Realizability of Fault Tolerant Graphs*

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Abstract: A connected graph G is optimal-κ if the connectivity $\kappa(G) = \delta(G)$, where $\delta(G)$ is the minimum degree of G. It is super-κ if every minimum vertex cut isolates a vertex. An optimal-κ graph G is m-optimal-κ if for any vertex set $S \subseteq V(G)$ with $|S| \leq m$, $G - S$ is still optimal-κ. The maximum integer of such m, denoted by $O_\kappa(G)$, is the vertex fault tolerance of G with respect to the property of optimal-κ. The concept of vertex fault tolerance with respect to the property of super-κ, denoted by $S_\kappa(G)$, is defined in a similar way. In a previous paper, we have proved that $\min\{\kappa_1(G)-\delta(G), \delta(G)-1\} \leq O_\kappa(G) \leq \delta(G)-1$ and $\min\{\kappa_1(G) - \delta(G) - 1, \delta(G) - 1\} \leq S_\kappa(G) \leq \delta(G) - 1$. We also have $S_\kappa(G) \leq O_\kappa(G) \leq \delta(G) - 1$. In this paper, we study the realizability problems concerning with the above three bounds. By construction, we proved that for any non-negative integers a, b, c with $a \leq b \leq c$, (i) there exists a graph G such that $\kappa_1(G) - \delta(G) = a$, $O_\kappa(G) = b$, and $\delta(G) - 1 = c$; (ii) there exists a graph G with $\kappa_1(G) - \delta(G) - 1 = a$, $S_\kappa(G) = b$, and $\delta(G)-1 = c$; (iii) there exists a graph G such that $S_\kappa(G) = a$, $O_\kappa(G) = b$ and $\delta(G)-1 = c$.

Keywords: fault tolerance, maximally connected, super-connected, super connectivity, realizability.

1 Introduction

Throughout this paper, all graphs are simple and finite.

We use a simple connected graph $G = (V, E)$ to model an interconnection network, where V is the set of processors and E is the set of communication links in the network. For a vertex set $U \subseteq V(G)$, $G[U]$ is the subgraph of G induced by U; $N_G(U) = \{v \in V(G) \setminus U \mid v \text{ is adjacent to some vertex in } U\}$ is the *neighborhood* of U; $N_G[U] = N_G(U) \cup U$ is the *the closed neighborhood of* U. If U has exactly one vertex v, we use $N_G(v)$ instead of $N_G(\{v\})$ etc. If U has exactly k vertices, we say that U is a *k-set* of G. The *degree* of a vertex v in G is $d_G(v) = |N_G(v)|$. Denote by $\delta(G)$ the minimum degree of G. When the graph under consideration is obvious, we use $N(U)$, δ etc. instead of $N_G(U)$, $\delta(G)$ etc. Sometimes, we use a graph itself to represent its vertex set. For example, $N(C) = N(V(C))$, where C is a subgraph of G.

A vertex subset $S \subseteq V(G)$ is a *vertex cut* of G if $G - S$ is disconnected. The cardinality of a minimum vertex cut of a non-complete graph G is called the *connectivity* of G, denoted by $\kappa(G)$, and for complete graphs, the connectivity $\kappa(G)$ is denoted by $|V(G)| - 1$. The connectivity $\kappa(G)$ of G is an important measurement for fault-tolerance of the network. In general, the larger $\kappa(G)$ is, the more reliable the network is. Since $\kappa(G) \leq \delta(G)$, a connected graph G with $\kappa(G) = \delta(G)$ is said to be *maximally connected* (or *optimal-κ* for short), there are many studies on this subject (see a survey in [10] for example). In the design of network topology, graphs of high symmetry are often used because they usually have many desirable properties. For instance, edge transitive graphs are maximally connected. One might be interested in more refined indices of reliability. Network reliability is one of the major factors in designing the topology of an interconnection network. As more refined network reliability index than connectivity, super connectivity was proposed in [1, 2].

It is *super-connected* (*super-κ* for short) if every minimum vertex cut of G isolates a vertex. A super-κ graph is clearly optimal-κ. In recent years, there are many studies on this subject (see [9, 14] for example). It has been shown that a network is more reliable if it is super-connected [4, 5, 16]. Some important families of interconnection networks have been proven to be super-connected [4, 5, 16].

In [11], Hong and Meng first proposed the concept of edge fault tolerance for super edge connected graphs, which was generalized to vertex fault tolerance for optimal-κ and super-κ graphs in [12]. An optimal-κ (resp. super-κ) graph G is *m-optimal-κ* (resp. *m-super-κ*) if $G - S$ is still optimal-κ (resp. super-κ) for any vertex set $S \subseteq V(G)$ with $|S| \leq m$. The maximum integer of such m, denoted by $O_\kappa(G)$

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(resp. $S_\kappa(G)$), is the *vertex fault tolerance with respect to the property of optimal-κ* (resp. *super-κ*). Denote by $n_\delta(G)$ the number of vertices with degree $\delta(G)$ in G. In [12], the authors showed that

Theorem 1.1 ([12])**.** *Let G be an optimal-κ graph with minimum degree $\delta(G)$ and super connectivity $\kappa_1(G)$. Suppose $n_\delta(G) \geq \delta(G)$. Then* $\min\{\kappa_1(G) - \delta(G), \delta(G) - 1\} \leq O_\kappa(G) \leq \delta(G) - 1$.

Theorem 1.2 ([12])**.** *Let G be a super-κ graph with $n_\delta(G) \geq \delta(G)$. Then* $\min\{\kappa_1(G) - \delta(G) - 1, \delta(G) - 1\} \leq S_\kappa(G) \leq \delta(G) - 1$.

In the above two theorems, $\kappa_1(G)$ is the super connectivity of G which was first proposed by Fàbrega and Fiol [7, 8](1-extra connectivity is used in their paper). A graph is *non-trivial* if it contains at least two vertices. A *super cut* of G is a vertex cut S of G such that each component of $G - S$ is non-trivial. The *super connectivity* $\kappa_1(G)$ is the minimum cardinality of all 1-extra cuts.

By $\kappa(G) \leq \delta(G)$ and the observation that a super-κ graph is also optimal-κ, the following theorem is ready to see, without requiring $n_\delta(G) \geq \delta(G)$.

Theorem 1.3. *Let G be a super-κ graph. Then $S_\kappa(G) \leq O_\kappa(G) \leq \delta(G) - 1$.*

The following realizability problem is natural.

Open Problem 1.4. *For any integer k between the upper and the lower bounds, is there a graph G satisfying $O_\kappa(G) = k$ or $S_\kappa(G) = k$?*

This problem is exactly the one studied in this paper. In Section 2, we obtain three realizability theorems by construction (Theorems 2.1, 2.2, and 2.3). The following observation will be used frequently in the proofs.

Observation 1.5. *Let G be a connected graph. If $n_\delta(G) \geq \delta(G)$, then $\delta(G-S) \leq \delta(G)$ for any $S \subseteq V(G)$ with $|S| \leq \delta(G) - 1$.*

For more information on connectivity of graphs, we refer the reader to survey articles by Fàbrega and Fiol [6], Mader [13], Oellermann [15], and Hellwig and Volkmann [10]. For terminology not given here, we refer [3] for references.

2 Main Results

Theorem 2.1. *For any non-negative integers a, b, c with $a \leq b \leq c$, there exists a graph G such that $\kappa_1(G) - \delta(G) = a$, $O_\kappa(G) = b$ and $\delta(G) - 1 = c$.*

Proof. In the case that $a = b$, a graph G satisfying Theorem 2.1 can be constructed as in Figure 1, where B and D are two complete graphs on n_b vertices, n_b is sufficiently large; C consists of $n_c = a + c + 1$ vertices, each of which is adjacent to every vertex in $B \cup D$; A is an independent set of $n_a = c + 1$ vertices, each having degree n_a. It can be seen that $\delta(G) = n_\delta(G) = \kappa(G) = n_a$, $\kappa_1(G) = n_c$. Hence $\kappa_1(G) - \delta(G) = a$ and $\delta(G) - 1 = c$. By Theorem 1.1, $O_\kappa(G) \geq \min\{\kappa_1 - \delta, \delta - 1\} = a$. On the other hand, let S be a subset of C with $|S| = a+1$. Then $\kappa(G - S) \leq |C \setminus S| = |C| - |S| = c < n_a = \delta(G - S)$, i.e., $G - S$ is not optimal-κ. Hence $O_\kappa(G) < a + 1$. It follows that $O_\kappa(G) = a = b$.

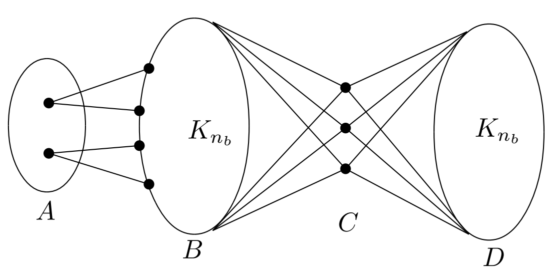

Figure 1: An illustration of graph G with $\kappa_1(G) - \delta(G) = a$, $O_\kappa(G) = b$ and $\delta(G) - 1 = c$, where $a = b$. In this example, $a = b = c = 1$.

Next, suppose $a+1 \leq b \leq c$. The constructed graph G is illustrated in Figure 2, where C is a complete graph on $a+c+1$ vertices; B is an independent set on $(2c+2) \cdot \binom{a+c+1}{b-a}$ vertices; there are $\binom{a+c+1}{b-a}$ subsets of C containing $b-a$ vertices, in which each subset corresponds to a distinct set of $(2c+2)$ vertices in B, and the edge set between them forms a complete bipartite graph (for instance, in Figure 2, $\{u_1, u_2\}$ is a subset of C containing $b-a=2$ vertices, which corresponds to the upper $2c+2=6$ vertices of B, and the edge set between them is a complete bipartite graph $K_{2,6}$; $\{u_1, u_3\}$ corresponds to the middle six vertices of B; $\{u_2, u_3\}$ corresponds to the lower six vertices of B); A is a complete graph on $a+c+1-b$ vertices, and the edge set between A and B forms a complete bipartite graph; D and E are symmetric to B and A, respectively. Next, we prove that G is as desired.

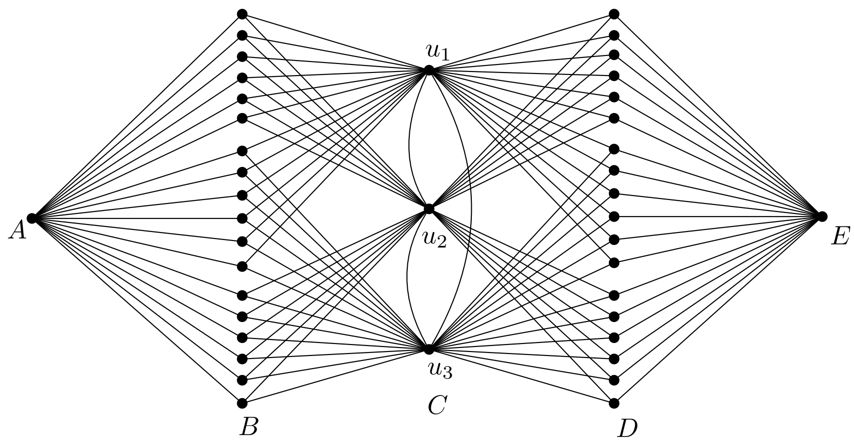

Figure 2: An illustration of graph G with $\kappa_1(G)-\delta(G)=a$, $O_\kappa(G)=b$ and $\delta(G)-1=c$, where $a+1\leq b$. In this example, $a=0$ and $b=c=2$.

(i) $\delta(G)-1=c$.

Every vertex in A and E has degree $|A|-1+|B|=a+c-b+(2c+2)\cdot\binom{a+c+1}{b-a}$. Every vertex in B and D has degree $|A|+b-a=c+1$. For any vertex $u\in C$, u is contained in $\binom{a+c}{b-a-1}$ subsets with order $b-a$. By the construction of G, u is adjacent to $(2c+2)\cdot\binom{a+c}{b-a-1}$ vertices in B. Since D is symmetric to B, we have $d_G(u)=2(2c+2)\cdot\binom{a+c}{b-a-1}+|C|-1$ for $u\in C$. Then it is easy to see that $\delta(G)=c+1$, which is reached by vertices in B and D.

From the proof of (i), we see that $n_\delta(G)=|B\cup D|=2(2c+2)\cdot\binom{a+c+1}{b-a}>c+1=\delta(G)$. Hence by Observation 1.5, we have

$$\delta(G-S)\leq\delta(G) \text{ for any } S\subseteq V(G) \text{ with } |S|\leq c. \tag{1}$$

(ii) $\kappa_1(G)-\delta(G)=a$.

Since C is a super cut of G, we have $\kappa_1(G)\leq|C|=a+c+1=a+\delta$. Suppose $\kappa_1(G)<a+\delta$, we will derive a contradiction.

Let S be a minimum super cut of G. Then $|S|=\kappa_1(G)$ and $G-S$ has at least two nontrivial components. Since C is complete, $C-S$ is contained in exactly one component of $G-S$. Thus there must exist a nontrivial component of $G-S$ disjoint from C, say X. Since X is a component of $G-S$, we have $N(X)\subseteq S$. Hence

$$|N(X)|\leq|S|=\kappa_1(G)<a+\delta=a+c+1. \tag{2}$$

Furthermore, by the structure of G, we see that X lies completely to the left or completely to the right of C. Assume, without loss of generality, that X lies to the left of C, that is,

$$X\subseteq A\cup B. \tag{3}$$

Then $X\cap A\neq\emptyset$, since otherwise $X\subseteq B$ is an independent set, contradicting the fact that X induces a non-trivial connected subgraph. It follows that

$$A\cup B\subseteq N[X]. \tag{4}$$

In fact, we can prove that

$$N[X]=A\cup B\cup C. \tag{5}$$

It is clear that $N[X] \subseteq A \cup B \cup C$. Suppose there exists a vertex $u \in C$ such that $u \notin N[X]$. Then $N(u) \cap X = \emptyset$. It follows that $X \subseteq A \cup B - N(u) \cap B$, and thus

$$|X| \leq |A \cup B| - |N(u) \cap B|. \tag{6}$$

By (4), we have $|A \cup B| \leq |X| + |N(X)|$. Combining this with (6), we have

$$|N(X)| \geq |N(u) \cap B| = (2c+2) \cdot \binom{a+c}{b-a-1} \geq 2c+2 \tag{7}$$

(recall $b \geq a+1$). Inequalities (2) and (7) yield $c+1 < a$, contradicting $a \leq c$. Hence equation (5) is proved.

By assumption (3) and equation (5), we have

$$|N(X)| = |N[X]| - |X| \geq |A \cup B \cup C| - |A \cup B| = |C| = a+c+1,$$

which contradicts inequality (2). Thus (ii) is proved.

(iii) $O_\kappa(G) = b$.

Let $S \subseteq C$ be a vertex set with $|S| = b+1$. We first show that

$$\delta(G-S) = a+c+1-b. \tag{8}$$

For each vertex $u \in A \cup E$, $d_{G-S}(u) = d_G(u) = a+c-b+(2c+2)\cdot\binom{a+c+1}{b-a} > a+c+1-b$. For each vertex $u \in C$, $d_{G-S}(u) = d_G(u) - |S| = 2(2c+2)\cdot\binom{a+c}{b-a-1} + a+c-b-1 > a+c+1-b$. For each vertex $u \in B \cup D$, when S is deleted, its degree is decreased by at most $b-a$ (since every vertex in $B \cup D$ is adjacent to exactly $b-a$ vertices of C). Furthermore, if u is in a subset of $2c+2$ vertices of B which corresponds to a $(b-a)$-subset of S (recall the construction of G and notice that $|S| = b+1 > b-a$), the degree of u is decreased by exactly $b-a$. Thus $d_{G-S}(u) \geq d_G(u) - (b-a) = a+c+1-b$ and equality can be reached. Then equation (8) follows.

Combining equation (8) with $\kappa(G-S) \leq |C-S| = a+c+1-(b+1) = a+c-b$, we see that $G-S$ is not optimal-κ. Hence

$$O_\kappa(G) \leq b. \tag{9}$$

Suppose $O_\kappa(G) < b$. Then there exists a vertex set $S \subseteq V(G)$ such that $|S| \leq b$ and $G-S$ is not optimal-κ. Let S_1 be a minimum vertex cut of $G-S$. Then $|S_1| = \kappa(G-S) < \delta(G-S)$. As a consequence, every component of $G-S-S_1$ is non-trivial. Let X be a component of $(G-S)-S_1$ such that $X \cap C = \emptyset$ (such X exists by the same reason as in the proof of (ii)). Then

$$|N_{G-S}(X)| \leq |S_1| < \delta(G-S) \leq \delta(G) = c+1, \tag{10}$$

where the second inequality follows from (1) by noticing that $|S| \leq b \leq c$. Assume that X lies to the left of C, that is,

$$X \subseteq A \cup B - S. \tag{11}$$

Similar to the proof of (5), it can be proved that

$$N_{G-S}[X] = A \cup B \cup C - S. \tag{12}$$

In fact, since X induces a non-trivial connected subgraph of G, $A \cup B - S \subseteq N_{G-S}[X]$, and if there exists a vertex $u \in C - S$ such that $u \notin N_{G-S}[X]$, then similar to the proof of (7), we have

$$|N_{G-S}(X)| \geq |N_{G-S}(u) \cap B| \geq |N_G(u) \cap B| - |S| \geq (2c+2) - b \geq c+2, \tag{13}$$

which contradicts (10).

By assumption (11) and equation (12), we have

$$|N_{G-S}(X)| = |N_{G-S}[X]| - |X| \geq |A \cup B \cup C - S| - |A \cup B - S| = |C - S|. \tag{14}$$

We consider two cases:

Case 1. $|S \cap C| \leq b-a$.

In this case, there exist $\binom{a+c+1-|S\cap C|}{(b-a)-|S\cap C|}$ distinct $(b-a)$-sets in C containing $S\cap C$. Let Y be such a $(b-a)$-set containing $S\cap C$, and $Z=N(Y)\cap B$. According to the structure of G, we see that $|Z|=2c+2>b\geq|S|$, and thus there is a vertex $u\in Z\setminus S$. Since u is adjacent to every vertex in $S\cap C$, we have

$$\delta(G-S)\leq d_{G-S}(u)\leq d_G(u)-|S\cap C|=c+1-|S\cap C|. \tag{15}$$

Combining inequalities (10), (14) and (15), we see that

$$|C|=|C-S|+|C\cap S|\leq|N_{G-S}(X)|+c+1-\delta(G-S)<c+1\leq a+c+1=|C|,$$

a contradiction.

Case 2. $|S\cap C|>b-a$.

In this case, there is a $(b-a)$-set Y contained in $S\cap C$. Let $Z=N(Y)\cap B$. Then $|Z|=2c+2>b=|S|$, and thus there is a vertex $u\subseteq Z\setminus S$. Similar to the above

$$\delta(G-S)\leq d_{G-S}(u)\leq d_G(u)-(b-a)=c+1-(b-a)=a+c+1-b. \tag{16}$$

Combining inequalities (10), (14) and (16), we have

$$a+c+1-b\geq\delta(G-S)>|N_{G-S}(X)|\geq|C-S|\geq|C|-|S|\geq a+c+1-b,$$

a contradiction.

In both cases, we derive contradictions, which implies the equality in (9). Thus (iii) is proved. □

Theorem 2.2. *For any non-negative integers a,b,c with $a\leq b\leq c$, there exists a graph G such that $\kappa_1(G)-\delta(G)-1=a$, $S_\kappa(G)=b$ and $\delta(G)-1=c$.*

Proof. The graph constructed in this theorem is similar to that in Theorem 2.1. The difference here is that C has $a+c+2$ vertices and $|B|=|D|=(2c+2)\cdot\binom{a+c+2}{b-a}$, since there are $\binom{a+c+2}{b-a}$ distinct $(b-a)$-subsets of C. The proof is also similar to that of Theorem 2.1, using super-κ instead of optimal-κ. This is feasible from the two clear observations: a graph G is optimal-κ if and only if $\kappa_1(G)\geq\delta(G)$ and is super-κ if and only if $\kappa_1(G)\geq\delta(G)+1$. □

In the following, we consider the realizability problem for Theorem 1.3

Theorem 2.3. *For any non-negative integers a,b,c with $a\leq b\leq c$, there exists a graph G such that $S_\kappa(G)=a$, $O_\kappa(G)=b$ and $\delta(G)-1=c$.*

We prove the theorem by distinguishing the case $a+1\leq b$ (Lemma 2.4) and the case $a=b$ (Lemma 2.5), since the graph G constructed differs a lot in these two cases.

Lemma 2.4. *For any non-negative integers a,b,c with $a+1\leq b\leq c$, there exists a graph G such that $S_\kappa(G)=a$, $O_\kappa(G)=b$ and $\delta(G)-1=c$.*

Proof. When $b=a+1$, the graph G is as in Theorem 2.2, taking a,b,c in Theorem 2.2 to be $a-1,a,c$ of this theorem, respectively. By Theorem 2.2, we have $S_k(G)=a$ and $\delta(G)-1=c$. By a similar argument as in the proof of (iii) in Theorem 2.1, it can be proved that $O_\kappa(G)=a+1=b$.

When $b\geq a+2$, the graph G is illustrated in Figure 3, where C is a complete graph on $c+1$ vertices; B is an independent set on $(2c+2)\cdot\binom{c+1}{b-a-1}$ vertices; there are $\binom{c+1}{b-a-1}$ subsets of C with exactly $b-a-1$ vertices (recall that $b\geq a+2$, hence such subset is not empty), each subset corresponds to $2c+2$ vertices in B, and the edge set between them forms a complete bipartite graph; A is a complete graph on $a+c+2-b$ vertices, and the edge sets between A and B forms a complete bipartite graph; D and E are symmetric to B and A, respectively; F is a complete graph on $a+1$ vertices, each of which is adjacent to every vertex in $A\cup C\cup E$.

(i) $\delta(G)-1=c$.

Let u be a vertex of G. By the construction of G, we see that

$$d_G(u)=\begin{cases} 2a+c-b+2+(2c+2)\cdot\binom{c+1}{b-a-1}, & u\in A\cup E,\\ c+1, & u\in B\cup D,\\ a+c+1+2(2c+2)\cdot\binom{c}{b-a-2}, & u\in C,\\ 3c+3a-2b+5, & u\in F.\end{cases} \tag{17}$$

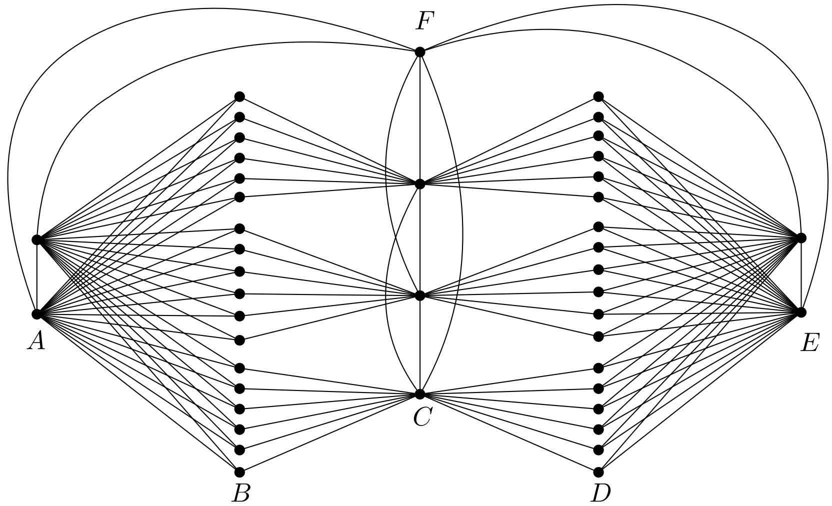

Figure 3: An illustration of graph G with $S_\kappa(G) = a$, $O_\kappa(G) = b$ and $\delta(G) - 1 = c$, where $b \geq a + 2$. In this example, $a = 0$, $b = c = 2$.

Hence $\delta(G) = c + 1$, which is reached by vertices in $B \cup D$. (i) is proved.

By the proof of (i), we have $n_\delta(G) = |B \cup D| = 2(2c + 2) \cdot \binom{c+1}{b-a-1} > c + 1 = \delta(G)$. By Observation 1.5, $\delta(G - S) \leq \delta(G)$ for any subset $S \subseteq V(G)$ with $|S| \leq c$, i.e., inequality (1) holds.

(ii) $S_\kappa(G) = a$.

By (17), we have $\delta(G - F) = c + 1$. Since $\kappa_1(G - F) \leq |C| = c + 1$, we see that $G - F$ is not super-κ. Hence $S_\kappa(G) \leq |F| - 1 = a$.

Suppose $S_\kappa(G) < a$. Then there exists a vertex set $S \subseteq V(G)$ with $|S| \leq a$ such that $G - S$ is not super-κ. Let S_1 be a minimum vertex cut of $G - S$ such that each component of $(G - S) - S_1$ is non-trivial. Let X be a non-trivial component of $G - S$ which is disjoint from $C \cup F$ (such X exists because $C \cup F$ is complete). Since $G - S$ is not super-κ, we have $|S_1| = \kappa_1(G - S) = \kappa(G - S) \leq \delta(G - S)$. Hence

$$|N_{G-S}(X)| \leq |S_1| \leq \delta(G - S) \leq \delta(G) = c + 1, \tag{18}$$

where the third inequality holds by (1).

Suppose X lies to the left of $C \cup F$, that is,

$$X \subseteq A \cup B - S. \tag{19}$$

Similar to the proof of (5), it can be proved that

$$N_{G-S}[X] = A \cup B \cup C \cup F - S. \tag{20}$$

In fact, as X induces a non-trivial connected subgraph, $A \cup B \cup F - S \subseteq N[X]$. If there is a vertex $u \in C - S$ such that $u \notin N_{G-S}[X]$, then by the same line as in proving (7), we have

$$|N_{G-S}(X)| \geq |F - S| + |N_{G-S}(u) \cap B|. \tag{21}$$

Combining this with (18), we arrive at a contradiction that

$$|F - S| \leq c + 1 - (2c + 2) \cdot \binom{c}{b - a - 2} + a \leq a - c - 1 < 0.$$

By (18), (19) and (20), we have

$$\begin{aligned} c + 1 &\geq |N_{G-S}(X)| = |N_{G-S}[X]| - |X| \\ &\geq |A \cup B \cup C \cup F - S| - |A \cup B - S| = |C \cup F - S| \\ &\geq |C| + |F| - |S| \geq c + 1 + a + 1 - a = c + 2, \end{aligned}$$

a contradiction. Thus (ii) is proved.

(iii) $O_\kappa(G) = b$.

Let S_0 be a vertex subset of C with order $b-a$. Set $S = S_0 \cup F$. Then $G - S$ is not optimal-κ, since $\delta(G-S) = \delta(G) - (b-a-1) = a+c-b+2$ (recall that each vertex in B has $b-a-1$ neighbors in C) and $\kappa(G-S) \le |C - S_0| = a+c-b+1$. Thus $O_\kappa(G) \le |S| - 1 = b$.

Suppose $O_\kappa(G) < b$. Then there exists a vertex set S with $|S| \le b$ such that $G-S$ is not optimal-κ. Let S_1 be a minimum vertex cut of $G-S$. Then

$$|S_1| = \kappa(G-S) < \delta(G-S). \tag{22}$$

As a consequence, every component of $(G-S)-S_1$ is non-trivial. Consider a non-trivial component X of $(G-S)-S_1$ which is disjoint from $C \cup F$. Suppose X lies to the left of $C \cup F$. Similar to the above, it can be proved that $N_{G-S}[X] = A \cup B \cup C \cup F - S$, and thus

$$\begin{aligned}\delta(G-S) > |S_1| &\ge |N_{G-S}(X)| = |N_{G-S}[X]| - |X| \\ &\ge |A \cup B \cup C \cup F - S| - |A \cup B - S| = |C \cup F - S| \qquad (23)\\ &\ge |C| + |F| - |S| \ge a+c-b+2. \qquad (24)\end{aligned}$$

Case 1. $|S \cap C| \le b-a-1$.

Similar to Case 1 in the proof of Theorem 2.1, by considering $(b-a-1)$-sets in C containing $S \cap C$, we have inequality (15). Combing inequalities (15), (22), and (23), we have

$$|C \cup F| = |C \cup F - S| + |C \cap S| + |F \cap S| < \delta(G-S) + \big(c+1-\delta(G-S)\big) + |F| = a+c+2,$$

which contradicts that $|C \cup F| = |C| + |F| = a+c+2$.

Case 2. $|S \cap C| > b-a-1$.

Similar to the proof of (16), it can be proved that $\delta(G-S) \le a+c-b+2$, contradicting inequality (24). □

Lemma 2.5. *For any non-negative integers a, c with $a \le c$, there exists a graph G such that $S_\kappa(G) = O_\kappa(G) = a$ and $\delta(G) - 1 = c$.*

Proof. The case $a = 0$ is illustrated in Figure 4, where D is an independent set on $c+2$ vertices, each vertex in D has $c+2$ neighbors in A; B is a complete graph on $c+1$ vertices; A and C are two complete graphs containing large enough number of vertices; the edges between B and $A \cup C \cup \{u\}$ form a complete bipartite graph.

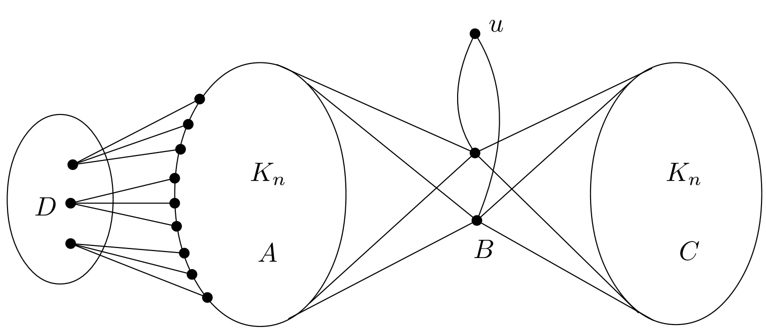

Figure 4: An illustration of graph G with $S_\kappa(G) = O_\kappa(G) = a = 0$ and $\delta(G) - 1 = c$. In this example, $c = 1$.

It is easy to see that $\kappa_1(G) = |B \cup \{u\}| = c+2$ and $\delta(G) = d(u) = c+1$. Hence G is super-κ and thus $S_\kappa(G) \ge 0$. By noticing that $G-u$ is not optimal-κ since $\kappa(G-u) = c+1$ and $\delta(G-u) = c+2$, we have $O_\kappa(G) = 0$. Hence $S_\kappa(G) = O_\kappa(G) = 0$ and $\delta(G) - 1 = c$.

In the case that $a = c$, the graph in Theorem 2.2 satisfies the requirement (by setting $b = c$).

Next, we consider the case that $1 \le a < c$. In this case $c \ge 2$. The constructed graph G is illustrated in Figure 5, where B is an independent set on c vertices; C is a complete graph on c vertices; E is a complete graph on $a+1$ vertices; A and D are two complete graphs containing large enough number of vertices. Each vertex $x \in E$ is joined to a set A_x of $\lfloor c/2 \rfloor$ vertices in A and a set D_x of $\lceil c/2 \rceil$ vertices in D. For $x, y \in E$ with $x \neq y$, $A_x \cap A_y = D_x \cap D_y = \emptyset$. The vertex u has degree $c+1$ and has at least one neighbor in each A_x. Each vertex $x \in B$ is joined to a set $\widetilde{A}_x$ of c vertices in A, and for $x, y \in B$ with $x \neq y$, $\widetilde{A}_x \cap \widetilde{A}_y = \emptyset$. The edge set between $A \cup D$ and C forms a complete bipartite graph and the edge set between B and C forms a perfect matching.

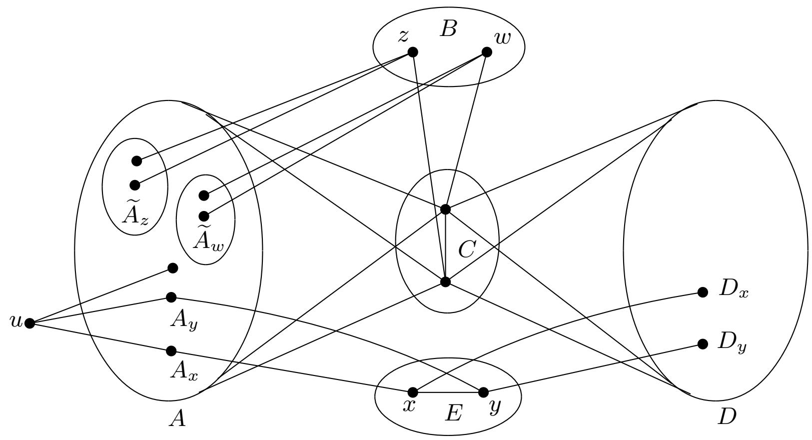

Figure 5: A graph with $S_\kappa(G) = O_\kappa(G) = a, \delta(G) - 1 = c$. In this example, $a = 1, c = 2$.

(i) $\delta(G) - 1 = c$.

Let x be a vertex of $G \setminus \{u\}$. If $x \in A$, then $d(x) \geq |A| - 1 + |C| > c + 1$. If $x \in B$, then $d(x) = c + 1$. If $x \in C$, then $d(x) = 1 + |A| + |D| + |C| - 1 > c + 1$. If $x \in D$, then $d(x) \geq |D| - 1 + |C| > c + 1$. If $x \in E$, then $d(x) = \lfloor c/2 \rfloor + \lceil c/2 \rceil + |E| - 1 = a + c \geq c + 1$. The vertex u has degree $d(u) = c + 1$. It follows that $\delta(G) = c + 1$, which is reached by vertices in $B \cup \{u\}$ and vertices in E if $a = 1$.

(ii) $S_\kappa(G) = O_\kappa(G) = a$.

Let S_0 be a minimum super cut of G. We claim that

Claim. $\kappa_1(G) = a + c + 1$, and S_0 has the structure that either $S_0 = C \cup E$, or $[S_0 \subseteq C \cup E \cup N(E)$ and $c = 2]$, or $[S_0 \subseteq C \cup E \cup (N(E) \cap A)$ and $c \leq 3]$.

Since $C \cup E$ is a super cut of G, we have

$$\kappa_1(G) \leq |C \cup E| = a + c + 1. \tag{25}$$

Since $|D|$ is sufficiently large, we have $D \not\subseteq S_0$, and thus there exists a non-trivial component X of $G - S_0$ intersecting D. We consider two cases.

Case 1. $X \cap A \neq \emptyset$.

In this case, $A \cup C \cup D \subseteq N[X] \subseteq X \cup S_0$. Let Y be another non-trivial component of $G - S_0$ disjoint from X. Then $Y \subseteq B \cup E \cup \{u\}$. But by the connectedness of Y, we can only have $Y \subseteq E$. Thus

$$\kappa_1(G) = |S_0| \geq |N(Y)| = (\lfloor c/2 \rfloor + \lceil c/2 \rceil)|Y| + |E \setminus Y| = a + 1 + (c-1)|Y|.$$

Combining this with inequality (25) and the assumption that Y is non-trivial, we have $2 \leq |Y| \leq c/(c-1)$. It follows that $c = 2$, $a = 1$ (since $1 \leq a < c$), $\kappa_1(G) = a + c + 1$, $Y = E$ (since $|Y| = 2 = a + 1 = |E|$), and $S_0 = N(Y) = N(E)$.

Case 2. $X \cap A = \emptyset$.

In this case, $X \cap C = \emptyset$, since otherwise $A \subseteq N(X)$ and thus $\kappa_1(G) = |S_0| \geq |N(X)| \geq |A| > \kappa_1(G)$, a contradiction. Then we see that $X \subseteq D \cup E$. Denote

$$F = \{v \in E \setminus X : N(v) \cap X = \emptyset\}.$$

Then $N(F) \cap D \subseteq D \setminus X$. It follows from $c \geq 2$ and

$$\begin{aligned}
a + c + 1 \quad & \geq \kappa_1(G) \geq |N(X)| \\
& = |C| + |D \setminus X| + |N(X) \cap E| + |N(X) \cap A| \\
& \geq c + |N(F) \cap D| + |E \setminus (F \cup X)| + |X \cap E| \cdot \lfloor c/2 \rfloor \\
& = c + |F| \cdot \lceil c/2 \rceil + |E| - |F| - |E \cap X| + |X \cap E| \cdot \lfloor c/2 \rfloor \\
& = a + c + 1 + |F|(\lceil c/2 \rceil - 1) + |X \cap E|(\lfloor c/2 \rfloor - 1)
\end{aligned}$$

that $\kappa_1(G) = a + c + 1$ and all the above equalities holds, which implies $N(F) \cap D = D \setminus X$ and

$$|F|(\lceil c/2 \rceil - 1) = 0 = |X \cap E|(\lfloor c/2 \rfloor - 1). \tag{26}$$

If $F = X \cap E = \emptyset$, then $X = D$ and $S_0 = N(X) = C \cup E$.

If $F = \emptyset$ and $X \cap E \neq \emptyset$, then $D \subseteq X$ and $S_0 = N(X) \subseteq E \cup (N(E) \cap A)$. Also, by (26), $c \leq 3$.

If $F \neq \emptyset$, then $S_0 = N(X) = C \cup (D \setminus X) \cup \big(N(X \cap E) \setminus X\big) = C \cup \big(N(F) \cap D\big) \cup \big(N(X \cap E) \setminus X\big) \subseteq C \cup E \cup N(E)$. In this case, by (26), $c = 2$. The claim is proved.

By noting that $G - E$ is not optimal-κ since $\kappa(G - E) = c < c + 1 = \delta(G - E)$, we have $O_\kappa(G) \leq |E| - 1 = a$. By Theorem 1.3, $S_\kappa(G) \leq O_\kappa(G) \leq a$. Next, we show that $S_\kappa(G) \geq a$.

Suppose, to the contrary, that $S_\kappa(G) < a$. Then there exists a vertex subset S with $|S| \leq a$ such that $G - S$ is not super-κ. Let S_1 be a minimum super cut of $G - S$. Then $|S_1| = \kappa_1(G - S) = \kappa(G - S) \leq \delta(G - S) \leq \delta(G) = c + 1$. Since $S \cup S_1$ is a super cut of G, we have $\kappa_1(G) \leq |S \cup S_1| = |S| + |S_1| \leq a + c + 1$. Combining this with the claim, we see that all inequalities in the deduction become equalities. In particular,

$$|S| = a, \ \delta(G - S) = \delta(G), \text{ and } \kappa_1(G) = \kappa_1(G - S) + a. \tag{27}$$

It follows that S is contained in some minimum super cut of G. Furthermore, if $N(S)$ contains some vertex of minimum degree, then $\delta(G - S)$ would be strictly less than $\delta(G)$, contradicting (25). Hence

$$N(S) \text{ does not contain any vertex of minimum degree.} \tag{28}$$

As a consequence, $C \cap S = \emptyset$ since $B \subseteq N(C)$ and every vertex in B has minimum degree. Then it follows from the claim that either $S \subseteq E$ or $[S \subseteq E \cup N(E)$ and $c = 2]$, or $[S \subseteq E \cup \big(N(E) \cap A\big)$ and $c \leq 3]$.

In fact, if $S \subseteq E$, then by $|S| = a$ (see (27)) and $|E| = a + 1$, we have $|E \setminus S| = 1$. Let x be the unique vertex in $E \setminus S$. Then $\delta(G - S) \leq d_{G-S}(x) = \lceil c/2 \rceil + \lfloor c/2 \rfloor = c < \delta(G)$, contradicting (27). So, we may further assume $S \setminus E \neq \emptyset$, and thus either $[S \cap N(E) \neq \emptyset$ and $c = 2]$ or $[S \cap \big(N(E) \cap A\big) \neq \emptyset$ and $c \leq 3]$.

If $S \cap N(E) \neq \emptyset$ and $c = 2$, then we have $a = 1$ (as $1 \leq a < c = 2$), and $|E| = 2$, and for any vertex $x \in E$, $d(x) = c + 1 = \delta(G)$. But this implies that S is adjacent to a vertex having minimum degree, contradicting (28). If $S \cap \big(N(E) \cap A\big) \neq \emptyset$ and $c \leq 3$. Then each vertex in E has exactly one neighbor in A. By the construction of G, we have $(N(E) \cap A) \subseteq N(u)$, which also implies S is adjacent to a vertex (which is u) with minimum degree, contradicting (28).

(ii) is proved. □

Theorem 2.3 follows directly from Lemma 2.4 and Lemma 2.5.

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