E$ be a multi-valued mapping defined by

$$M(f(x),g(y)) = f(x)-g(y).$$

Then for $\lambda = 1$,

$$\begin{aligned}\|[H(A,B)+M(f,g)](x)\| &= \|A(x)+B(x)+f(x)-g(x)\|\\ &= \max_{t\in[0,1]} |e^{x(t)}+e^{-x(t)}+(x(t)+1-\sin(x(t)))| > 0,\end{aligned}$$

which implies that $0 \notin [H(A,B)+M(f,g)](E)$ and thus M is not $H(\cdot,\cdot)$-co-accretive with respect to A, B, f and g.

Note 3.1. Throughout rest of the paper, whenever we use M is $H(\cdot,\cdot)$-co-accretive, means that $H(A,B)$ is symmetric cocoercive with respect to A and B with constants μ and γ, respectively and $M(f,g)$ is symmetric accretive with respect to f and g with constants α and β, respectively.

Theorem 3.1. Let $A, B, f, g : E \to E$ and $H : E\times E \to E$ be single-valued mappings. Let $M : E\times E \rightrightarrows 2^E$ be an $H(\cdot,\cdot)$-co-accretive mapping with respect to A, B, f and g. Let A be η-expansive and B be σ-Lipschitz continuous and let $\alpha>\beta, \mu>\gamma$ and $\eta>\sigma$. Then the mapping $[H(A,B)+\lambda M(f,g)]^{-1}$ is single-valued, for all $\lambda>0$.

Proof. For any given $u\in E$, let $x, y \in [H(A,B)+\lambda M(f,g)]^{-1}(u)$. It follows that

$$\frac{1}{\lambda}(u-H(A(x),B(x))) \in M(f(x),g(x)),$$

and

$$\frac{1}{\lambda}(u - H(A(y), B(y))) \in M(f(y), g(y)).$$

Since M is $H(\cdot,\cdot)$-co-accretive with respect to A, B, f and g, we have

$$\begin{aligned}
(\alpha-\beta)\|x-y\|^q &\leq \langle \frac{1}{\lambda}(u - H(A(x), B(x))) - \frac{1}{\lambda}(u - H(A(y), B(y))), J_q(x-y)\rangle \\
&= -\frac{1}{\lambda}\langle H(A(x), B(x)) - (H(A(y), B(y))), J_q(x-y)\rangle \\
&= -\frac{1}{\lambda}\langle H(A(x), B(x)) - H(A(y), B(x)), J_q(x-y)\rangle \\
&\quad - \frac{1}{\lambda}\langle H(A(y), B(x)) - H(A(y), B(y)), J_q(x-y)\rangle \\
&\leq -\frac{\mu}{\lambda}\|A(x) - A(y)\|^q + \frac{\gamma}{\lambda}\|B(x) - B(y)\|^q. \qquad (3.1)
\end{aligned}$$

Since A is η-expansive and B is σ-Lipschitz continuous, thus (3.1) becomes

$$0 \leq (\alpha-\beta)\|x-y\|^q \leq -\frac{\mu\eta^q}{\lambda}\|x-y\|^q + \frac{\gamma\sigma^q}{\lambda}\|x-y\|^q,$$

which implies that

$$0 \leq [(\alpha-\beta) + \frac{(\mu\eta^q - \gamma\sigma^q)}{\lambda}]\|x-y\|^q \leq 0.$$

Since $\alpha > \beta, \mu > \gamma, \eta > \sigma$ and $\lambda > 0$, it follows that $x = y$ and so $[H(A, B) + \lambda M(f, g)]^{-1}$ is single-valued. This completes the proof.

Definition 3.6. Let $A, B, f, g : E \to E$ and $H : E \times E \to E$ be single-valued mappings. Let $M : E \times E \rightrightarrows 2^E$ be an $H(\cdot,\cdot)$-co-accretive mapping with respect to A, B, f and g. The resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)} : E \to E$ is defined by

$$R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) = [H(A, B) + \lambda M(f, g)]^{-1}(u),\ \forall u \in E,\ \lambda > 0.$$

Theorem 3.2. Let $A, B, f, g : E \to E$ and $H : E \times E \to E$ be single-valued mappings. Suppose $M : E \times E \rightrightarrows 2^E$ is an $H(\cdot,\cdot)$-co-accretive mapping with respect to A, B, f and g. Let A be η-expansive and B be σ-Lipschitz continuous such that $\alpha > \beta, \mu > \gamma$ and $\eta > \sigma$. Then the resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)} : E \to E$ is Lipschitz continuous with constant θ, that is,

$$\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\| \leq \theta\|u - v\|, \quad \forall\ u, v \in E \text{ and } \lambda > 0,$$

where $\theta = \dfrac{1}{\lambda(\alpha-\beta) + (\mu\eta^q - \gamma\sigma^q)}$.

Proof. Let u, v be any given points in E. It follws that

$$R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) = [H(A, B) + \lambda M(f, g)]^{-1}(u)$$

and

$$R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v) = [H(A,B) + \lambda M(f,g)]^{-1}(v)$$

and so

$$\frac{1}{\lambda}(u - H(A(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)), B(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)))) \in M(f(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)), g(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)))$$

and

$$\frac{1}{\lambda}(v - H(A(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)), B(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)))) \in M(f(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)), g(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v))).$$

Since M is symmetric accretive with respect to f and g, we have

$$\begin{aligned}
(\alpha - \beta)\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\|^q \leq \langle &\frac{1}{\lambda}(u - H(A(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)), B(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)))) \\
&- \frac{1}{\lambda}(v - H(A(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)), B(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)))), \\
&J_q(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v))\rangle \\
\leq \frac{1}{\lambda}&\langle u - v, J_q(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v))\rangle \\
&- \frac{1}{\lambda}\langle H(A(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)), B(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u))) \\
&- H(A(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)), B(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u))), \\
&J_q(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v))\rangle \\
&- \frac{1}{\lambda}\langle H(A(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)), B(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u))) \\
&- H(A(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)), B(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v))), \\
&J_q(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v))\rangle.
\end{aligned}$$

Since H is symmetric cocoercive with respect to A and B, A is η-expansive and B is σ-Lipschitz continuous, we have

$$\begin{aligned}
(\alpha - \beta)\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\|^q \leq &\frac{1}{\lambda}\langle u - v, J_q(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v))\rangle \\
&- \frac{1}{\lambda}(\mu\eta^q - \gamma\sigma^q)\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\|^q,
\end{aligned}$$

or

$$\begin{aligned}
(\alpha - \beta) + \frac{1}{\lambda}(\mu\eta^q - \gamma\sigma^q)&\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\|^q \\
&\leq \frac{1}{\lambda}\langle u - v, J_q(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v))\rangle.
\end{aligned}$$

It follows that

$$\langle u - v, J_q(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v))\rangle \geq [\lambda(\alpha - \beta) + (\mu\eta^q - \gamma\sigma^q)]\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\|^q,$$

$$\|u - v\|\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\|^{q-1} \geq [\lambda(\alpha - \beta) + (\mu\eta^q - \gamma\sigma^q)]\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\|^q.$$

That is,

$$\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) - R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\| \leq \theta\|u-v\|,$$

where $\theta = \dfrac{1}{\lambda(\alpha-\beta)+(\mu\eta^q-\gamma\sigma^q)}$. This completes the proof.

4 Graph convergence for the $H(\cdot,\cdot)$-co-accretive mapping

In this section, we study the graph convergence for the $H(\cdot,\cdot)$-co-accretive mapping.

Let $M : E\times E \rightrightarrows 2^E$ be a multi-valued mapping. The graph of M is defined by

$$graph(M) = \{((x,y),z) : z \in M(x,y)\}.$$

Definition 4.1. Let $A, B, f, g : E \to E$ and $H : E\times E \to E$ be single-valued mappings. Let $M_n, M : E\times E \rightrightarrows 2^E$ be $H(\cdot,\cdot)$-co-accretive mappings, for $n = 0, 1, 2, \cdots$. The sequence M_n is said to be graph convergence to M, denoted by $M_n \underrightarrow{G} M$, if for every $((f(x), g(x)), z) \in graph(M)$ there exists a sequence $((f(x_n), g(x_n)), z_n) \in graph(M_n)$ such that

$$f(x_n) \to f(x), g(x_n) \to g(x) \text{ and } z_n \to z \text{ as } n \to \infty.$$

Remark 4.1. If $f = I$ and $g \equiv 0$ then Definition 4.1 reduces to Definition 3.1 of [29].

Theorem 4.1. Let $M_n, M : E\times E \rightrightarrows 2^E$ be $H(\cdot,\cdot)$-co-accretive mappings with respect to A, B, f and g and $H : E\times E \to E$ be a single-valued mapping such that

(i) $H(A,B)$ is ξ_1-Lipschitz continuous with respect to A and ξ_2-Lipschitz continuous with respect to B;

(ii) f is τ-expansive mapping.

Then $M_n \underrightarrow{G} M$ if and only if

$$R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}(u) \to R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u), \forall u \in E, \lambda > 0,$$

where $R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)} = [H(A,B)+\lambda M_n(f,g)]^{-1}, R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)} = [H(A,B)+\lambda M(f,g)]^{-1}$.

Proof. Suppose that $M_n \underrightarrow{G} M$. For any given $x \in E$, let

$$z_n = R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}(x), z = R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(x).$$

It follows that $z = (H(A,B)+\lambda M(f,g))^{-1}(x)$,

$$\text{i.e., } \frac{1}{\lambda}[x - H(A(z),B(z))] \in M(f(z),g(z))$$

or

$$((f(z), g(z)), \frac{1}{\lambda}[x - H(A(z), B(z))]) \in graph(M).$$

By the definition of graph convergence, there exists a sequence $\{(f(z_n'), g(z_n')), y_n'\} \in graph(M_n)$ such that

$$f(z_n') \to f(z), g(z_n') \to g(z) \text{ and } y_n' \to \frac{1}{\lambda}[x - H(A(z), B(z))] \text{ as } n \to \infty. \tag{4.1}$$

Since $y_n' \in M_n(f(z_n'), g(z_n'))$, we have

$$H(A(z_n'), B(z_n')) + \lambda y_n' \in [H(A, B) + \lambda M_n(f, g)](z_n')$$

and so

$$z_n' = R^{H(\cdot,\cdot)}_{\lambda, M_n(\cdot,\cdot)}[H(A(z_n'), B(z_n')) + \lambda y_n'].$$

Now using the Lipschitz continuity of $R^{H(\cdot,\cdot)}_{\lambda, M_n(\cdot,\cdot)}$, we have

$$\begin{aligned}
\|z_n - z\| &\le \|z_n - z_n'\| + \|z_n' - z\| \\
&= \|R^{H(\cdot,\cdot)}_{\lambda, M_n(\cdot,\cdot)}(x) - R^{H(\cdot,\cdot)}_{\lambda, M_n(\cdot,\cdot)}[H(A(z_n'), B(z_n')) + \lambda y_n']\| + \|z_n' - z\| \\
&\le \theta\|x - H(A(z_n'), B(z_n')) - \lambda y_n'\| + \|z_n' - z\| \\
&\le \theta[\|x - H(A(z), B(z)) - \lambda y_n'\| + \|H(A(z), B(z)) - H(A(z_n'), B(z_n'))\|] \\
&\quad + \|z_n' - z\|,
\end{aligned} \tag{4.2}$$

where $\theta = \frac{1}{\lambda(\alpha-\beta)+(\mu\eta^q - \gamma\sigma^q)}$.

Since H is ξ_1-Lipschitz continuous with respect to A and ξ_2-Lipschitz continuous with respect to B, we have

$$\begin{aligned}
\|H(A(z), B(z)) - H(A(z_n'), B(z_n'))\| &\le \|H(A(z), B(z)) - H(A(z), B(z_n'))\| \\
&\quad + \|H(A(z), B(z_n')) - H(A(z_n'), B(z_n'))\| \\
&\le (\xi_1 + \xi_2)\|z - z_n'\|.
\end{aligned} \tag{4.3}$$

From (4.2) and (4.3), we have

$$\|z_n - z\| \le \theta\|x - H(A(z), B(z)) - \lambda y_n'\| + [1 + \theta(\xi_1 + \xi_2)]\|z - z_n'\|. \tag{4.4}$$

Since f is τ-expansive mapping, we have

$$\|f(z_n') - f(z)\| \ge \tau\|z_n' - z\| \ge 0. \tag{4.5}$$

Since $f(z_n') \to f(z)$ as $n \to \infty$. By (4.5), we have $z_n' \to z$ as $n \to \infty$.
Also from (4.1), $y_n' \to \frac{1}{\lambda}[x - H(A(z), B(z))]$ as $n \to \infty$.

It follows from (4.4) that

$$\|z_n - z\| \to 0 \text{ as n} \to \infty,$$

$$\text{i.e., } R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}(x) \to R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(x).$$

Conversely, suppose that

$$R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}(x) \to R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(x), \forall x \in E, \lambda > 0.$$

For any $((f(x), g(x)), y) \in graph(M)$, we have

$$y \in M(f(x), g(x)),$$

$$H(A(x), B(x)) + \lambda y \in [H(A, B) + \lambda M(f, g)](x)$$

and so

$$x = R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}[H(A(x), B(x)) + \lambda y].$$

Let $x_n = R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}[H(A(x), B(x)) + \lambda y]$, then
$\frac{1}{\lambda}[H(A(x), B(x)) - H(A(x_n), B(x_n)) + \lambda y] \in M_n(f(x_n), g(x_n))$.
Let $y_n = \frac{1}{\lambda}[H(A(x), B(x)) - H(A(x_n), B(x_n)) + \lambda y]$.
Now,

$$\begin{aligned}
\|y_n - y\| &= \|\frac{1}{\lambda}[H(A(x), B(x)) - H(A(x_n), B(x_n)) + \lambda y] - y\| \\
&= \frac{1}{\lambda}\|H(A(x), B(x)) - H(A(x_n), B(x_n))\| \\
&= \frac{(\xi_1 + \xi_2)}{\lambda}\|x_n - x\|. \qquad (4.6)
\end{aligned}$$

Since $R^{H(\cdot,\cdot)}_{\lambda,M_n(\cdot,\cdot)}(x) \to R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(x), \forall x \in E$, we know that $\|x_n - x\| \to 0$, thus (4.6) implies that $y_n \to y$ as $n \to \infty$ and so $M_n \underset{\to}{G} M$. This completes the proof.

5. System of generalized variational inclusions

Let for each $i = 1, 2, E_i$ be q_i-uniformly smooth Banach spaces with norm $\|.\|_i$. Let $A_i, B_i, f_i, g_i : E_i \to E_i$ be non linear mappings; let $F_i, H_i : E_1 \times E_2 \to E_i$ be non linear mappings; let $P_i : E_i \to E_i$ be single-valued mappings and let $Q_i : E_i \to 2^{E_i}$ be multi-valued mappings. Let $M_1 : E_1 \times E_1 \to 2^{E_1}$ be an $H_1(\cdot,\cdot)$-co-accretive mapping with respect to A_1, B_1, f_1 and g_1 and $M_2 : E_2 \times E_2 \to 2^{E_2}$ be an $H_2(\cdot,\cdot)$-co-accretive mapping with respect to A_2, B_2, f_2 and g_2. We consider the following system of generalized variational inclusions (in short SGVI):

Find $(x, y) \in E_1 \times E_2, u \in Q_1(x), v \in Q_2(y)$ such that

$$\begin{cases} 0 \in F_1(P_1(x), v) + M_1(f_1(x), g_1(x)); \\ 0 \in F_2(u, P_2(y)) + M_2(f_2(y), g_2(y)). \end{cases} \qquad (5.1)$$

Remark 5.1. For suitable choices of the mappings $A_1, A_2, B_1, B_2, f_1, f_2, g_1, g_2, G_1, G_2,$
$H_1, H_2, P_1, P_2, Q_1, Q_2, M_1, M_2$ and the spaces E_1, E_2, SGVI (5.1) reduces to various classes of system of

variational inclusions and system of variational inequalities, see for examples [10, 14-17, 21, 22-25, 27, 32, 33, 35-37, 38, 40].

Definition 5.1. A mapping $F : E_1 \times E_2 \to E_1$ is said to be (β, γ)-mixed Lipschitz continuous, if there exist constants $\beta > 0, \gamma > 0$ such that

$$\|F(x_1, y_1) - F(x_2, y_2)\|_1 \leq \beta \|x_1 - x_2\|_1 + \gamma \|y_1 - y_2\|_2, \forall\ x_1, x_2 \in E_1, y_1, y_2 \in E_2.$$

Lemma 5.1. For any $(x, y) \in E_1 \times E_2, u \in Q_1(x), v \in Q_2(y), (x, y)$ is a solution of SGVI (5.1) if and only if (x, y) satisfies

$$x = R^{H_1(\cdot,\cdot)}_{\lambda_1, M_1(\cdot,\cdot)}[H_1(A_1(x), B_1(x)) - \lambda_1 F_1(P_1(x), v)],$$

$$y = R^{H_2(\cdot,\cdot)}_{\lambda_2, M_2(\cdot,\cdot)}[H_2(A_2(y), B_2(y)) - \lambda_2 F_2(u, P_2(y))],$$

where $\lambda_1, \lambda_2 > 0$ are constants; $R^{H_1(\cdot,\cdot)}_{\lambda_1, M_1(\cdot,\cdot)}(x) \equiv [H_1(A_1, B_1) + \lambda_1 M_1(f_1, g_1)]^{-1}(x)$;
$R^{H_2(\cdot,\cdot)}_{\lambda_2, M_2(\cdot,\cdot)}(y) \equiv [H_2(A_2, B_2) + \lambda_2 M_2(f_2, g_2)]^{-1}(y), \forall\ x \in E_1, y \in E_2$.

Proof. Proof of the above lemma follows directly from the definition of resolvent operators $R^{H_1(\cdot,\cdot)}_{\lambda_1, M_1(\cdot,\cdot)}$ and $R^{H_2(\cdot,\cdot)}_{\lambda_2, M_2(\cdot,\cdot)}$.

Next, we prove the existence and uniqueness theorem for SGVI (5.1).

Theorem 5.1. Let for each $i = 1, 2, E_i$ be q_i-uniformly smooth Banach spaces with norm $\|.\|_i$. Let $A_i, B_i, f_i, g_i : E_i \to E_i$ be single-valued mappings such that A_i be η_i-expansive and B_i be σ_i-Lipschitz continuous. Let $H_i : E_1 \times E_2 \to E_i$ be symmetric cocoercive mappings with respect to A_i and B_i with constants μ_i and γ_i, respectively and (v_i, δ_i)-mixed Lipschitz continuous. Let $P_i : E_i \to E_i$ be single-valued mappings and $Q_i : E_i \to 2^{E_i}$ be $\mathcal{D}$-Lipschitz continuous multi-valued mappings with constants $\lambda_{\mathcal{D}_{Q_i}}$. Let $F_1 : E_1 \times E_2 \to E_1$ be ρ_1-strongly accretive mapping in the first argument, λ_{F_1}-Lipschitz continuous in the second argument and T_{F_1}-Lipschitz continuous with respect to P_1 in the first argument and let $F_2 : E_1 \times E_2 \to E_2$ be ρ_2-strongly accretive mapping in the second argument, λ_{F_2}-Lipschitz continuous in the first argument and S_{F_2}-Lipschitz continuous with respect to P_2 in the second argument. Let $M_1 : E_1 \times E_1 \to 2^{E_1}$ be an $H_1(\cdot,\cdot)$-co-accretive mapping with respect to A_1, B_1, f_1 and g_1 and $M_2 : E_2 \times E_2 \to 2^{E_2}$ be an $H_2(\cdot,\cdot)$-co-accretive mapping with respect to A_2, B_2, f_2 and g_2. Suppose that there exist constants $\lambda_1, \lambda_2 > 0$ satisfying

$$\begin{cases} L_1 = m_1 + \theta_2 \lambda_2 \lambda_{F_2} \lambda_{D_{Q_1}} < 1; \\ L_2 = m_2 + \theta_1 \lambda_1 \lambda_{F_1} \lambda_{D_{Q_2}} < 1, \end{cases} \tag{5.2}$$

where
$m_1 = \theta_1[\sqrt[q_1]{1 - 2q_1(\mu_1 \eta_1^{q_1} - \gamma_1 \sigma_1^{q_1}) + C_{q_1}(v_1 + \delta_1)^{q_1}} + \sqrt[q_1]{1 - 2\lambda_1 q_1 \rho_1 + C_{q_1} \lambda_1^{q_1} T_{F_1}^{q_1}}]$;
$m_2 = \theta_2[\sqrt[q_2]{1 - 2q_2(\mu_2 \eta_2^{q_2} - \gamma_2 \sigma_2^{q_2}) + C_{q_2}(v_2 + \delta_2)^{q_2}} + \sqrt[q_2]{1 - 2\lambda_2 q_2 \rho_2 + C_{q_2} \lambda_2^{q_2} S_{F_2}^{q_2}}]$;

$\theta_1 = \dfrac{1}{\lambda_1(\alpha_1-\beta_1)+(\mu_1\eta_1^q-\gamma_1\sigma_1^q)}; \theta_2 = \dfrac{1}{\lambda_2(\alpha_2-\beta_2)+(\mu_2\eta_2^q-\gamma_2\sigma_2^q)}$.
Then SGVI (5.1) has a unique solution.

Proof. For each $i = 1, 2$, it follows that for $(x, y) \in E_1 \times E_2$, the resolvent operators $R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}$ and $R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}$ are $\frac{1}{\lambda_1(\alpha_1-\beta_1)+(\mu_1\eta_1^{q_1}-\gamma_1\sigma_1^{q_1})}$ and $\frac{1}{\lambda_2(\alpha_2-\beta_2)+(\mu_2\eta_2^{q_2}-\gamma_2\sigma_2^{q_2})}$ -Lipschitz continuous respectively.

Now, we define a mapping $N : E_1 \times E_2 \to E_1 \times E_2$ by

$$N(x, y) = (T(x, y), S(x, y)), \forall\ (x, y) \in E_1 \times E_2; \tag{5.3}$$

where $T : E_1 \times E_2 \to E_1$ and $S : E_1 \times E_2 \to E_2$ are defined by

$$T(x, y) = R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}[H_1(A_1(x), B_1(x)) - \lambda_1 F_1(P_1(x), v)], \lambda_1 > 0; \tag{5.4}$$

$$S(x, y) = R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}[H_2(A_2(y), B_2(y)) - \lambda_2 F_2(u, P_2(y))], \lambda_2 > 0. \tag{5.5}$$

For any $(x_1, y_1), (x_2, y_2) \in E_1 \times E_2$, using (5.4) and (5.5) and Lipschitz continuity of $R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}$ and $R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}$, we have

$$\begin{aligned}
\|T(x_1, y_1) - T(x_2, y_2)\|_1 &= \|R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}[H_1(A_1(x_1), B_1(x_1)) - \lambda_1 F_1(P_1(x_1), v_1)] \\
&\quad - R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}[H_1(A_1(x_2), B_1(x_2)) - \lambda_1 F_1(P_1(x_2), v_2)]\|_1 \\
&\le \theta_1\|H_1(A_1(x_1), B_1(x_1)) - H_1(A_1(x_2), B_1(x_2)) \\
&\quad - \lambda_1(F_1(P_1(x_1), v_1)) - F_1(P_1(x_2), v_1)\|_1 \\
&\quad + \theta_1\lambda_1\|F_1(P_1(x_2), v_1) - F_1(P_1(x_2), v_2)\|_1 \\
&\le \theta_1\|H_1(A_1(x_1), B_1(x_1)) - H_1(A_1(x_2), B_1(x_2)) - (x_1 - x_2)\|_1 \\
&\quad + \|(x_1 - x_2) - \lambda_1(F_1(P_1(x_1), v_1)) - F_1(P_1(x_2), v_1)\|_1 \\
&\quad + \theta_1\lambda_1\|F_1(P_1(x_2), v_1) - F_1(P_1(x_2), v_2)\|_1.
\end{aligned} \tag{5.6}$$

Since H_1 is symmetric cocoercive with respect to A_1 and B_1 with constants μ_1 and γ_1, respectively and (v_1, δ_1)-mixed Lipschitz continuous, then using Lemma 2.1, we have

$$\begin{aligned}
\|H_1(A_1(x_1), B_1(x_1)) &- H_1(A_1(x_2), B_1(x_2)) - (x_1 - x_2)\|_1^{q_1} \\
&\le \|x_1 - x_2\|_1^{q_1} - 2q_1\langle H_1(A_1(x_1), B_1(x_1)) - H_1(A_1(x_2), B_1(x_2)), \\
&\quad J_{q_1}(x_1 - x_2)\rangle_1 + C_{q_1}\|H_1(A_1(x_1), B_1(x_1)) - H_1(A_1(x_2), B_1(x_2))\|_1^{q_1} \\
&\le \|x_1 - x_2\|_1^{q_1} - 2q_1(\mu_1\|A_1(x_1) - A_1(x_2)\|_1^{q_1} \\
&\quad - \gamma_1\|B_1(x_1) - B_1(x_2)\|_1^{q_1}) + C_{q_1}(v_1 + \delta_1)^{q_1}\|x_1 - x_2\|_1^{q_1}.
\end{aligned}$$

Since A_1 is η_1-expansive and B_1 is σ_1-Lipschitz continuous, we have

$$\begin{aligned}
\|H_1(A_1(x_1), B_1(x_1)) - H_1(A_1(x_2), B_1(x_2)) - (x_1 - x_2)\|_1^{q_1} \\
\le [1 - 2q_1(\mu_1\eta_1^{q_1} - \gamma_1\sigma_1^{q_1}) + C_{q_1}(v_1 + \delta_1)^{q_1}]\|x_1 - x_2\|_1^{q_1},
\end{aligned}$$

which implies that

$$\|H_1(A_1(x_1), B_1(x_1)) - H_1(A_1(x_2), B_1(x_2)) - (x_1 - x_2)\|_1$$
$$\leq \sqrt[q_1]{[1 - 2q_1(\mu_1\eta_1^{q_1} - \gamma_1\sigma_1^{q_1}) + C_{q_1}(v_1 + \delta_1)^{q_1}]}\|x_1 - x_2\|_1. \tag{5.7}$$

Again, since F_1 is ρ_1-strongly accretive and T_{F_1}-Lipschitz continuous with respect to P_1 in the first argument, then using Lemma 2.1, we have

$$\begin{aligned}
&\|(x_1 - x_2) - \lambda_1(F_1(P_1(x_1), v_1)) - F_1(P_1(x_2), v_1)\|_1^{q_1} \\
&\leq \|x_1 - x_2\|_1^{q_1} - 2\lambda_1 q_1 \langle F_1(P_1(x_1), v_1)) - F_1(P_1(x_2), v_1), \\
&\quad J_{q_1}(x_1 - x_2)\rangle_1 + C_{q_1}\lambda_1^{q_1}\|F_1(P_1(x_1), v_1)) - F_1(P_1(x_2), v_1)\|_1^{q_1} \\
&\leq (1 - 2\lambda_1 q_1\rho_1 + C_{q_1}\lambda_1^{q_1}T_{F_1}^{q_1})\|x_1 - x_2\|_1^{q_1},
\end{aligned}$$

which implies that

$$\|(x_1 - x_2) - \lambda_1(F_1(P_1(x_1), v_1)) - F_1(P_1(x_2), v_1)\|_1 \leq \sqrt[q_1]{1 - 2\lambda_1 q_1\rho_1 + C_{q_1}\lambda_1^{q_1}T_{F_1}^{q_1}}\|x_1 - x_2\|_1 \tag{5.8}$$

Also, F_1 is λ_{F_1}-Lipschitz continuous in the second argument and Q_2 is $\mathcal{D}$-Lipschitz continuous with constant $\lambda_{D_{Q_2}}$, we have

$$\begin{aligned}
\|F_1(P_1(x_2), v_1)) - F_1(P_1(x_2), v_2)\|_1 &\leq \lambda_{F_1}\|v_1 - v_2\| \leq \lambda_{F_1}\mathcal{D}(Q_2(y_1), Q_2(y_2)) \\
&\leq \lambda_{F_1}\lambda_{D_{Q_2}}\|y_1 - y_2\|_2.
\end{aligned} \tag{5.9}$$

From (5.6),(5.7),(5.8) and (5.9), we have

$$\begin{aligned}
\|T(x_1, y_1) - T(x_2, y_2)\|_1 &\leq \theta_1[\sqrt[q_1]{1 - 2q_1(\mu_1\eta_1^{q_1} - \gamma_1\sigma_1^{q_1}) + C_{q_1}(v_1 + \delta_1)^{q_1}} \\
&\quad + \sqrt[q_1]{1 - 2\lambda_1 q_1\rho_1 + C_{q_1}\lambda_1^{q_1}T_{F_1}^{q_1}}]\|x_1 - x_2\|_1 \\
&\quad + \theta_1\lambda_1\lambda_{F_1}\lambda_{D_{Q_2}}\|y_1 - y_2\|_2.
\end{aligned} \tag{5.10}$$

Now

$$\begin{aligned}
\|S(x_1, y_1) - S(x_2, y_2)\|_2 &= \|R^{H_2(\cdot,\cdot)}_{\lambda_2, M_2(\cdot,\cdot)}[H_2(A_2(y_1), B_2(y_1)) - \lambda_2 F_2(u_1, P_2(y_1))] \\
&\quad - R^{H_2(\cdot,\cdot)}_{\lambda_2, M_2(\cdot,\cdot)}[H_2(A_2(y_2), B_2(y_2)) - \lambda_2 F_2(u_2, P_2(y_2))]\|_2 \\
&\leq \theta_2\|H_2(A_2(y_1), B_2(y_1)) - H_2(A_2(y_2), B_2(y_2)) \\
&\quad - \lambda_2(F_2(u_1, P_2(y_1))) - F_2(u_1, P_2(y_2))\|_2 \\
&\quad + \theta_2\lambda_2\|F_2(u_1, P_2(y_2)) - F_2(u_2, P_2(y_2))\|_2 \\
&\leq \theta_2[\|H_2(A_2(y_1), B_2(y_1)) - H_2(A_2(y_2), B_2(y_2)) - (y_1 - y_2)\|_2 \\
&\quad + \|(y_1 - y_2) - \lambda_2(F_2(u_1, P_2(y_1))) - F_2(u_1, P_2(y_2))\|_2] \\
&\quad + \theta_2\lambda_2\|F_2(u_1, P_2(y_2)) - F_2(u_2, P_2(y_2))\|_2.
\end{aligned} \tag{5.11}$$

Since H_2 is symmetric cocoercive with respect to A_2 and B_2 with constants μ_2 and γ_2, respectively and

(v_2,δ_2)-mixed Lipschitz continuous, then using Lemma 2.1, we have

$$\begin{aligned}
\|H_2(A_2(y_1),B_2(y_1)) &- H_2(A_2(y_2),B_2(y_2)) - (y_1-y_2)\|_2^{q_2} \\
&\le \|y_1-y_2\|_2^{q_2} - 2q_2\langle H_2(A_2(y_1),B_2(y_1)) - H_2(A_2(y_2),B_2(y_2)), \\
&\quad J_{q_2}(y_1-y_2)\rangle_2 + C_{q_2}\|H_2(A_2(y_1),B_2(y_1)) - H_2(A_2(y_2),B_2(y_2))\|_2^{q_2} \\
&\le \|y_1-y_2\|_2^{q_2} - 2q_2(\mu_2\|A_2(y_1)-A_2(y_2)\|_2^{q_2} \\
&\quad - \gamma_2\|B_2(y_1)-B_2(y_2)\|_2^{q_2}) + C_{q_2}(v_2+\delta_2)^{q_2}\|y_1-y_2\|_2^{q_2}.
\end{aligned}$$

Since A_2 is η_2-expansive and B_2 is σ_2-Lipschitz continuous, we have

$$\begin{aligned}
\|H_2(A_2(y_1),B_2(y_1)) &- H_2(A_2(y_2),B_2(y_2)) - (y_1-y_2)\|_2^{q_2} \\
&\le [1-2q_2(\mu_2\eta_2^{q_2}-\gamma_2\sigma_2^{q_2}) + C_{q_2}(v_2+\delta_2)^{q_2}]\|y_1-y_2\|_2^{q_2},
\end{aligned}$$

which implies that

$$\begin{aligned}
\|H_2(A_2(y_1),B_2(y_1)) &- H_2(A_2(y_2),B_2(y_2)) - (y_1-y_2)\|_2 \\
&\le \sqrt[q_2]{[1-2q_2(\mu_2\eta_2^{q_2}-\gamma_2\sigma_2^{q_2}) + C_{q_2}(v_2+\delta_2)^{q_2}]}\|y_1-y_2\|_2. \qquad (5.12)
\end{aligned}$$

Also, F_2 is ρ_2-strongly accretive and S_{F_2}-Lipschitz continuous with respect to P_2 in the second argument, then using Lemma 2.1, we have

$$\begin{aligned}
\|(y_1-y_2) &- \lambda_2(F_2(u_1,P_2(y_1)) - F_2(u_1,P_2(y_2))\|_2^{q_2} \\
&\le \|y_1-y_2\|_2^{q_2} - 2\lambda_2 q_2\langle F_2(u_1,P_2(y_1))) - F_2(u_1,P_2(y_2)), \\
&\quad J_{q_2}(y_1-y_2)\rangle_2 + C_{q_2}\lambda_2^{q_2}\|F_2(u_1,P_2(y_1)) - F_2(u_1,P_2(y_2))\|_2^{q_2} \\
&\le (1-2\lambda_2 q_2\rho_2 + C_{q_2}\lambda_2^{q_2}S_{F_2}^{q_2})\|y_1-y_2\|_2^{q_2},
\end{aligned}$$

which implies that

$$\|(y_1-y_2)-\lambda_2(F_2(u_1,P_2(y_1))-F_2(u_1,P_2(y_2))\|_2 \le \sqrt[q_2]{1-2\lambda_2 q_2\rho_2 + C_{q_2}\lambda_2^{q_2}S_{F_2}^{q_2}}\|y_1-y_2\|_2. \qquad (5.13)$$

Also, F_2 is λ_{F_2}-Lipschitz continuous in the first argument and Q_1 is $\mathcal{D}$-Lipschitz continuous with constant $\lambda_{D_{Q_1}}$, we have

$$\begin{aligned}
\|F_2(u_1,P_2(y_2)) - F_2(u_2,P_2(y_2))\|_2 &\le \lambda_{F_2}\|u_1-u_2\|_1 \le \lambda_{F_2}\mathcal{D}(Q_1(x_1),Q_1(x_2)) \\
&\le \lambda_{F_2}\lambda_{D_{Q_1}}\|x_1-x_2\|_1. \qquad (5.14)
\end{aligned}$$

From (5.11),(5.12),(5.13) and (5.14), we have

$$\begin{aligned}
\|S(x_1,y_1) - S(x_2,y_2)\|_2 &\le \theta_2[\sqrt[q_2]{1-2q_2(\mu_2\eta_2^{q_2}-\gamma_2\sigma_2^{q_2}) + C_{q_2}(v_2+\delta_2)^{q_2}} \\
&\quad + \sqrt[q_2]{1-2\lambda_2 q_2\rho_2 + C_{q_2}\lambda_2^{q_2}S_{F_2}^{q_2}}]\|y_1-y_2\|_2 \\
&\quad + \theta_2\lambda_2\lambda_{F_2}\lambda_{D_{Q_1}}\|x_1-x_2\|_1. \qquad (5.15)
\end{aligned}$$

From (5.10) and (5.15), we have

$$\begin{aligned}
\|T(x_1,y_1) - T(x_2,y_2)\|_1 + \|S(x_1,y_1) - S(x_2,y_2)\|_2 &\le L_1\|x_1-x_2\|_1 + L_2\|y_1-y_2\|_2 \\
&\le \max\{L_1,L_2\}(\|x_1-x_2\|_1 + \|y_1-y_2\|_2), \qquad (5.16)
\end{aligned}$$

where

$$\begin{cases} L_1 = m_1 + \theta_2\lambda_2\lambda_{F_2}\lambda_{D_{Q_1}}; \\ L_2 = m_2 + \theta_1\lambda_1\lambda_{F_1}\lambda_{D_{Q_2}}, \end{cases} \tag{5.17}$$

and

$m_1 = \theta_1[\sqrt[q_1]{1 - 2q_1(\mu_1\eta_1^{q_1} - \gamma_1\sigma_1^{q_1}) + C_{q_1}(v_1 + \delta_1)^{q_1}} + \sqrt[q_1]{1 - 2\lambda_1 q_1\rho_1 + C_{q_1}\lambda_1^{q_1}T_{F_1}^{q_1}}];$
$m_2 = \theta_2[\sqrt[q_2]{1 - 2q_2(\mu_2\eta_2^{q_2} - \gamma_2\sigma_2^{q_2}) + C_{q_2}(v_2 + \delta_2)^{q_2}} + \sqrt[q_2]{1 - 2\lambda_2 q_2\rho_2 + C_{q_2}\lambda_2^{q_2}S_{F_2}^{q_2}}],$

Now, we define the norm $\|.\|_*$ on $E_1 \times E_2$ by

$$\|(x, y)\|_* = \|x\|_1 + \|y\|_2, \forall (x, y) \in E_1 \times E_2. \tag{5.18}$$

We see that $(E_1 \times E_2, \|.\|_*)$ is a Banach space. Hence, from (5.3), (5.16) and (5.18), we have

$$\|N(x_1, y_1) - N(x_2, y_2)\|_* \leq \max\{L_1, L_2\}\|(x_1, y_1) - (x_2, y_2)\|_*. \tag{5.19}$$

Since $\max\{L_1, L_2\} < 1$ by condition (5.2), then from (5.19), it follows that N is a contraction mapping. Hence by Banach Contraction Principle, there exists a unique point $(x, y) \in E_1 \times E_2$ such that

$$N(x, y) = (x, y);$$

which implies that

$$x = R^{H_1(\cdot,\cdot)}_{\lambda_1, M_1(\cdot,\cdot)}[H_1(A_1(x), B_1(x)) - \lambda_1 F_1(P_1(x), v)],$$

$$y = R^{H_2(\cdot,\cdot)}_{\lambda_2, M_2(\cdot,\cdot)}[H_2(A_2(y), B_2(y)) - \lambda_2 F_2(u, P_2(y))].$$

Then by Lemma 5.1, (x, y) is a unique solution of SGVI (5.1). This completes the proof.

For $i = 1, 2$; let $M_{in} : E_1 \times E_2 \to 2^{E_i}$ be $H_i(\cdot,\cdot)$-co-accretive mappings for $n = 0, 1, 2, \cdots$. Based on Lemma (5.1), we suggest the following iterative algorithm for finding an approximate solution for SGVI (5.1).

Algorithm 5.1. For any $(x_0, y_0) \in E_1 \times E_2$, compute $(x_n, y_n) \in E_1 \times E_2$ by the following iterative scheme:

$$x_{n+1} = R^{H_1(\cdot,\cdot)}_{\lambda_1, M_{1n}(\cdot,\cdot)}[H_1(A_1(x_n), B_1(x_n)) - \lambda_1 F_1(P_1(x_n), v_n)], \tag{5.20}$$

$$y_{n+1} = R^{H_2(\cdot,\cdot)}_{\lambda_2, M_{2n}(\cdot,\cdot)}[H_2(A_2(y_n), B_2(y_n)) - \lambda_2 F_2(u_n, P_2(y_n))], \tag{5.21}$$

where $n = 0, 1, 2, \cdots; \lambda_1, \lambda_2 > 0$ are constants.

Theorem 5.2. Let for each $i = 1, 2, A_i, B_i, f_i, g_i, H_i, F_i, P_i, Q_i, M_i$ be the same as in Theorem 5.1, $M_{in}, M_i : E_1 \times E_2 \to 2^{E_i}$ be $H_i(\cdot,\cdot)$-co-accretive mappings such that $M_{in} \underset{\to}{G} M_i$ and the condition (5.2) of Theorem 5.1 holds. Then approximate solution (x_n, y_n) generated by Algorithm 5.1 converges strongly to unique solution (x, y) of SGVI (5.1).

Proof. By Algorithm 5.1, there exists a unique solution (x, y) of SGVI (5.1). It follows from Algorithm 5.1 and Theorem 3.2 that

$$\begin{aligned}
\|x_{n+1} - x\|_1 &= \|R^{H_1(\cdot,\cdot)}_{\lambda_1, M_{1n}(\cdot,\cdot)}[H_1(A_1(x_n), B_1(x_n)) - \lambda_1 F_1(P_1(x_n), v_n)] \\
&\quad - R^{H_1(\cdot,\cdot)}_{\lambda_1, M_1(\cdot,\cdot)}[H_1(A_1(x), B_1(x)) - \lambda_1 F_1(P_1(x), v)]\|_1 \\
&\leq \|R^{H_1(\cdot,\cdot)}_{\lambda_1, M_{1n}(\cdot,\cdot)}[H_1(A_1(x_n), B_1(x_n)) - \lambda_1 F_1(P_1(x_n), v_n)] \\
&\quad - R^{H_1(\cdot,\cdot)}_{\lambda_1, M_{1n}(\cdot,\cdot)}[H_1(A_1(x), B_1(x)) - \lambda_1 F_1(P_1(x), v)]\|_1 \\
&\quad + \|R^{H_1(\cdot,\cdot)}_{\lambda_1, M_{1n}(\cdot,\cdot)}[H_1(A_1(x), B_1(x)) - \lambda_1 F_1(P_1(x), v)] \\
&\quad - R^{H_1(\cdot,\cdot)}_{\lambda_1, M_1(\cdot,\cdot)}[H_1(A_1(x), B_1(x)) - \lambda_1 F_1(P_1(x), v)]\|_1, \qquad (5.22)
\end{aligned}$$

and

$$\begin{aligned}
\|y_{n+1} - y\|_2 &= \|R^{H_2(\cdot,\cdot)}_{\lambda_2, M_{2n}(\cdot,\cdot)}[H_2(A_2(y_n), B_2(y_n)) - \lambda_2 F_2(u_n, P_2(y_n))] \\
&\quad - R^{H_2(\cdot,\cdot)}_{\lambda_2, M_2(\cdot,\cdot)}[H_2(A_2(y), B_2(y)) - \lambda_2 F_2(u, P_2(y))]\|_2 \\
&\leq \|R^{H_2(\cdot,\cdot)}_{\lambda_2, M_{2n}(\cdot,\cdot)}[H_2(A_2(y_n), B_2(y_n)) - \lambda_2 F_2(u_n, P_2(y_n))] \\
&\quad - R^{H_2(\cdot,\cdot)}_{\lambda_2, M_{2n}(\cdot,\cdot)}[H_2(A_2(y), B_2(y)) - \lambda_2 F_2(u, P_2(y))]\|_2 \\
&\quad + \|R^{H_2(\cdot,\cdot)}_{\lambda_2, M_{2n}(\cdot,\cdot)}[H_2(A_2(y), B_2(y)) - \lambda_2 F_2(u, P_2(y))] \\
&\quad - R^{H_2(\cdot,\cdot)}_{\lambda_2, M_2(\cdot,\cdot)}[H_2(A_2(y), B_2(y)) - \lambda_2 F_2(u, P_2(y))]\|_2. \qquad (5.23)
\end{aligned}$$

Following the similar arguments from (5.6)-(5.10), we have

$$\begin{aligned}
&\|R^{H_1(\cdot,\cdot)}_{\lambda_1, M_{1n}(\cdot,\cdot)}[H_1(A_1(x_n), B_1(x_n)) - \lambda_1 F_1(P_1(x_n), v_n)] - R^{H_1(\cdot,\cdot)}_{\lambda_1, M_{1n}(\cdot,\cdot)}[H_1(A_1(x), B_1(x)) \\
&\qquad - \lambda_1 F_1(P_1(x), v)]\|_1 \leq m_1 \|x_n - x\|_1 + \theta_1 \lambda_1 \lambda_{F_1} \lambda_{D_{q_2}} \|y_n - y\|_2, \qquad (5.24)
\end{aligned}$$

and using the same arguments from (5.11)-(5.15), we have

$$\begin{aligned}
&\|R^{H_2(\cdot,\cdot)}_{\lambda_2, M_{2n}(\cdot,\cdot)}[H_2(A_2(y_n), B_2(y_n)) - \lambda_2 F_2(u_n, P_2(y_n))] - R^{H_2(\cdot,\cdot)}_{\lambda_2, M_{2n}(\cdot,\cdot)}[H_2(A_2(y), B_2(y)) \\
&\qquad - \lambda_2 F_2(u, P_2(y))]\|_2 \leq m_2 \|y_n - y\|_2 + \theta_2 \lambda_2 \lambda_{F_2} \lambda_{D_{q_1}} \|x_n - x\|_1. \qquad (5.25)
\end{aligned}$$

By Theorem 4.1, we have

$$R^{H_1(\cdot,\cdot)}_{\lambda_1, M_{1n}(\cdot,\cdot)}[H_1(A_1(x), B_1(x)) - \lambda_1 F_1(P_1(x), v)] \to R^{H_1(\cdot,\cdot)}_{\lambda_1, M_1(\cdot,\cdot)}[H_1(A_1(x), B_1(x)) - \lambda_1 F_1(P_1(x), v)], \qquad (5.26)$$

and

$$R^{H_2(\cdot,\cdot)}_{\lambda_2, M_{2n}(\cdot,\cdot)}[H_2(A_2(y), B_2(y)) - \lambda_2 F_2(u, P_2(y))] \to R^{H_2(\cdot,\cdot)}_{\lambda_2, M_2(\cdot,\cdot)}[H_2(A_2(y), B_2(y)) - \lambda_2 F_2(u, P_2(y))]. \qquad (5.27)$$

Let

$$b_n = R^{H_1(\cdot,\cdot)}_{\lambda_1, M_{1n}(\cdot,\cdot)}[H_1(A_1(x), B_1(x)) - \lambda_1 F_1(P_1(x), v)] - R^{H_1(\cdot,\cdot)}_{\lambda_1, M_1(\cdot,\cdot)}[H_1(A_1(x), B_1(x)) - \lambda_1 F_1(P_1(x), v)], \qquad (5.28)$$

and

$$c_n = R^{H_2(\cdot,\cdot)}_{\lambda_2, M_{2n}(\cdot,\cdot)}[H_2(A_2(y), B_2(y)) - \lambda_2 F_2(u, P_2(y))] - R^{H_2(\cdot,\cdot)}_{\lambda_2, M_2(\cdot,\cdot)}[H_2(A_2(y), B_2(y)) - \lambda_2 F_2(u, P_2(y))]. \qquad (5.29)$$

Then

$$b_n, c_n \to 0 \text{ as } n \to \infty \qquad (5.30)$$

From (5.22),(5.23),(5.24),(5.25),(5.28) and (5.29), we have

$$\begin{aligned}\|x_{n+1}-x\|_1+\|y_{n+1}-y\|_2 &\leq L_1\|x_n-x\|_1+L_2\|y_n-y\|_2+b_n+c_n\\ &\leq \{L_1,L_2\}(\|x_n-x\|_1+\|y_n-y\|_2)+b_n+c_n. \end{aligned}\tag{5.31}$$

Since $(E_1\times E_2,\|.\|_*)$ is a Banach space defined by (5.18), then we have

$$\begin{aligned}\|(x_{n+1},y_{n+1})-(x,y)\|_* = \|(x_{n+1}-x),(y_{n+1}-y)\|_* &= \|x_{n+1}-x\|_1+\|y_{n+1}-y\|_2\\ &\leq max\{L_1,L_2\}(\|(x_n,y_n)-(x,y)\|_*)+b_n+c_n \end{aligned}\tag{5.32}$$

From (5.2) and (5.30), (5.32) implies that

$$\|(x_{n+1},y_{n+1})-(x,y)\|_*\to 0 \text{ as } n\to\infty.$$

Thus $\{(x_n,y_n)\}$ converges strongly to the unique solution (x,y) of SGVI (5.1). This completes the proof.

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