Applications of soft union sets in h-hemiregular and h-$intra$-hemiregular hemirings

Jianming Zhan[a], Kar Ping Shum[b,*]

[a] Department of Mathematics, Hubei University for Nationalities,

Enshi, Hubei Province, 445000, China

[b] Institute of Mathematics, Yunnan University,

Kunming, 650091, China

Abstract: The aim of this paper is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. In order to provide these soft algebraic structures, we introduce the concepts of soft union h-bi-ideals and soft union h-$quasi$-ideals of hemirings by means of soft-intersection-union sum and soft-intersection-union product. Some related properties are obtained. Finally, we investigate some characterizations of h-hemiregular and h-$intra$-hemiregular hemirings by using some kinds of soft union h-ideals.

Keywords: Soft set; soft-intersection-union(product); SU-h-bi-ideal; SU-h-$quasi$-ideal; (h-hemiregular semirings; h-intra-hemiregular) hemirings.

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1 Introduction

In order to model vagueness and uncertainty, Molodstov [25] introduced soft set theory and it has received much attention since its inception. Nowadays, many related concepts with soft sets, especially soft set operations, have undergone tremendous studies. Ali [3–5] proposed many new operations on soft sets. Maji [22] discussed further soft set theory. Sezgin [29] also investigated some new operations on soft sets. In the same time, this theory has been proven useful in many different fields such as decision making [7, 8, 10, 12, 23, 27], data analysis [32, 38], forecasting and so on. Recently, the algebraic structures of soft sets have been studied increasingly, such as, soft rings [1], soft groups [2], soft semirings [11], soft BCK/BCI-algebras [13, 15], soft ordered semigroups [14, 35], soft mappings [24], soft equality [26].

As a generalization of rings, semirings have been found useful for solving problems in different areas of applied mathematics and information sciences, since the structures of a semiring provides an algebraic framework for modelling and studying the key factors in these applied areas. We know that ideals in semirings do not in general coincide with the ideals of rings. For this reason, the usage of ideals in

E-mail address: zhanjianming@hotmail.com(J. Zhan), kpshum@ynu.edu.cn (K.P. Shum)

semirings is somewhat limited. By a hemiring, we mean a special semiring with a zero and with a commutative addition. For hemirings, for more details, see [9, 16–21, 31, 33, 34, 37].

Recently, Çagman and Sezgin proposed the concepts of soft intersection and soft union theory in algebraic structures, see [6, 28, 30]. This theory was put forward an important research direction for soft set theory. In [36], we investigated some characterizations of h-hemiregular hemirings by means of soft union left(right) h-ideals of hemirings. As a continuation of this paper, we organize the present paper as follows. In section 2, we first recall some basic definitions and results on soft sets and hemirings. Then in sections 3 and 4, we introduce the concepts of soft union h-*bi*-ideals and soft union h-*quasi*-ideals, respectively. In section 5, we investigate some characterizations of h-hemiregular hemirings by means of soft union h-*bi*-ideals(h-*quasi*-ideals). Finally, we consider some characterizations of h-*intra*-hemiregular hemirings in section 6.

2 Preliminaries

A semiring is an algebraic system $(S, +, \cdot)$ consisting of a non-empty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that $(S, +)$ and $(S, \cdot)$ are semigroups and the following distributive laws:

$a(b + c) = ab + ac$ and $(a + b)c = ac + bc$

are satisfied for all $a, b, c \in S$.

By zero of a semiring $(S, +, \cdot)$, we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = 0$ for all $x \in S$. A semiring with zero and a commutative semigroup $(S, +)$ is called a hemiring. For the sake of simplicity, we shall write ab for $a \cdot b (a, b \in S)$.

A subhemiring of a hemiring S is a subset A of S closed under addition and multiplication. A subset A of S is called a left(right) ideal of S if A is closed under addition and $SA \subseteq A (AS \subseteq A)$. A subset A is called an ideal if it is both a left ideal and a right ideal. A subset B of S is called a *bi*-ideal of S if B is closed under addition and multiplication such that $BSB \subseteq B$. A subset Q of S is called a *quasi*-ideal of S if Q is closed under addition and $SQ \cap QS \subseteq Q$.

A subhemiring(left ideal, right ideal, ideal, *bi*-ideal) A of S is called an h-subhemiring(left h-ideal, right h-ideal, h-ideal, h-*bi*-ideal) of S, respectively, if for any $x, z \in S, a, b \in A$, and $x + a + z = b + z$ implies $x \in A$.

The h-closure $\overline{A}$ of a subset A of S is defined as

$$\overline{A} = \{x \in S | x + a + z = b + z \text{ for some } a, b \in A, \ z \in S\}.$$

A *quasi*-ideal Q of S is called an h-*quasi*-ideal of S if $\overline{SQ} \cap \overline{QS} \subseteq Q$ and for any $x, z \in S$ and $a, b \in Q$ from $x + a + z = b + z$, it follows $x \in Q$.

From now on, S is a hemiring, U is an initial universe, E is a set of parameters, $P(U)$ is the power set of U and $A, B, C \subseteq E$.

Definition 2.1 [25] *A soft set f_A over U is defined as $f_A : E \to P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$. Here f_A is also called an approximate function. A soft set over U can be represented by the set of ordered pairs $f_A = \{(x, f_A(x)) | x \in E, f_A(x) \in P(U)\}$.*

It is clear to see that a soft set is a parameterized family of subsets of U. Note that the set of all soft sets over U will be defined by $S(U)$.

Definition 2.2 [7] *Let $f_A, f_B \in S(U)$. Then*

(i) f_A is called soft subset of f_B and denoted by $f_A \tilde{\subseteq} f_B$ if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.

(ii) The union of f_A and f_B, is denoted by $f_A \tilde{\cup} f_B = f_{A\cup B}$, where $f_{A\cup B}(x) = f_A(x) \cup f_B(x)$ for all $x \in E$.

(iii) The intersection of f_A and f_B, is denoted by $f_A \tilde{\cap} f_B = f_{A\cap B}$, where $f_{A\cap B}(x) = f_A(x) \cap f_B(x)$ for all $x \in E$.

(iv) The $\vee$-product of f_A and f_B, is denoted by $f_A \vee f_B = f_{A\vee B}$, where $f_{A\vee B}(x, y) = f_A(x) \cup f_B(y)$ for all $(x, y) \in E \times E$.

(v) The $\wedge$-product of f_A and f_B, is denoted by $f_A \wedge f_B = f_{A\wedge B}$, where $f_{A\wedge B}(x, y) = f_A(x) \cap f_B(y)$ for all $(x, y) \in E \times E$.

Definition 2.3 [6] *Let $f_A, f_B \in S(U)$, Ψ be a function from A to B. Then the anti-image of f_A under Ψ, denoted by $\Psi^*(f_A)$ is a soft set over U by*

$$(\Psi^*(f_A))(b) = \begin{cases} \cap\{f_A(a) | a \in A \text{ and } \Psi(a) = b\}, & if\ \Psi^{-1}(b) \neq \emptyset, \\ \emptyset & otherwise, \end{cases}$$

for all $b \in B$. And the soft pre-image of f_B under Ψ, denoted by $\Psi^{-1}(f_B)$, is a soft set over U by $(\Psi^{-1}(f_B))(a) = f_B(\Psi(a))$ for all $a \in A$.

Definition 2.4 [28] *Let $f_A \in S(U)$ and $\alpha \subseteq U$. Then, lower α-inclusion of f_A, denoted by $L(f_A; \alpha)$, is defined as $L(f_A; \alpha) = \{x \in A | f_A(x) \subseteq \alpha\}$.*

Definition 2.5 [36] *Let $f_S, g_S \in S(U)$. Then*

(1) Soft-intersection-union sum $f_S \oplus g_S$ is defined by

$$(f_S \oplus g_S)(x) = \bigcap_{\substack{x+a_1+b_1+z \\ =a_2+b_2+z}} (f_S(a_1) \cup f_S(a_2) \cup g_S(b_1) \cup g_S(b_2))$$

and $(f_S \oplus g_S)(x) = U$ if x cannot be expressed as $x + a_1 + b_1 + z = a_2 + b_2 + z$.

(2) Soft-intersection-union product $f_S \diamondsuit g_S$ is defined by

$$(f_S \diamondsuit g_S)(x) = \bigcap_{x+\sum_{i=1}^{m} a_i b_i+z=\sum_{j=1}^{n} a'_j b'_j+z} (f_S(a_i) \cup f_S(a'_j) \cup g_S(b_i) \cup g_S(b'_j))$$

for all $i = 1, 2, ..., m;\ \ j = 1, 2, ..., n$,

and $(f_S \diamondsuit g_S)(x) = U$ if x cannot be expressed as $x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z$.

Definition 2.6 [28] *Let A be a subset of S. We denote by $\mathcal{S}_{A^C}$ the soft characteristic function of the complement of A and define as*

$$\mathcal{S}_{A^C}(x) = \begin{cases} \emptyset & if\ x \in A, \\ U & if\ x \in S \backslash A. \end{cases}$$

Definition 2.7 [36](i) *A soft set f_S over U is called a soft union hemiring(briefly, SU-hemiring) of S if*

(SU_1) $f_S(x+y) \subseteq f_S(x) \cup f_S(y)$, *for all* $x, y \in S$;

(SU_2) $f_S(xy) \subseteq f_S(x) \cup f_S(y)$, *for all* $x, y \in S$;

(SU_3) $f_S(x) \subseteq f_S(a) \cup f_S(b)$ *with* $x + a + z = b + z$ *for all* $x, a, b, z \in S$.

(ii) *A soft set f_S over U is called a soft union left(right) h-ideal of S over U(briefly, SU-left(right) h-ideal) if it satisfies (SU_1), (SU_3) and*

(SU_4) $f_S(xy) \subseteq f_S(y) (f_S(xy) \subseteq f_S(x))$ *for all* $x, y \in S$.

A soft set f_S over U is called an SU-h-ideal of S over U if it is both an SU-left h-ideal and an SU-right h-ideal of S over U.

It is easy to see that if $f_S(x) = \emptyset$ for all $x \in S$, then f_S is an SU-hemiring(left h-ideal, right h-ideal, s-ideal) of S over U. We denote such a kind of SU-hemiring(left h-ideal, right h-ideal, h-ideal) by $\tilde{\theta}$ [36].

Proposition 2.8 [36] *Let $A \subseteq S$. Then A is an h-subhemiring(left h-ideal, right h-ideal, h-ideal) of S if and only if $\mathcal{S}_{A^C}$ is an SU-hemiring(left h-ideal, right h-ideal, h-ideal) of S over U.*

Theorem 2.9 [36] *Let f_S be a soft set over U. Then*

(1) f_S is an SU-hemiring of S over U if and only if it satisfies (SU_3) and

(SU_5) $f_S \oplus f_S \tilde{\supseteq} f_S$;

(SU_6) $f_S \diamondsuit f_S \tilde{\supseteq} f_S$.

(2) f_S is an SU-left(right) h-ideal of S over U if and only if it satisfies (SU_3), (SU_5) and

(SU_7) $\tilde{\theta} \diamondsuit f_S \tilde{\supseteq} f_S$ $\quad (f_S \diamondsuit \tilde{\theta} \tilde{\supseteq} f_S)$.

3 SU-h-bi-ideals

In this section, we introduce the concept of soft union h-bi-ideals and investigate some related properties.

Definition 3.1 *A soft set f_S over U is called a soft union h-bi-ideal (briefly, SU-h-bi-ideal) of S over U if it satisfies (SU_1), (SU_2), (SU_3) and*

(SU_8) $f_S(xyz) \subseteq f_S(x) \cup f_S(z)$ *for all* $x, y, z \in S$.

Example 3.2 *Assume that $U = D_2 = \{< x, y > | x^2 = y^2 = e,\ xy = yx\} = \{e, x, y, yx\}$, Dihedral group, is the universal set. Let $S = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ be the hemiring of non-negative integers modulo 4.*

Define a soft set f_S over U by $f_S(0) = \{y\}$, $f_S(1) = f_S(3) = \{e, y, yx\}$ and $f_S(2) = \{y, yx\}$.

One can easily check that f_S is an SU-h-bi-ideal of S over U.

Theorem 3.3 *Let f_S be a soft set over U. Then f_S is an SU-h-bi-ideal of S over U if and only if it satisfies (SU_3), (SU_5), (SU_6) and*

(SU_9) $f_S \diamondsuit \tilde{\theta} \diamondsuit f_S \tilde{\supseteq} f_S$

Proof. By Theorem 2.9, we know that the conditions (SU_1), (SU_2), (SU_3) are equivalent to the conditions (SU_3), (SU_5) and (SU_6).

Assume that f_S is an *SU-h-bi*-ideal of S over U. Let $x \in S$. If $(f_S \diamondsuit \tilde{\theta} \diamondsuit f_S)(x) = U$, then it is clear that $f_S \diamondsuit \tilde{\theta} \diamondsuit f_S \tilde{\supseteq} f_S$. Otherwise, let $x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z$ for all $i = 1, 2, ..., m; j = 1, 2, ..., n$. Thus,

$$
\begin{aligned}
(f_S \diamondsuit \tilde{\theta} \diamondsuit f_S)(x) &= ((f_S \diamondsuit \tilde{\theta}) \diamondsuit f_S)(x) \\
&= \bigcap_{x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} ((f_S \diamondsuit \tilde{\theta})(a_i) \cup (f_S \diamondsuit \tilde{\theta})(a'_j) \cup f_S(b_i) \cup f_S(b'_j)) \\
&= \bigcap_{\substack{x + \sum_{i=1}^{m} a_i b_i + z \\ = \sum_{j=1}^{n} a'_j b'_j + z}} \Big(\bigcap_{\substack{a_i + \sum_{k=1}^{m_i} a_{i_k} b_{i_k} + z_1 \\ = \sum_{l=1}^{n_j} a'_{j_l} b'_{j_l} + z_1}} (f_S(a_{i_k}) \cup f_S(a'_{j_l}) \cup \tilde{\theta}(b_{i_k}) \cup \tilde{\theta}(b'_{j_l})) \\
&\qquad \bigcap_{\substack{a'_j + \sum_{k=1}^{m_i} a_{i_k} b_{i_k} + z_2 \\ = \sum_{l=1}^{n_j} a'_{j_l} b'_{j_l} + z_2}} (f_S(a_{i_k}) \cup f_S(a'_{j_l}) \cup \tilde{\theta}(b_{i_k}) \cup \tilde{\theta}(b'_{j_l})) \cup f_S(b_i) \cup f_S(b'_j)) \\
&= \bigcap_{x + \sum_{i=1}^{m'} a_i c_i b_i + z' = \sum_{j=1}^{n} a'_j c'{}_j b'_j + z'} (f_S(a_i) \cup f_S(a'_j) \cup f_S(b_i) \cup f_S(b'_j)) \\
&\supseteq \bigcap_{x + \sum_{i=1}^{m'} a_i c_i b_i + z' = \sum_{j=1}^{n} a'_j c'_j b'_j + z'} (f_S(\sum_{i=1}^{m'} a_i c_i b_i) \cup f_S(\sum_{j=1}^{n} a'_j c'_j b'_j)) \\
&\supseteq \bigcap_{x + \sum_{i=1}^{m'} a_i c_i b_i + z' = \sum_{j=1}^{n} a'_j c'_j b'_j + z'} (f_S(x)) \\
&= f_S(x),
\end{aligned}
$$

which implies, $f_S \diamondsuit \tilde{\theta} \diamondsuit f_S \tilde{\supseteq} f_S$. This proves that ($SU_9$) holds.

Conversely, assume that the given conditions hold. Let $x, y, z \in S$, we have

$$
\begin{aligned}
f_S(xyz) &\subseteq (f_S\diamondsuit\tilde{\theta}\diamondsuit f_S)(xyz) = (f_S\diamondsuit(\tilde{\theta}\diamondsuit f_S))(xyz)\\
&= \bigcap_{xy+\sum\limits_{i=1}^{m} a_ib_i+z'=\sum\limits_{j=1}^{n} a'_jb'_j+z'} (f_S(a_i)\cup f_S(a'_j)\cup(\tilde{\theta}\diamondsuit f_S)(b_i)\cup(\tilde{\theta}\diamondsuit f_S)(b'_j))\\
&\subseteq f_S(0)\cup f_S(x)\cup(\tilde{\theta}\diamondsuit f_S)(0)\cup(\tilde{\theta}\diamondsuit f_S)(yz)\\
&= f_S(x)\cup \bigcap_{yz+\sum\limits_{i=1}^{m} c_id_i+z''=\sum\limits_{j=1}^{n} c'_jd'_j+z''} (\tilde{\theta}(c_i)\cup\tilde{\theta}(c'_j)\cup f_S(d_i)\cup f_S(d'_j))\\
&= f_S(x)\cup\tilde{\theta}(0)\cup\tilde{\theta}(y)\cup f_S(0)\cup f_S(z)\\
&= f_S(x)\cup f_S(z).
\end{aligned}
$$

Thus, (SU_8) holds. This proves that f_S is an *SU*-*h*-*bi*-ideal of S over U. $\square$

The following proposition is obvious.

Proposition 3.4 *A non-empty subset A of S is an h-bi-ideal of S if and only if the soft subset f_S defined by*

$$f_S(x) = \begin{cases} \alpha & if\ x\in S\backslash A,\\ \beta & if\ x\in S,\end{cases}$$

is an SU-h-bi-ideal of S over U, where $\alpha,\beta\subseteq U$ such that $\alpha\supseteq\beta$.

Corollary 3.5 *Let A be a non-empty subset of S. Then A is an h-bi-ideal of S if and only if $\mathcal{S}_{A^C}$ is an SU-h-bi-ideal of S over U.*

Theorem 3.6 *(i) Let f_S be a soft set over U and $\alpha\subseteq U$ such that $\alpha\in \mathrm{I}_m(f_S)$. If f_S is an SU-h-bi-ideal of S over U, then $L(f_S;\alpha)$ is an h-bi-ideal of S.*

(ii) Let f_S be a soft set over U, $L(f_S;\alpha)$ a lower h-bi-ideal of f_S for each $\alpha\subseteq U$ and $\mathrm{I}_m(f_S)$ an ordered set by inclusion. Then f_S is an h-bi-ideal of S over U.

Proof. (i) Since $f_S(x)=\alpha$ for some $x\in S$, $\emptyset = L(f_S;\alpha)\subseteq S$. Let $x,z\in L(f_S;\alpha)$ and $y\in S$, then $f_S(x)\subseteq\alpha$ and $f_S(z)\subseteq\alpha$. Then

$$f_S(x+z)\subseteq f_S(x)\cup f_S(z)\subseteq\alpha\cup\alpha=\alpha,$$
$$f_S(xz)\subseteq f_S(x)\cup f_S(z)\subseteq\alpha\cup\alpha=\alpha,$$
$$f_S(xyz)\subseteq f_S(x)\cup f_S(z)\subseteq\alpha\cup\alpha=\alpha,$$

which implies, $x+z$, xz and $xyz\in L(f_S;\alpha)$.

Now, let $x,z\in S$ and $a,b\in L(f_S;\alpha)$ with $x+a+z=b+z$, then $f_S(a)\subseteq\alpha$ and $f_S(b)\subseteq\alpha$. Thus $f_S(x)\subseteq f_S(a)\cup f_S(b)\subseteq\alpha\cup\alpha=\alpha$, which implies, $x\in L(f_S;\alpha)$. Hence, $L(f_S;\alpha)$ is an *h*-*bi*-ideal of S.

(ii) Let $x,y,z\in S$ be such that $f_S(x)=\alpha_1$ and $f_S(z)=\alpha_2$, where $\alpha_1\subseteq\alpha_2$. Then $x\in L(f_S;\alpha_1)$ and $z\in L(f_S;\alpha_2)$, and so $x\in L(f_S;\alpha_2)$. Since $L(f_S;\alpha_2)$ is an *h*-*bi*-ideal of S for any $\alpha\subseteq U$, $x+z,xz,xyz\in L(f_S;\alpha_2)$. Hence, we have the following equalities

$$f_S(x+z)\subseteq\alpha_2\subseteq\alpha_1\cup\alpha_2=f_S(x)\cup f_S(z),$$
$$f_S(xz)\subseteq\alpha_2\subseteq\alpha_1\cup\alpha_2=f_S(x)\cup f_S(z),$$
$$f_S(xyz)\subseteq\alpha_2\subseteq\alpha_1\cup\alpha_2=f_S(x)\cup f_S(z).$$

Now, let $x, z, a, b \in S$ with $x+a+z = b+z$ be such that $f_S(a) = \alpha_1$ and $f_S(b) = \alpha_2$, where $\alpha_1 \subseteq \alpha_2$, then $a \in L(f_S; \alpha_1)$ and $b \in L(f_S; \alpha_2)$ and so $a \in L(f_S; \alpha_2)$. Since $L(f_S; \alpha_2)$ is an h-*bi*-ideal of S for each $\alpha \subseteq U$, $x \in L(f_S; \alpha_2)$. Then $f_S(x) \subseteq \alpha_2 \subseteq \alpha_1 \cup \alpha_2 = f_S(a) \cup f_S(b)$. Therefore, f_S is an SU-h-*bi*-ideal of S over U. □

Proposition 3.7 *Let f_{S_1} and f_{S_2} be two SU-h-bi-ideals over U. Then so is $f_{S_1} \vee f_{S_2}$ over U.*

Proof. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in S_1 \times S_2$. Then

$$
\begin{aligned}
(i) f_{S_1 \vee S_2}((x_1, y_1) + (x_2, y_2)) &= f_{S_1 \vee S_2}(x_1 + x_2, y_1 + y_2) \\
&= f_{S_1}(x_1 + x_2) \cup f_{S_2}(y_1 + y_2) \\
&\subseteq (f_{S_1}(x_1) \cup f_{S_1}(x_2)) \cup (f_{S_2}(y_1) \cup f_{S_2}(y_2)) \\
&= (f_{S_1}(x_1) \cup f_{S_2}(y_1)) \cup (f_{S_1}(x_2) \cup f_{S_2}(y_2)) \\
&= f_{S_1 \vee S_2}(x_1, y_1) \cup f_{S_1 \vee S_2}(x_2, y_2)
\end{aligned}
$$

$$
\begin{aligned}
(ii) f_{S_1 \vee S_2}((x_1, y_1)(x_2, y_2)) &= f_{S_1 \vee S_2}(x_1 x_2, y_1 y_2) \\
&= f_{S_1}(x_1 x_2) \cup f_{S_2}(y_1 y_2) \\
&\subseteq (f_{S_1}(x_1) \cup f_{S_1}(x_2)) \cup (f_{S_2}(y_1) \cup f_{S_2}(y_2)) \\
&= (f_{S_1}(x_1) \cup f_{S_2}(y_1)) \cup (f_{S_1}(x_2) \cup f_{S_2}(y_2)) \\
&= f_{S_1 \vee S_2}(x_1, y_1) \cup f_{S_1 \vee S_2}(x_2, y_2)
\end{aligned}
$$

$$
\begin{aligned}
(iii) f_{S_1 \vee S_2}((x_1, y_1)(x_2, y_2)(x_3, y_3)) &= f_{S_1 \vee S_2}(x_1 x_2 x_3, y_1 y_2 y_3) \\
&= f_{S_1}(x_1 x_2 x_3) \cup f_{S_2}(y_1 y_2 y_3) \\
&\subseteq (f_{S_1}(x_1) \cup f_{S_1}(x_3)) \cup (f_{S_2}(y_1) \cup f_{S_2}(y_3)) \\
&= (f_{S_1}(x_1) \cup f_{S_2}(y_1)) \cup (f_{S_1}(x_3) \cup f_{S_2}(y_3)) \\
&= f_{S_1 \vee S_2}(x_1, y_1) \cup f_{S_1 \vee S_2}(x_3, y_3)
\end{aligned}
$$

(iv) Let $(x_1, y_1), (z_1, z_2), (a_1, a_2), (b_1, b_2) \in S_1 \times S_2$ be such that $(x_1, y_1) + (a_1, a_2) + (z_1, z_2) = (b_1, b_2) + (z_1, z_2)$, and so $x_1 + a_1 + z_1 = b_1 + z_1$ and $y_1 + a_2 + z_2 = b_2 + z_2$. Then

$$
\begin{aligned}
f_{S_1 \vee S_2}(x_1, y_1) &= f_{S_1}(x_1) \cup f_{S_2}(y_1) \\
&\subseteq (f_{S_1}(a_1) \cup f_{S_1}(b_2)) \cup (f_{S_2}(a_2) \cup f_{S_2}(b_2)) \\
&= (f_{S_1}(a_1) \cup f_{S_2}(a_2)) \cup (f_{S_1}(b_1) \cup f_{S_2}(b_2)) \\
&= f_{S_1 \vee S_2}(a_1, a_2) \cup f_{S_1 \vee S_2}(b_1, b_2)
\end{aligned}
$$

Thus, $f_{S_1} \vee f_{S_2}$ is an SU-h-*bi*-ideal of $S_1 \times S_2$ over U. □

Remark 3.8 *Note that if f_{S_1} and f_{S_2} are two SU-h-bi-ideals over U, then $f_{S_1} \wedge f_{S_2}$ is not always an SU-h-bi-ideal over U as shown in the following example:*

Example 3.9 *Assume that $U = S_4$, symmetric group, is the universal set. Let $S_1 = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ be the hemiring of non-negative integers modulo 4 and the hemiring $S_2 = \left\{ \begin{pmatrix} x & y \\ x & y \end{pmatrix} \mid x, y \in \mathbb{Z}_2 = \{0, 1\} \right\}$, 2×2 matrices with $\mathbb{Z}_2$ terms.*

Defined two SU-h-bi-ideals f_{S_1} and f_{S_2} over U by

$f_{S_1}(0)=\{(1234)\}, f_{S_1}(2)=\{(1234),(1324),(12)\}$
and $f_{S_1}(1)=f_{S_1}(3)=\{(1234),(1324),(12),(14),(12)(34)\}$.
$f_{S_2}\left(\left(\begin{array}{cc}0&0\\0&0\end{array}\right)\right)=\{e\},\ f_{S_2}\left(\left(\begin{array}{cc}0&1\\0&1\end{array}\right)\right)=\{e,(13)\},\ f_{S_2}\left(\left(\begin{array}{cc}1&0\\1&0\end{array}\right)\right)=\{e,(13),(14)\}$ *and* $f_{S_2}\left(\left(\begin{array}{cc}1&1\\1&1\end{array}\right)\right)=\{e,(14)\}$.

We obtain that

$$f_{S_1\wedge S_2}\left(\left(2,\left(\begin{array}{cc}1&0\\1&0\end{array}\right)\right)+\left(3,\left(\begin{array}{cc}0&1\\0&1\end{array}\right)\right)\right)$$
$$=f_{S_1\wedge S_2}\left(1,\left(\begin{array}{cc}1&1\\1&1\end{array}\right)\right),$$
$$=f_{S_1}(1)\cap f_{S_2}\left(\left(\begin{array}{cc}1&1\\1&1\end{array}\right)\right)$$
$$=\{(1234),(1324),(12),(14),(12)(34)\}\cap\{e,(14)\}$$
$$=\{(14)\},$$

but

$$f_{S_1\wedge S_2}\left(2,\left(\begin{array}{cc}1&0\\1&0\end{array}\right)\right)$$
$$=f_{S_1}(2)\cap f_{S_2}\left(\left(\begin{array}{cc}1&0\\1&0\end{array}\right)\right)$$
$$=\{(1234),(1324),(12)\}\cap\{e,(13),(14)\}$$
$$=\emptyset,$$

and

$$f_{S_1\wedge S_2}\left(\left(3,\left(\begin{array}{cc}0&1\\0&1\end{array}\right)\right)\right)$$
$$=f_{S_1}(3)\cap f_{S_2}\left(\left(\begin{array}{cc}0&1\\0&1\end{array}\right)\right)$$
$$=\{(1234),(1324),(12),(14),(12)(34)\}\cap\{e,(13)\}$$
$$=\emptyset,$$

which implies, $f_{S_1\wedge S_2}\left(\left(2,\left(\begin{array}{cc}1&0\\1&0\end{array}\right)\right)\right)\cup f_{S_1\wedge S_2}\left(\left(3,\left(\begin{array}{cc}0&1\\0&1\end{array}\right)\right)\right)=\emptyset.$

Hence

$$f_{S_1\wedge S_2}\left(\left(2,\left(\begin{array}{cc}1&0\\1&0\end{array}\right)\right)+\left(3,\left(\begin{array}{cc}0&1\\0&1\end{array}\right)\right)\right)\nsubseteq f_{S_1\wedge S_2}\left(2,\left(\begin{array}{cc}1&0\\1&0\end{array}\right)\right)\cup f_{S_1\wedge S_2}\left(3,\left(\begin{array}{cc}0&1\\0&1\end{array}\right)\right).$$

Thus, $f_{S_1}\wedge f_{S_2}$ *is not an SU-h-bi-ideal of* $S_1\times S_2$ *over* U.

Now, we give the following proposition.

Proposition 3.10 *If f_S and h_S are two SU-h-bi-ideals over U, then so is $f_S\tilde{\cup}h_S$.*

Theorem 3.11 *Let $f_S \in S(U)$ and h_S an SU-h-bi-ideal of S over U. Then $f_S\diamondsuit h_S$ and $h_S\diamondsuit f_S$ are SU-h-bi-ideals of S over U.*

Proof. For any $x, y \in S$, we have

$(i)(f_S\diamondsuit h_S)(x) \cup (f_S\diamondsuit h_S)(y)$

$$= \bigcap_{x+\sum_{i=1}^{m} a_ib_i+z_1=\sum_{j=1}^{n} a'_jb'_j+z_1} (f_S(a_i) \cup f_S(a'_j) \cup h_S(b_i) \cup h_S(b'_j))$$

$$\cup \bigcap_{x+\sum_{i=1}^{p} c_id_i+z_2=\sum_{j=1}^{q} c'_jd'_j+z_2} (f_S(c_i) \cup f_S(c'_j) \cup h_S(d_i) \cup h_S(d'_j))$$

$$= \bigcap_{\substack{x+\sum_{i=1}^{m} a_ib_i+z_1\\=\sum_{j=1}^{n} a'_jb'_j+z_1}} (\bigcap_{\substack{x+\sum_{i=1}^{p} c_id_i+z_2\\=\sum_{j=1}^{q} c'{}_jd'{}_j+z_2}} \begin{array}{l}(f_S(a_i) \cup f_S(a'_j) \cup h_S(b_i) \cup h_S(b'_j)\cup\\ f_S(c_i) \cup f_S(c'_j) \cup h_S(d_i) \cup h_S(d'_j))\end{array}$$

$$\supseteq \bigcap_{x+y+\sum_{i=1}^{k} x_iy_i+z_1+z_2=\sum_{j=1}^{l} x'_jy'_j+z_1+z_2} (f_S(x_i) \cup f_S(x'_j) \cup h_S(y_i) \cup h_S(y'_j))$$

$(k = \max\{m, p\}, l = \max\{n, q\}, x_iy_i = a_ib_i + c_id_i, x'_jy'_j = a'_jb'_j + c'_jd'_j)$

$= (f_S\diamondsuit h_S)(x + y)$.

(ii) Let $x, a, b, z \in S$ with $x + a + z = b + z$. Then it is similar to check that $(f_S\diamondsuit h_S)(a) \cup (f_S\diamondsuit h_S)(a) \supseteq (f_S\diamondsuit h_S)(x)$.

$(iii)(f_S\diamondsuit h_S)\diamondsuit(f_S\diamondsuit h_S) = f_S\diamondsuit(h_S\diamondsuit(f_S\diamondsuit h_S))\tilde{\supseteq}f_S\diamondsuit(h_S\diamondsuit(\tilde{\theta}\diamondsuit h_S)) = f_S\diamondsuit(h_S\diamondsuit\tilde{\theta}\diamondsuit h_S)\tilde{\supseteq}f_S\diamondsuit h_S$. (since $h_S\diamondsuit\tilde{\theta}\diamondsuit h_S\tilde{\supseteq}h_S$)

$(iv)(f_S\diamondsuit h_S)\diamondsuit\tilde{\theta}\diamondsuit(f_S\diamondsuit h_S) = f_S\diamondsuit(h_S\diamondsuit(\tilde{\theta}\diamondsuit f_S)\tilde{\supseteq}f_S\diamondsuit(h_S\diamondsuit\tilde{\theta}\diamondsuit h_S)\tilde{\supseteq}f_S\diamondsuit h_S$.

(since $\tilde{\theta}\diamondsuit h_S\tilde{\supseteq}h_S$). Thus, $f_S\diamondsuit h_S$ is an *SU-h-bi*-ideal of S over U. Similarly, we can prove that $h_S\diamondsuit f_S$ is also an *SU-h-h*-ideal of S over U. $\square$

4 *SU-h-quasi*-ideals

In this section, we introduce the concept of soft union *h-quasi*-ideals and investigate some related properties.

Definition 4.1 *A soft set f_S over U is called a soft union h-quasi-ideal (briefly, SU-h-quasi-ideal) of S over U if it satisfies (SU_1), (SU_3) and (SU_{10}) $(f_S\diamondsuit\tilde{\theta})\tilde{\cup}(\tilde{\theta}\diamondsuit f_S)\tilde{\supseteq}f_S$.*

Example 4.2 *Assume that $U = \mathbb{Z}^-$ the set of all negative integers, is the universal set. Let the hemiring $S = \left\{\begin{pmatrix} x & y \\ y & x \end{pmatrix} \mid x, y \in \mathbb{Z}_2\right\}$, 2×2 matrices with $\mathbb{Z}_2$ terms, be the set of parameters.*

Define a soft set f_S over U by $f_S\left(\left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right)\right) = \{-1\}$, $f_S\left(\left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right)\right) = \{-1,-2\}$,
$f_S\left(\left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)\right) = f_S\left(\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)\right) = \{-1,-2,-3\}$.
Then, one can easily check that f_S is an SU-h-quasi-ideal of S over U.

From Definition 4.1 and Theorem 2.9, we have following theorem.

Theorem 4.3 *Let f_S be a soft set over U. Then f_S is an SU-h-quasi-ideal of S over U if and only if it satisfies (SU_3), (SU_5) and (SU_{10}).*

Proposition 4.4 *(i) Every SU-left(right) h-ideal of S over U is an SU-h-quasi-ideal of S over U.*
(ii) Every SU-h-quasi-ideal of S over U is an SU-h-bi-ideal of S over U.

Proof. We only show that *(ii)* holds. Let f_S be an *SU-h-quasi*-ideal of S over U. Then (SU_3) and (SU_5) hold. Moreover, we have,
$f_S\diamondsuit f_S = (f_S\diamondsuit f_S)\tilde{\cup}(f_S\diamondsuit f_S)\tilde{\supseteq}(f_S\diamondsuit\tilde{\theta})\tilde{\cup}(\tilde{\theta}\diamondsuit f_S)\tilde{\supseteq} f_S$.
This proves that (SU_6) holds. Finally, we have $f_S\diamondsuit\tilde{\theta}\diamondsuit f_S\tilde{\supseteq}\tilde{\theta}\diamondsuit\tilde{\theta}\diamondsuit f_S\tilde{\supseteq}\tilde{\theta}\diamondsuit f_S$ and $f_S\diamondsuit\tilde{\theta}\diamondsuit f_S\tilde{\supseteq} f_S\diamondsuit\tilde{\theta}\diamondsuit\tilde{\theta}\tilde{\supseteq} f_S\diamondsuit\tilde{\theta}$, which implies, $f_S\diamondsuit\tilde{\theta}\diamondsuit f_S\tilde{\supseteq}(\tilde{\theta}\diamondsuit f_S)\tilde{\cup}(f_S\diamondsuit\tilde{\theta})\tilde{\supseteq} f_S$. This proves that ($SU_9$) holds. It follows from Theorem 3.3 that f_S is an *SU-h-bi*-ideal of S. $\square$

Remark 4.5 *Note that the converse of Proposition 4.4 is not true as following example.*

Example 4.6 *Assume that $U = \mathbb{Z}^-$ the set of all negative integers, is the universal set. Let the hemiring* $S = \left\{\left(\begin{array}{cc} x & y \\ y & x \end{array}\right) \mid x,y \in \mathbb{Z}_2 = \{0,1\}\right\}$, *$2\times 2$ matrices with $\mathbb{Z}_2$ terms, be the set of parameters.*

Define a soft set f_S over U by $h_S\left(\left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right)\right) = \{-1\}$, $h_S\left(\left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right)\right) = \{-1,-2\}$,
$h_S\left(\left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)\right) = \{-1,-2,-3\}$, $h_S\left(\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)\right) = \{-1,-3\}$.
One can easily check that f_S is an SU-h-bi-ideal of S over U but it is not an SU-left or right h-ideal of S over U. In fact, $h_S\left(\left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right)\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)\right) = h_S\left(\left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right)\right) \not\subseteq \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)$.

Similar to Proposition 3.4 and Corollary 3.5, we have the following proposition.

Proposition 4.7 *(i) A non-empty subset A of S is an h-quasi-ideal of S if and only if the soft subset* f_S *defined by* $f_S(x) = \begin{cases} \alpha & if\ x \in S\backslash A, \\ \beta & if\ x \in A, \end{cases}$

is an SU-h-quasi-ideal of S over U, where $\alpha, \beta \in U$ such that $\alpha \supseteq \beta$.

ii) Let $A \subseteq S$. Then S is an h-quasi-ideal of S if and only if S_{A^C} is an SU-h-quasi-ideal of S over U.

Theorem 4.8 *(i) Let f_S be a soft set over U and $\alpha \subseteq U$ such that $\alpha \in \mathrm{I}_m(f_S)$. If f_S is an SU-h-quasi-ideal of S over U, then $L(f_S; \alpha)$ is an h-quasi-ideal of S.*

(ii) Let f_S be a soft set over U, $L(f_S; \alpha)$ a lower h-quasi-ideal of f_S for each $\alpha \subseteq U$ and $\mathrm{I}_m(f_S)$ an ordered set by inclusion. Then f_S is an h-quasi-ideal of S over U.

Proof. (i) Let $x, y \in L(f_S; \alpha)$. Then as in the proof of Theorem 3.6, we know $x + y \in L(f_S; \alpha)$. Also, let $x, z \in S$ and $a, b \in L(f_S; \alpha)$ with $x + a + z = b + z$, we know that $x \in L(f_S; \alpha)$.

Now, let $x \in \overline{S \cdot L(f_S; \alpha)} \cap \overline{L(f_S; \alpha) \cdot S}$, then there exist $s_1, s_2, t_1, t_2, z_1, z_2 \in S$ and $a_1, a_2, b_1, b_2 \in L(f_S; \alpha)$ such that $x + s_1a_1 + z_1 = s_2a_2 + z_1$ and $x + b_1t_1 + z_2 = b_2t_2 + z_2$ and so, $f_S(a_1) \subseteq \alpha, f_S(a_2) \subseteq \alpha, f_S(b_1) \subseteq \alpha$ and $f_S(b_2) \subseteq \alpha$. Then

$$\begin{aligned}
(\tilde{\theta} \diamondsuit f_S)(x) &= \bigcap_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} (\tilde{\theta}(a_i) \cup \tilde{\theta}(a'_j) \cup f_S(b_i) \cup f_S(b'_j)) \\
&\subseteq \tilde{\theta}(s_1) \cup \tilde{\theta}(s_2) \cup f_S(a_1) \cup f_S(a_2) \\
&= f_S(a_1) \cup f_S(a_2) \\
&\subseteq \alpha. \\
(f_S \diamondsuit \tilde{\theta})(x) &= \bigcap_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} (\tilde{\theta}(a_i) \cup \tilde{\theta}(a'_j) \cup f_S(b_i) \cup f_S(b'_j)) \\
&\subseteq \tilde{\theta}(t_1) \cup \tilde{\theta}(t_2) \cup f_S(b_1) \cup f_S(b_2) \\
&= f_S(b_1) \cup f_S(b_2) \\
&\subseteq \alpha.
\end{aligned}$$

Since f_S is an *SU-h-quasi*-ideal of S, we have $f_S(x) \subseteq (\tilde{\theta} \diamondsuit f_S)(x) \cup (f_S \diamondsuit \tilde{\theta})(x) \subseteq \alpha \cup \alpha = \alpha$, and so, $x \in L(f_S; \alpha)$. Hence $L(f_S; \alpha)$ is an *h-quasi*-ideal of S.

(ii) Let $x, y \in S$ be such that $f_S(x) = \alpha_1$ and $f_S(y) = \alpha_2$, where $\alpha_1 \subseteq \alpha_2$. Then as in the proof of Theorem 3.6, we know that $f_S(x + y) \subseteq f_S(x) \cup f_S(y)$. Also, let $a, b, x, z \in S$ with $x + a + z = b + z$, we know $f_S(x) \subseteq f_S(a) \cup f_S(b)$.
Let $a \in S$ such that $(f_S \diamondsuit \tilde{\theta})(a) = \alpha_1$ and $(\tilde{\theta} \diamondsuit f_S)(a) = \alpha_2$, where $\alpha_1 \subseteq \alpha_2$, then $a \in L(f_S \diamondsuit \tilde{\theta}; \alpha_1)$ and $a \in L(\tilde{\theta} \diamondsuit f_S; \alpha_2)$. Since $\alpha_1 \subseteq \alpha_2$, we have $a \in L(f_S \diamondsuit \tilde{\theta}; \alpha_2)$. We can deduce that $a \in \overline{S \cdot L(f_S; \alpha_2)} \cap \overline{L(f_S; \alpha_2) \cdot S}$. Since $L(f_S; \alpha)$ is an *h-quasi*-ideal of S for all $\alpha \subseteq U$, $a \in L(f_S; \alpha)$. Thus, $f_S(a) \subseteq \alpha_2 \subseteq \alpha_1 \cup \alpha_2 = (f_S \diamondsuit \tilde{\theta})(a) \cup (\tilde{\theta} \diamondsuit f_S)(a)$, to this result implies that $f_S \tilde{\subseteq} (f_S \diamondsuit \tilde{\theta}) \tilde{\cup} (\tilde{\theta} \diamondsuit f_S)$. Hence, f_S is an *SU-h-quasi*-ideal of S over U. □

Proposition 4.9 *Let f_S and h_S be two SU-h-quasi-ideals of S over U. Then $f_S \diamondsuit g_S$ is an SU-h-bi-ideal of S over U.*

Proof. This proposition is a consequence of Proposition 4.4 and Theorem 3.11. □

The following two propositions are obvious.

Proposition 4.10 *(1) Let f_S and h_S be an SU-right h-ideal and an SU-left h-ideal of S over U, respectively, then $f_S\tilde{\cup}h_S$ is an SU-h-quasi-ideal of S over U.*

(2) Let f_S and h_S are two SU-h-quasi-ideals of S over U. Then so is $f_S\tilde{\cup}h_S$.

Proposition 4.11 *(1) Let $f_{S_1}, f_{S_2} \in S(U)$ and Ψ be an isomorphism from S_1 toS_2. If f_{S_1} is an SU-h-quasi-ideal of S_1 over U. Then so is $\Psi(f_{S_1})$ of S_2 over U.*

(2) Let $f_{S_1}, f_{S_2} \in S(U)$ and Ψ be a homomorphism from S_1 to S_2. If f_{S_2} is an SU-h-quasi-ideal of S_2 over U. Then so is $\Psi^{-1}(f_{S_2})$ of S_1 over U.

5 h-hemiregular hemirings

In this section, we investigate some characterizations of h-hemiregular hemirings via *SU*-h-ideals, *SU*-h-*bi*-ideals and *SU*-h-*quasi*-ideals.

Definition 5.1 [37] *A hemiring S is called h-hemiregular if for each $a \in S$, there exist $x_1, x_2, z \in S$ such that $a + ax_1a + z = ax_2a + z$.*

Lemma 5.2 [37] *If A and B, are respectively, a right h-ideal and a left h-ideal of S, then $\overline{AB} \subseteq A \cap B$.*

Lemma 5.3 [37] *A hemiring S is h-hemiregular if and only if for any right h-ideal A and left h-ideal B, we have $\overline{AB} = A \cap B$.*

Theorem 5.4 [36] *For any hemiring S, the following statements are equivalent:*

(1) S is h-hemiregular;

(2) $f_S \diamondsuit g_S = f_S\tilde{\cup}g_S$ for any SU-right h-ideal f_S and any SU-left h-ideal g_S of S over U.

Lemma 5.5 [34] *Let S be a hemiring. Then the following statements are equivalent:*

(1) S is h-hemiregular;

(2) $B = \overline{BSB}$ for every h-bi-ideal B of S;

(3) $Q = \overline{QSQ}$ for every h-quasi-ideal Q of S.

Theorem 5.6 *For any hemiring S, the following conditions are equivalent:*

(1) S is h-hemiregular;

(2) $f_S = f_S \diamondsuit \tilde{\theta} \diamondsuit f_S$ for every SU-h-bi-ideal f_S of S over U;

(3) $f_S = f_S \diamondsuit \tilde{\theta} \diamondsuit f_S$ for every SU-h-quasi-ideal f_S of S over U.

Proof. (1)$\Longrightarrow$(2) Let f_S be an *SU*-h-*bi*-ideal of S over U. For any $x \in S$, there exist $a, a', z \in S$ such that $x + xax + z = xa'x + z$ since S is h-hemiregular. Then the following equalities hold:

$$
\begin{aligned}
(f_S\diamondsuit\tilde{\theta}\diamondsuit f_S)(x) &= \bigcap_{x+\sum_{i=1}^{m}a_ib_i+z=\sum_{j=1}^{n}a'_jb'_j+z} ((f_S\diamondsuit\tilde{\theta})(a_i)\cup(f_S\diamondsuit\tilde{\theta})(a'_j)\cup f_S(b_i)\cup f_S(b'_j))\\
&\subseteq (f_S\diamondsuit\tilde{\theta})(xa)\cup(f_S\diamondsuit\tilde{\theta})(xa')\cup f_S(x)\\
&= \bigcap_{xa+\sum_{i=1}^{m}a_ib_i+z=\sum_{j=1}^{n}a'_jb'_j+z} (f_S(a_i)\cup f_S(a'_j)\cup\tilde{\theta}(b_i)\cup\tilde{\theta}(b'_j))\\
&\cup \bigcap_{xa'+\sum_{i=1}^{m}a_ib_i+z=\sum_{j=1}^{n}a'_jb'_j+z} (f_S(a_i)\cup f_S(a'_j)\cup\tilde{\theta}(b_i)\cup\tilde{\theta}(b'_j))\cup f_S(x)\\
&\subseteq (f_S(xax)\cup f_S(xa'x))\cup(f_S(xax)\cup f_S(xa'x))\cup f_S(x)\\
&(xa+xaxa+za=xa'xa+za \text{ and } xa'+xaxa'+za'=xa'xa'+za')\\
&\subseteq f_S(x)
\end{aligned}
$$

This implies that $f_S\tilde{\subseteq}f_S\diamondsuit\tilde{\theta}\diamondsuit f_S$. Since f_S is an *SU-h-bi*-ideal of S over U, $f_S\diamondsuit\tilde{\theta}\diamondsuit f_S\tilde{\supseteq}f_S$. Hence $f_S\diamondsuit\tilde{\theta}\diamondsuit f_S=f_S$.

(2)$\Longrightarrow$(3) This part is straightforward by Proposition 4.4.

(3)$\Longrightarrow$(1) Let Q be any *h-quasi*-ideal of S. Then by Proposition 4.7(ii), S_{Q^C} is an *SU-h-quasi*-ideal of S over U. For any $x\in Q$ and $x\notin\overline{QSQ}$. We have $S_{Q^C}=\emptyset$. By our assumption, we have $(S_{Q^C}\diamondsuit\tilde{\theta}\diamondsuit S_{Q^C})(x)=\emptyset$. Since $x\notin\overline{QSQ}$, it is clear that there do not exist $a_1,a_2,b_1,b_2\in Q$ and $s_1,s_2,z\in S$ such that $x+a_1s_1b+z=a_2s_2b_2+z$, and so, $(S_{Q^C}\diamondsuit\tilde{\theta}\diamondsuit S_{Q^C})(x)=U$, a contradiction. Thus, we have proved that $Q\subseteq\overline{QSQ}$. Since Q is an *h-quasi*-ideal of S, $\overline{QSQ}\subseteq\overline{SQ}\cap\overline{QS}\subseteq Q$, and so, $Q=\overline{QSQ}$. Hence, it follows from Lemma 5.5 that S is h-hemiregular.

Theorem 5.7 *Let S be a hemiring. Then the following conditions are equivalent:*

(1) S is h-hemiregular;

(2) $f_S\tilde{\cup}g_S=f_S\diamondsuit g_S\diamondsuit f_S$ for every SU-h-bi-ideal f_S and every SU-h-ideal $g_S$of S over U;

SU-h-ideal g_S of S over U.

Proof. (1)$\Longrightarrow$(2) Let f_S and g_S be any *SU-h-bi*-ideal and any *SU-h*-ideal of S over U, respectively. For any $x\in S$, then there exist $a,a',z\in S$ such that $x+xax+z=xa'x+z$ since S is h-hemiregular. Then

$$
\begin{aligned}
(f_S\diamondsuit g_S\diamondsuit f_S)(x) &= \bigcap_{x+\sum_{i=1}^{m}a_ib_i+z=\sum_{j=1}^{n}a'_jb'_j+z} ((f_S\diamondsuit g_S)(a_i)\cup(f_S\diamondsuit g_S)(a'_j)\cup f_S(b_i)\cup f_S(b'_j))\\
&\subseteq (f_S\diamondsuit g_S)(xa)\cup(f_S\diamondsuit g_S)(xa')\cup f_S(x)\\
&= \bigcap_{xa+\sum_{i=1}^{m}a_ib_i+z=\sum_{j=1}^{n}a'_jb'_j+z} (f_S(a_i)\cup f_S(a'_j)\cup g_S(b_i)\cup g_S(b'_j))\cup
\end{aligned}
$$

we have

$$
\begin{aligned}
&\bigcap_{xa'+\sum_{i=1}^{m}a_ib_i+z=\sum_{j=1}^{n}a'_jb'_j+z} f_S(a_i)\cup f_S(a'_j)\cup g_S(b_i)\cup g_S(b'_j)\cup f_S(x)\\
&\subseteq (f_S(x)\cup g_S(axa))\cup g_S(a'xa))\cup(f_S(x)\cup g_S(axa'))\cup g_S(a'xa'))\cup f_S(x)\\
&(xa+xaxa+za=xa'xa+za \text{ and } xa'+xaxa'+za'=xa'xa'+za')\\
&\subseteq f_S(x)\cup g_S(x)\\
&= (f_S\tilde{\cup}g_S)(x),
\end{aligned}
$$

The above result implies that $f_S \diamondsuit g_S \diamondsuit f_S \tilde{\subseteq} f_S \tilde{\cup} g_S$.

Since f_S is an SU-h-bi-ideal of S over U, $f_S \diamondsuit g_S \diamondsuit f_S \tilde{\supseteq} f_S \diamondsuit \tilde{\theta} \diamondsuit f_S \tilde{\supseteq} f_S$. $(*)$

Since g_S is an SU-h-ideal of S over U, we have $f_S \diamondsuit g_S \diamondsuit f_S \tilde{\supseteq} \tilde{\theta} \diamondsuit g_S \diamondsuit \tilde{\theta} = (\tilde{\theta} \diamondsuit g_S) \diamondsuit \tilde{\theta} \tilde{\supseteq} g_S \diamondsuit \tilde{\theta} \tilde{\supseteq} g_S$. $(**)$

By$(*)$ and $(**)$, we have $f_S \diamondsuit g_S \diamondsuit f_S \tilde{\supseteq} f_S \tilde{\cup} g_S$. Hence, $f_S \diamondsuit g_S \diamondsuit f_S = f_S \tilde{\cup} g_S$.

(2)$\Longrightarrow$(3) This is straightforward by Proposition 4.4.

(3)$\Longrightarrow$(1) Since $\tilde{\theta}$ is an SU-h-ideal of S over U, by the assumption, we have $f_S = f_S \tilde{\cup} \tilde{\theta} = f_S \diamondsuit \tilde{\theta} \diamondsuit f_S$. It follows from Theorem 5.6 that S is h-hemiregular. □ □

Theorem 5.8 *Let S be a hemiring. Then the following conditions are equivalent:*

(1)S is h-hemiregular;

(2) $f_S \tilde{\cup} g_S \tilde{\supseteq} f_S \diamondsuit g_S$ for every SU-h-bi-ideal f_S and every SU-left h-ideal g_S of S over U;

(3) $f_S \tilde{\cup} g_S \tilde{\supseteq} f_S \diamondsuit g_S$ for every SU-h-quasi-ideal f_S and every SU-left h-ideal g_S of S over U;

(4) $f_S \tilde{\cup} g_S \tilde{\supseteq} f_S \diamondsuit g_S$ for every SU-h-bi-ideal g_S and every SU-right h-ideal f_S of S over U;

(5) $f_S \tilde{\cup} g_S \tilde{\supseteq} f_S \diamondsuit g_S$ for every SU-h-quasi-ideal g_S and every SU-right h-ideal f_S of S over U;

(6) $f_S \tilde{\cup} g_S \tilde{\cup} h_S \tilde{\supseteq} f_S \diamondsuit g_S \diamondsuit h_S$ for every SU-h-bi-ideal g_S and every SU-right h-ideal f_S and every SU-left h-ideal h_S of S over U;

(7) $f_S \tilde{\cup} g_S \tilde{\cup} h_S \tilde{\supseteq} f_S \diamondsuit g_S \diamondsuit h_S$ for every SU-h-quasi-ideal g_S and every SU-right h-ideal f_S and every SU-left h-ideal h_S of S over U.

Proof. (1)$\Longrightarrow$(2) Let f_S and g_S be an SU-h-bi-ideal and an SU-left h-ideal of S over U, respectively. For any $x \in S$, there exists $a, a', z \in S$ such that $x + xax + z = xa'x + z$ since S is h-hemiregular. Hence, the following equalities hold.

$$\begin{aligned}(f_S \diamondsuit g_S)(x) &= \bigcap_{x+\sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} a'_j b'_j + z} ((f_S)(a_i) \cup (f_S)(a'_j) \cup g_S(b_i) \cup g_S(b'_j)) \\ &\subseteq f_S(x) \cup g_S(ax) \cup g_S(a'x) \\ &\subseteq f_S(x) \cup g_S(x) \\ &= (f_S \tilde{\cup} g_S)(x),\end{aligned}$$

This implies that $f_S \tilde{\cup} g_S \tilde{\supseteq} f_S \diamondsuit g_S$.

(2)$\Longrightarrow$(1) Let f_S and g_S be an SU-h-bi-ideal and an SU-left h-ideal of S over U, respectively. Then by Proposition 4.4, f_S is an SU-h-bi-ideal of S over U. By our assumption, we have $f_S \tilde{\cup} g_S \tilde{\supseteq} f_S \diamondsuit g_S \tilde{\supseteq} (f_S \tilde{\cup} \tilde{\theta}) \tilde{\cup} (\tilde{\theta} \tilde{\cup} f_S) \supseteq f_S \tilde{\cup} g_S$. Hence, $f_S \tilde{\cup} g_S = f_S \diamondsuit g_S$. It follows from Theorem 5.4 that S is h-hemiregular.

Similarly, we can show that (1)$\Longrightarrow$(3), (1)$\Longrightarrow$(4) and (1)$\Longrightarrow$(5).

(1)$\Longrightarrow$(6). Let f_S, g_S and h_S be any SU-right h-ideal, SU-h-bi-ideal, SU-left h-ideal of S over U, respectively. For any $x \in S$, there exist $a, a', z \in S$ such that $x + xax + z = xa'x + z$ since S is h-hemiregular, we have the following equalities:

$$\begin{aligned}
(f_S\diamondsuit g_S\diamondsuit h_S)(x) &= \bigcap_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z} ((f_S\diamondsuit g_S)(a_i)\cup(f_S\diamondsuit g_S)(a'_j)\cup h_S(b_i)\cup h_S(b'_j))\\
&\subseteq (f_S\diamondsuit g_S)(x)\cup h_S(ax)\cup h_S(a'x)\\
&= \bigcap_{xa+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z} (f_S(a_i)\cup f_S(a'_j)\cup g_S(b_i)\cup g_S(b'_j))\cup h_S(ax)\cup h_S(a'x)\\
&\subseteq f_S(xa)\cup f_S(xa')\cup g_S(x)\cup h_S(ax)\cup h_S(a'x)\\
&\subseteq f_S(x)\cup g_S(x)\cup h_S(x)\\
&= (f_S\tilde{\cup} g_S\tilde{\cup} h_S)(x).
\end{aligned}$$

This implies that $f_S\tilde{\cup} g_S\tilde{\cup} h_S\tilde{\supseteq} f_S\diamondsuit g_S\diamondsuit h_S$.

(6)$\Longrightarrow$(7) This part is straightforward by Proposition 4.4.

(7)$\Longrightarrow$(1) Let f_S and h_S be any SU-right h-ideal and any SU-left h-ideal of S over U, respectively. Since $\tilde{\theta}$ is an SU-h-*quasi*-ideal of S over U, we have
$f_S\tilde{\cup} h_S = f_S\tilde{\cup}\tilde{\theta}\tilde{\cup} h_S\tilde{\supseteq} f_S\diamondsuit\tilde{\theta}\diamondsuit h_S\tilde{\supseteq} f_S\diamondsuit h_S\tilde{\supseteq}(f_S\diamondsuit\tilde{\theta})\tilde{\cup}(\tilde{\theta}\diamondsuit h_S)\tilde{\supseteq} f_S\tilde{\cup} h_S$. Then $f_S\tilde{\cup} h_S = f_S\diamondsuit h_S$, and hence, it follows from Theorem 5.4 S is h-hemiregular. $\square$

Lemma 5.9 [36] *A hemiring S is h-hemiregular if and only if every SU-left(right) h-ideal of S is idempotent.*

Theorem 5.10 *Let f_S be an SU-h-quasi-ideal of an h-hemiregular hemiring S over U. Then $(\tilde{\theta}\diamondsuit f_S)\tilde{\cup}(f_S\diamondsuit\tilde{\theta}) = f_S$.*

Proof. Let f_S be any SU-h-*quasi*-ideal of S. Then $(\tilde{\theta}\diamondsuit f_S)\tilde{\cup}(f_S\diamondsuit\tilde{\theta})\tilde{\supseteq} f_S$. We show that $(\tilde{\theta}\diamondsuit f_S)\tilde{\cup}(f_S\diamondsuit\tilde{\theta})\tilde{\subseteq} f_S$. We know $f_S\tilde{\cup}(\tilde{\theta}\diamondsuit f_S)$ is an SU-left h-ideal of S over U. In fact,we have the following equalities.

$$\begin{aligned}
\tilde{\theta}\diamondsuit(f_S\tilde{\cup}(\tilde{\theta}\diamondsuit f_S)) &= (\tilde{\theta}\diamondsuit f_S)\tilde{\cup}(\tilde{\theta}\diamondsuit(\tilde{\theta}\diamondsuit f_S))\\
&= (\tilde{\theta}\diamondsuit f_S)\tilde{\cup}((\tilde{\theta}\diamondsuit\tilde{\theta})\diamondsuit f_S)\\
&= (\tilde{\theta}\diamondsuit f_S)\tilde{\cup}(\tilde{\theta}\diamondsuit f_S)\\
&= \tilde{\theta}\diamondsuit f_S\\
&\tilde{\subseteq} f_S\tilde{\cup}(\tilde{\theta}\diamondsuit f_S).
\end{aligned}$$

By Lemma 5.9, we can easily see that $f_S\tilde{\cup}(\tilde{\theta}\diamondsuit f_S)$ is idempotent. Then we have the following equalities.

$$\begin{aligned}
& f_S\tilde{\supseteq} f_S\tilde{\cup}(\tilde{\theta}\diamondsuit f_S) = (f_S\tilde{\cup}(\tilde{\theta}\diamondsuit f_S))\diamondsuit(f_S\tilde{\cup}(\tilde{\theta}\diamondsuit f_S))\\
&= ((f_S\tilde{\cup}(\tilde{\theta}\diamondsuit f_S))\diamondsuit f_S)\tilde{\cup}((f_S\tilde{\cup}(\tilde{\theta}\diamondsuit f_S))\diamondsuit(\tilde{\theta}\diamondsuit f_S))\\
&= ((f_S\diamondsuit f_S)\tilde{\cup}((\tilde{\theta}\diamondsuit f_S)\diamondsuit f_S))\tilde{\cup}((f_S\diamondsuit(\tilde{\theta}\diamondsuit f_S))\tilde{\cup}((\tilde{\theta}\diamondsuit f_S)\diamondsuit(\tilde{\theta}\diamondsuit f_S)))\\
&= ((f_S\diamondsuit f_S)\tilde{\cup}((\tilde{\theta}\diamondsuit(f_S\diamondsuit f_S))\tilde{\cup}((f_S\diamondsuit(\tilde{\theta}\diamondsuit f_S))\tilde{\cup}((\tilde{\theta}\diamondsuit f_S)^2)\\
&\tilde{\supseteq}((\tilde{\theta}\diamondsuit f_S)\tilde{\cup}(\tilde{\theta}\diamondsuit f_S))\tilde{\cup}((\tilde{\theta}\diamondsuit(\tilde{\theta}\diamondsuit f_S))\tilde{\cup}((\tilde{\theta}\diamondsuit f_S)^2)\\
&\tilde{\supseteq}((\tilde{\theta}\diamondsuit f_S)\tilde{\cup}(\tilde{\theta}\diamondsuit f_S)\tilde{\cup}(\tilde{\theta}\diamondsuit f_S)\tilde{\cup}(\tilde{\theta}\diamondsuit f_S)\\
&\tilde{\supseteq}\tilde{\theta}\diamondsuit f_S,
\end{aligned}$$

This implies that $f_S\tilde{\supseteq}\tilde{\theta}\diamondsuit f_S$.

Similarly, we can prove that $f_S \tilde{\supseteq} f_S \diamondsuit \tilde{\theta}$, and so, $(\tilde{\theta} \diamondsuit f_S) \tilde{\cup} (f_S \diamondsuit \tilde{\theta}) \tilde{\subseteq} f_S$. Thus, $(\tilde{\theta} \diamondsuit f_S) \tilde{\cup} (f_S \diamondsuit \tilde{\theta}) = f_S$. □

Theorem 5.11 *Let $f_S \in S(U)$ and S an h-hemiregular hemiring. Then the following statements are equivalent:*

(1) f_S is an SU-h-quasi-ideal of S over U;

(2) f_S may be presented in the form $f_S = g_S \diamondsuit h_S$, where g_S is an SU-right h-ideal and h_S is an SU-left h-ideal of S over U.

Proof. (1)$\Longrightarrow$(2) By Theorem 5.7, we have $f_S = f_S \diamondsuit \tilde{\theta} \diamondsuit f_S$, for any *SU-h-quasi*-ideal f_S of S over U. Then, we have $f_S = f_S \diamondsuit \tilde{\theta} \diamondsuit f_S = f_S \diamondsuit (\tilde{\theta} \diamondsuit \tilde{\theta}) \diamondsuit f_S = (f_S \diamondsuit \tilde{\theta}) \diamondsuit (\tilde{\theta} \diamondsuit f_S)$.

We know that $f_S \diamondsuit \tilde{\theta}$ and $\tilde{\theta} \diamondsuit f_S$ are an *SU*-right h-ideal and an *SU*-left h-ideal of S over U, respectively.

(2)$\Longrightarrow$(1) By Proposition 4.10, we know that $f_S = g_S \tilde{\cup} h_S$ is an *SU-h-quasi*-ideal of S over U. Since S is h-hemiregular, by Theorem 5.4, we know that $g_S \diamondsuit h_S = g_S \tilde{\cup} h_S$, then $f_S = g_S \diamondsuit h_S$. □

6 h-intra-hemiregular hemirings

In this section, we investigate some characteristics of the *h-intra*-hemiregular hemirings by mean of *SU-h*-ideals, *S*U-*h-bi*-ideals and *SU-h-quasi*-ideals.

Definition 6.1 [34] *A hemiring S is said to be a h-intra-hemiregular hemiring if for each $x \in S$, there exist $a_i, a'_i, b_j, b'_j, z \in S$ such that $x + \sum_{i=1}^{m} a_i x^2 a'_i + z = \sum_{j=1}^{n} b_j x^2 b'_j + z$. Equivalently,*

(1) $x \in \overline{Sx^2S}, \forall x \in S$;

(2) $A \subseteq \overline{SA^2S}, \forall A \subseteq S$.

Lemma 6.2 [34] *Let S be a hemiring. Then the following statements are equivalent:*

(1) S is h-intra-hemiregular;

(2) $L \cap R \subseteq \overline{LR}$ for every left h-ideal L and every right h-ideal R of S.

Theorem 6.3 *For any hemiring S, the following statements are equivalent:*

(1) S is h-intra-hemiregular;

(2) $f_S \tilde{\cup} g_S \tilde{\supseteq} f_S \diamondsuit g_S$ for every SU-left h-ideal f_S and every SU-right h-ideal of S over U.

Proof. (1) $\Rightarrow$ (2) Let f_S and g_S be any *SU*-left h-ideal and any *SU*-right h-ideal of S, respectively. For any $x \in S$, there exist $a_i, a'_i, b_j, b'_j, z \in S$ such that $x + \sum_{i=1}^{m} a_i x^2 a'_i + z = \sum_{j=1}^{n} b_j x^2 b'_j + z$.

Then we have the following properties:

$$
\begin{aligned}
(f_S \diamondsuit g_S)(x) &= \bigcap_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} (f_S(a_i) \cup f_S(a'_j) \cup g_S(b_i) \cup g_S(b'_j)) \\
&\subseteq f_S(a_i x) \cup f_S(b_j x) \cup g_S(x a'_i) \cup g_S(x b'_j) \\
&\subseteq f_S(x) \cup g_S(x) \\
&= (f_S \tilde{\cup} g_S)(x),
\end{aligned}
$$

Thus, this leads to $f_S \diamondsuit g_S \tilde{\subseteq} f_S \tilde{\cup} g_S$.

(2) $\Rightarrow$ (1) Let L and R be any left h-ideal and right h-ideal of S, respectively. Then by Proposition 2.8, S_{L^c} and S_{R^c} are an SU-left h-ideal and an SU-right h-ideal of S over U, respectively. If there exists $a \in L \cap R$ such that $a \notin \overline{LR}$, then there exist $a_1, a_2 \in L, b_1, b_2 \in R$ and $z \in S$ such that $a + a_1 b_1 + z = a_2 b_2 + z$. Then we have $(S_{L^c} \diamondsuit S_{R^c})(a) = U$. Since $a \in L \cap R$, $a \in L$ and $a \in R$, and so $S_{L^c}(a) = S_{R^c}(a) = \emptyset$, by our assumptions, we have the following equality: $(S_{L^c} \diamondsuit S_{R^c})(a) = S_{L^c}(a) \cup S_{R^c}(a) = \emptyset$, contradiction. This means that $L \cap R \subseteq \overline{LR}$. It follows from Lemma 6.2 that S is h-*intra*-hemiregular. $\square$

Lemma 6.4 [34] *Let S be a hemiring. Then the following statements are equivalent:*

(1) S is both h-hemiregular and h-intra-hemiregular;

(2) $B = \overline{B^2}$ for every h-bi-ideal B of S;

(3) $Q = \overline{Q^2}$ for every h-quasi-ideal Q of S.

Theorem 6.5 *Let S be a hemiring. Then the following statements are equivalent:*

(1) S is both h-hemiregular and h-intra-hemiregular;

(2) $f_S = f_S \diamondsuit f_S$ for every SU-h-bi-ideal f_S of S over U (that is, every SU-h-bi-ideal of S over U is idempotent);

(3) $f_S = f_S \diamondsuit f_S$ for every SU-h-quasi-ideal f_S of S over U (that is, every SU-h-quasi-ideal of S over U is idempotent).

Proof. (1) $\Rightarrow$ (2) Let f_S be any SU-h-bi-ideal of S over U. Then it is easy to see that $f_S \diamondsuit_S \tilde{\supseteq} f_S$. For any $x \in S$, then there exist $a_1, a_2, p_i, p'_i, q_j, q'_j, z \in S$ such that

$$
\begin{aligned}
& x + \sum_{j=1}^{n} (x a_2 q_j x)(x q'_j a_1 x) + \sum_{j=1}^{n} (x a_1 q_j x)(x q'_j a_2 x) + \sum_{i=1}^{m} (x a_1 p_i x)(x p'_i a_1 x) \\
& \quad + \sum_{i=1}^{m} (x a_2 p_i x)(x p'_i a_2 x) + z \\
& = \sum_{i=1}^{m} (x a_2 p_i x)(x p'_i a_1 x) + \sum_{i=1}^{m} (x a_1 p_i x)(x p'_i a_2 x) + \sum_{j=1}^{n} (x a_1 q_j x)(x q'_j a_1 x) \\
& \quad + \sum_{j=1}^{n} (x a_2 q_j x)(x q'_j a_2 x) + z.
\end{aligned}
$$

Hence, we deduce the following equality.

$$\begin{aligned}(f_S \diamondsuit f_S)(x) &= \bigcap_{x+\sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} a'_j b'_j + z} (f_S(a_i) \cup f_S(a'_j) \cup f_S(b_i) \cup f_S(b'_j)) \\ &\subseteq f_S(xa_2q_jx) \cup f_S(xq'_ja_1x) \cup f_S(xa_1q_jx) \cup f_S(xq'_ja_2x) \\ &\quad \cup f_S(xa_1p_ix) \cup f_S(xp'_ia_1x) \cup f_S(xa_2p_ix) \cup f_S(xp'_ia_2x) \\ &\subseteq f_S(x),\end{aligned}$$

This leads to $f_S \diamondsuit f_S \tilde{\subseteq} f_S$ and so $f_S \diamondsuit f_S = f_S$.

(2) $\Rightarrow$ (3) This part is straightforward by Proposition 4.4.

(3) $\Rightarrow$ (1) Let Q be any *h-quasi*-ideal of S. Then $\overline{Q^2} \subseteq Q$ always holds. To show that $Q \subseteq \overline{Q^2}$. If there exists $x \in Q$ and $x \notin \overline{Q^2}$, then there do not exist $a_1, a_2, b_1, b_2 \in Q$ and $z \in S$ such that $x + a_1b_1 + z = a_2b_2 + z$, and so $(S_{Q^c} \diamondsuit S_{Q^c})(x) = U$. Since Q is an *h-quasi*-ideal of S, by Proposition 4.7, S_{Q^c} is an *SU-h-quasi*-ideal of S over U, and so $S_{Q^c}(x) = \emptyset$. Thus, by our assumption, we have $(S_{Q^c} \diamondsuit S_{Q^c})(x) = S_{Q^c}(x) = \emptyset$, a contradiction. Thus, we have proved that $Q \subseteq \overline{Q^2}$. Then $Q = \overline{Q^2}$. It follows from Lemma 6.4 that S is both h-hemiregular and *h-intra*-hemiregular. $\square$

Theorem 6.6 *Let S be a hemiring. Then the following statements are equivalent:*

(1) S is both h-hemiregular and h-intra-hemiregular;

(2) $f_S \tilde{\cup} g_S \tilde{\supseteq} f_S \diamondsuit g_S$ for all SU-h-bi-ideals f_S and g_S of S over U;

(3) $f_S \tilde{\cup} g_S \tilde{\supseteq} f_S \diamondsuit g_S$ for every SU-h-bi-ideal f_S and every SU-h-quasi-ideal g_S of S over U;

(4) $f_S \tilde{\cup} g_S \tilde{\supseteq} f_S \diamondsuit g_S$ for every SU-h-quasi-ideal f_S and every SU-h-bi-ideal g_S of S over U;

(5) $f_S \tilde{\cup} g_S \tilde{\supseteq} f_S \diamondsuit g_S$ for all SU-h-quasi-ideals f_S and g_S of S over U.

Proof. (1)$\Rightarrow$(2) Let f_S and g_S be any two *SU-h-bi*-ideals of S over U. For any $x \in S$,there exist $a_1, a_2, p_i, p'_i, q_j, q'_j, z \in S$ such that

$$\begin{aligned}&x + \sum_{j=1}^{n}(xa_2q_jx)(xq'_ja_1x) + \sum_{j=1}^{n}(xa_1q_jx)(xq'_ja_2x) + \sum_{i=1}^{m}(xa_1p_ix)(xp'_ia_1x) \\ &\quad + \sum_{i=1}^{m}(xa_2p_ix)(xp'_ia_2x) + z \\ &= \sum_{i=1}^{m}(xa_2p_ix)(xp'_ia_1x) + \sum_{i=1}^{m}(xa_1p_ix)(xp'_ia_2x) + \sum_{j=1}^{n}(xa_1q_jx)(xq'_ja_1x) \\ &\quad + \sum_{j=1}^{n}(xa_2q_jx)(xq'_ja_2x) + z.\end{aligned}$$

Then we have

$$
\begin{aligned}
(f_S \diamondsuit f_S)(x) &= \bigcap_{x+\sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} a'_j b'_j + z} (f_S(a_i) \cup f_S(a'_j) \cup g_S(b_i) \cup g_S(b'_j)) \\
&\subseteq f_S(xa_2q_jx) \cup f_S(xa_1q_jx) \cup f_S(xa_1p_ix) \cup f_S(xa_2p_ix) \\
&\quad \cup g_S(xq'_ja_1x) \cup g_S(xq'_ja_2x) \cup g_S(xp'_ia_1x) \cup g_S(xp'_ia_2x) \\
&\subseteq f_S(x) \cup g_S(x) \\
&= (f_S \tilde{\cup} g_S)(x),
\end{aligned}
$$

This implies that $f_S \diamondsuit g_S \tilde{\subseteq} f_S \tilde{\cup} g_S$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are clear.

(5) $\Rightarrow$ (1) Let Q be any *h-quasi-ideal* of S over U. Then by Proposition 4.7, S_{Q^c} is an SU-h-*quasi*-ideal of S over U, and so $S_{Q^c} = \emptyset$. Thus, by our assumption, we have $(S_{Q^c} \diamondsuit S_{Q^c})(x) \tilde{\subseteq} S_{Q^c} \tilde{\cup} S_{Q^c} = \emptyset \cup \emptyset = \emptyset$, and so $(S_{Q^c} \diamondsuit S_{Q^c})(x) = \emptyset$.

If there exists $x \in Q$ and $x \notin \overline{Q^2}$, then there do not exist $a_1, a_2, b_1, b_2 \in Q$ and $z \in S$ such that $x + a_1b_1 + z = a_2b_2 + z$, and so, $(S_{Q^c} \diamondsuit S_{Q^c})(x) = U$, a contradiction. This means that $Q \subseteq \overline{Q^2}$. Since $Q \supseteq \overline{Q^2}$ always holds.

Thus, we deduce that $Q = \overline{Q^2}$. It follows from Lemma 6.4 that S is both h-hemiregular and h-*intra*-hemiregular. $\square$

7 Conclusions

In order to give a foundation for providing a soft algebraic tool in considering varios problems related with the uncertainties, we now investigate some characteristics of h-hemiregular and h-*intra*-hemiregular hemirings by using the SU-h-ideals, SU-h-*bi*-ideals and SU-h-*quasi*-ideals. In our future study of soft hemirings, we will try to apply the above new soft hemirings to some other fields such as decision making, data analysis and forecasting and so on.

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