Neighbor sum distinguishing edge colorings of graphs with small maximum average degree *

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Abstract

A proper edge-k-coloring of a graph G is an assignment of k colors $1, 2, \cdots, k$ to the edges of G such that no two adjacent edges receive the same color. A neighbor sum distinguishing edge-k-coloring of G is a proper edge-k-coloring of G such that for each edge $uv \in E(G)$, the sum of colors taken on the edges incident with u is different from the sum of colors taken on the edges incident with v. By $ndi_{\sum}(G)$, we denote the smallest value k in such a coloring of G. The maximum average degree of G is $mad(G) = \max\{2|E(H)|/|V(H)|\}$, where the maximum is taken over all the non-empty subgraphs H of G. In this paper, we obtain that if G is a graph without isolated edges and $mad(G) < 8/3$, then $ndi_{\sum}(G) \leq k$ where $k = \max\{\Delta(G) + 1, 6\}$. It partially confirms the conjecture proposed by Flandrin et al.

Keywords: proper edge coloring; neighbor sum distinguishing edge coloring; maximum average degree

1 Introduction

In this paper, all graphs considered are finite, simple and undirected. The terminology and notation used but undefined in this paper can be found in [1]. Let $G = (V, E)$ be a graph. We use $V(G), E(G), \Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, maximum degree and minimum degree of G, respectively. Let $d_G(v)$ or simply $d(v)$, denote the degree of a vertex v in G. A vertex v is called a k-vertex (resp. k^--vertex, or k^+-vertex) if $d(v) = k$ (resp. $d(v) \leq k$, or $d(v) \geq k$). A vertex is called a leaf of G if $d(v) = 1$. A 2-vertex is called bad if it is adjacent to a 2-vertex, otherwise we call it good. A 5-vertex is called bad if it is adjacent to four bad 2-vertices, otherwise we call it good. The girth of a graph G is the length of a smallest cycle in G, and we denote it by $g(G)$. The maximum average degree of G is $mad(G) = \max\{2|E(H)|/|V(H)|\}$, where the maximum is taken over all the non-empty subgraphs H of G.

A proper edge-k-coloring of a graph G is an assignment of k colors $1, 2, \cdots, k$ to the edges of G such that no two adjacent edges receive the same color. Let c be a proper edge-k-coloring

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of G. By $w(v)$ (resp. $S(v)$), we denote the sum (resp. set) of colors taken on the edges incident with v, i.e. $w(v) = \sum_{uv \in E(G)} c(uv)$ (resp. $S(v) = \{c(uv) \mid uv \in E(G)\}$). We call the coloring c such that $w(u) \neq w(v)$ (resp. $S(u) \neq S(v)$) for each edge $uv \in E(G)$ a neighbor sum distinguishing (resp. neighbor distinguishing) edge-k-coloring of G. For simplicity, we use *nsd-k*-coloring (resp. *nd-k*-coloring) to denote the neighbor sum distinguishing (resp. neighbor distinguishing) edge-k-coloring of G. By $ndi_{\sum}(G)$ (resp. $ndi(G)$), we denote the smallest value k such that G has an *nsd-k*-coloring (resp. *nd-k*-coloring) of G.

Obviously, a graph G has a neighbor sum distinguishing (neighbor distinguishing) coloring if and only if G has no isolated edges (we call it normal). Apparently, for any normal graph G, $ndi(G) \leq ndi_{\sum}(G)$. In 2002, Zhang et al. [2] proposed the following conjecture.

Conjecture 1.1. [2] *If G is a normal graph with at least 6 vertices, then $ndi(G) \leq \Delta(G)+2$.*

Balister et al. [3] proved Conjecture 1.1 for bipartite graphs and for graphs G with $\Delta(G) = 3$. If G is bipartite planar with maximum degree $\Delta(G) \geq 12$, Conjecture 1.1 was confirmed by Edwards et al. [4]. Hatami [5] showed that if G is a normal graph and $\Delta(G) > 10^{20}$, then $ndi(G) \leq \Delta(G) + 300$. Akbari et al. [6] proved that $ndi(G) \leq 3\Delta(G)$ for any normal graph. Wang et al. [7], [8] confirmed Conjecture 1.1 for sparse graphs and K_4-minor free graphs. More precisely, in [7] they showed that if G is a normal graph and $mad(G) < 5/2$, then $ndi(G) \leq \Delta(G) + 1$. Furthermore, $ndi(G) = \Delta(G) + 1$ if and only if G has two adjacent maximum degree vertices. Recently, Hocquard et al. [9] proved that for every normal graph with $\Delta(G) \geq 5$ and $mad(G) < 13/5$, we have $ndi(G) \leq \Delta(G)+1$. Later, in [10] they proved that if G is a normal graph with $\Delta(G) \geq 5$ and $mad(G) < 3 - 2/\Delta(G)$, then $ndi(G) \leq \Delta(G) + 1$.

Recently, Flandrin et al. [11] studied the neighbor sum distinguishing colorings of cycles, trees, complete graphs and complete bipartite graphs. Based on these examples, they proposed the following conjecture.

Conjecture 1.2. [11] *If G is a connected graph on at least 3 vertices and $G \neq C_5$, then $ndi_{\sum}(G) \leq \Delta(G) + 2$.*

Flandrin et al. [11] also proved that for each connected graph G with maximum degree $\Delta \geq 2$, we have $ndi_{\sum}(G) \leq \lceil (7\Delta - 4)/2 \rceil$. Dong et al. [12] considered the neighbor sum distinguishing colorings of planar graphs and showed that if G is a normal planar graph, then $ndi_{\sum}(G) \leq \max\{2\Delta(G) + 1, 25\}$. In [13], Dong et al. proved that if G is a normal graph and $mad(G) \leq 5/2$, then $ndi_{\sum}(G) \leq k$ where $k = \max\{\Delta(G) + 1, 6\}$. Other results on graph coloring problems are referred to [15, 16, 17].

In this paper, we will prove the following results.

Theorem 1.3. *Let G be a normal graph. If $mad(G) < \frac{8}{3}$, then $ndi_{\sum}(G) \leq k$ where $k = \max\{\Delta(G)+1, 6\}$.*

Corollary 1.4. *Let G be a normal graph. If $mad(G) < \frac{8}{3}, \Delta(G) \geq 5$, then $ndi_{\sum}(G) \leq \Delta(G)+1$.*

In [14], the authors obtained that $mad(G) < 2g/(g-2)$ if G is a planar graph with girth g. The following corollary is obvious.

Corollary 1.5. *Let G be a normal planar graph. If $g(G) \geq 8$ and $\Delta(G) \geq 5$, then $ndi_{\sum}(G) \leq \Delta(G)+1$.*

We note that if G contains two adjacent vertices of maximum degree, then $ndi_{\sum}(G) \geq \Delta(G)+1$. So the bound $\Delta(G)+1$ in Corollary 1.4 is sharp. Furthermore, Corollary 1.4 implies a result of Hocquard et al. [9] about the neighbor distinguishing coloring of sparse graphs.

2 Proof of Theorem 1.3

Firstly, we give two lemmas obtained by Dong et al. in [13], all the elements in each set are integers.

Lemma 2.1. [13] *Let S_1, S_2 be two sets and $S_3 = \{\alpha+\beta \mid \alpha \in S_1, \beta \in S_2, \alpha \neq \beta\}$.*
(i) *If $|S_1| = 2$ and $|S_2| = 3$, then $|S_3| \geq 3$.*
(ii) *If $|S_1| = 2$ and $|S_2| = 4$, then $|S_3| \geq 4$.*
(iii) *If $|S_1| = |S_2| = 2$ and $S_1 \neq S_2$, then $|S_3| \geq 3$.*

Lemma 2.2. [13] *Let S be a set of size $k+1$. If $S_1 = \{\sum\limits_{i=1}^{k} x_i \mid x_i \in S, x_i \neq x_j$ if $1 \leq i < j \leq k\}$, then $|S_1| \geq k+1$.*

Let $k = \max\{\Delta(G)+1, 6\}$ and $[k] = \{1, 2, \cdots, k\}$. Suppose to the contrary that G is a counterexample to Theorem 1.3, such that $|E(G)|$ is minimum. By the choice of G, it is clear that G is connected and any normal subgraph G' has an *nsd-k*-coloring c. We use $w(v)$ and $S(v)$ to denote the sum and the set of colors taken on the edges incident with v in the coloring c of G', i.e. $w(v) = \sum_{v\in e, e\in E(G')} c(e)$ and $S(v) = \{c(e) \mid v \in e, e \in E(G')\}$. In the following, we will extend c to the whole graph G.

Let H be the graph obtained by removing all the leaves of G. Obviously, H is a connected graph and $mad(H) < 8/3$. In the following, we give some properties of H.

Claim 2.3. *H has the following properties:*
(i) [13] *$\delta(H) \geq 2$, where $\delta(H)$ is the minimum degree of H.*

(ii) [13] *Let* $v \in V(H)$ *such that* $d_H(v) = 2$, *then* $d_G(v) = 2$.

(iii) *Let* $uvxy$ *be a path in* H *such that* $d_H(v) = d_H(x) = 2$, *then* $d_G(u) = d_H(u)$ *and* $d_G(y) = d_H(y)$.

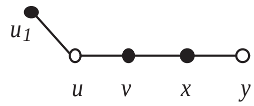

Figure 2.1: Illustration of Claim 2.3(iii)

Proof. (iii) Let $uvxy$ be a path in H such that $d_H(v) = d_H(x) = 2$. By Claim 2.3(ii), $d_G(v) = d_G(x) = 2$. By contradiction suppose $d_G(u) \neq d_H(u)$ (it follows from Claim 2.3(i) and construction of H that $d_G(u) \geq 3$). Hence there exits at least one 1-vertex adjacent to u in G, say u_1. Consider $G' = G \backslash \{vx\}$. By the minimality of G, G' admits an *nsd-k*-coloring c. If $c(uv) \neq c(xy)$, then we color vx with a color distinct from $c(uv), c(xy), w(u)-w(v), w(y)-w(x)$, then we obtain an *nsd-k*-coloring of G. Otherwise, we permute the colors assigned to uu_1 and uv. The obtained coloring is still an *nsd-k*-coloring of G'. We then extend this coloring to G as previously. This is a contradiction. □

Claim 2.4. *Let* $u \in V(H), d_H(u) = l, uu_i \in E(H), i = 1, 2, \cdots, l$.

(i) [13] *If* $l = 2$, *then* u *is adjacent to at most one* 2*-vertex.*

(ii) (a) [13] *If* $l = 3$ *and* $d_H(u) < d_G(u)$, *then* u *is adjacent to at most one* 2*-vertex.*

(b) *If* $l = 3$, *then* u *is not adjacent to any bad* 2*-vertex. Furthermore,* u *is adjacent to at most one good* 2*-vertex.*

(iii) *If* $l = 4$, *then* u *is adjacent to at most one bad* 2*-vertex. Furthermore, if* u *is adjacent to one bad* 2*-vertex, then* u *is adjacent to at most two good* 2*-vertices.*

(iv) *If* $l \geq 5$ *and* u *is adjacent to* $(l-1)$ *bad* 2*-vertices, then* u *is adjacent to at most* $(l-1)$ 2*-vertices.*

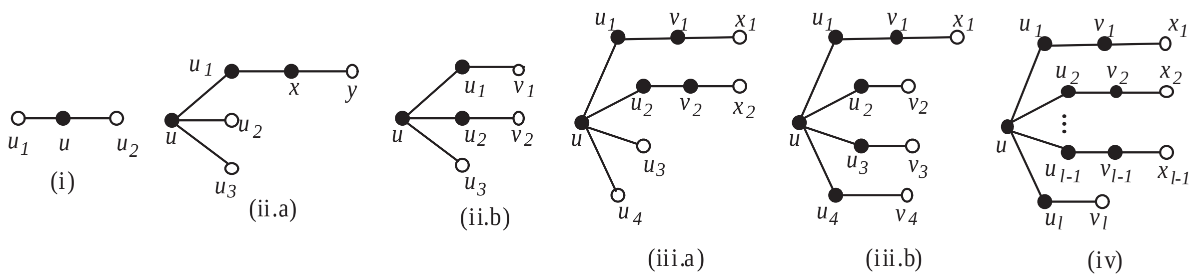

Figure 2.2: Illustration of Claim 2.4

Proof. (ii) (b) Firstly, we prove that u is not adjacent to any bad 2-vertex. Suppose to the contrary that u_1 is a bad 2-vertex. Let x be the other neighbor of u_1 with $d_H(x) = 2$, and $xy \in E(H)$, $y \neq u_1$. By Claim 2.3 (iii), $d_G(u) = d_H(u) = 3$. Consider the graph $G' = G\backslash\{uu_1\}$, then G' admits an *nsd-k*-coloring c. Color uu_1 with a color α in $S = [k]\backslash(\{c(xy)\}\bigcup\{c(uu_2), c(uu_3)\}\bigcup\{w(u_2)-w(u)\}\bigcup\{w(u_3)-w(u)\})$. Recolor u_1x with a color distinct from $\alpha, c(xy), w(y) - c(xy), c(uu_2) + c(uu_3)$ and we obtain an *nsd-k*-coloring of G, a contradiction.

In the following, we prove that u is adjacent to at most one good 2-vertex. Suppose to the contrary that $d_H(u_1) = d_H(u_2) = 2$, $v_iu_i \in E(H), v_i \neq u, i = 1, 2$.

Case 1 $d_G(u) > d_H(u) = 3$.

By Claim 2.4 (ii) (a), this claim holds.

Case 2 $d_G(u) = d_H(u) = 3$.

Subcase 2.1 $k \geq 7$. Consider the graph $G' = G\backslash\{uu_1, uu_2\}$, then G' has an *nsd-k*-coloring c. Let $S_i = [k]\backslash(\{c(u_iv_i)\}\bigcup\{c(uu_3)\}\bigcup\{w(v_i) - w(u_i)\}\bigcup\{w(u_{3-i}) - c(uu_3)\}), i = 1, 2$, then $|S_i| \geq 3, i = 1, 2$. By Lemma 2.1 (ii), we can choose $\alpha_i \in S_i, i = 1, 2$ such that $\alpha_1 \neq \alpha_2$, $\alpha_1 + \alpha_2 + c(uu_3) \neq w(u_3)$. We obtain an *nsd-k*-coloring of G, which is a contradiction.

Subcase 2.2 $k = 6$. From the above discussion, $d_G(v_i) \geq 3, i = 1, 2$. If $d_G(v_1) = 5$, then u_1 can be distinguished from v_1 under an arbitrary proper edge coloring of G. Consider $G' = G\backslash\{uu_1, uu_2\}$, then G' has an *nsd*-6-coloring c. The colors in $\{c(u_1v_1)\}\bigcup\{c(uu_3)\}\bigcup\{c(u_2v_2)-c(uu_3)\}$ are forbidden for uu_1. Let $S_1 = [6]\backslash(\{c(u_1v_1)\}\bigcup\{c(uu_3)\}\bigcup\{c(u_2v_2)-c(uu_3)\})$, $S_2 = [6]\backslash(\{c(u_2v_2)\}\bigcup\{c(uu_3)\}\bigcup\{w(v_2)-w(u_2)\}\bigcup\{c(u_1v_1)-c(uu_3)\})$, then $|S_1| \geq 3, |S_2| \geq 2$. By Lemma 2.1 (ii), we can choose $\alpha_i \in S_i, i = 1, 2$ such that $\alpha_1 \neq \alpha_2$, $\alpha_1 + \alpha_2 + c(uu_3) \neq w(u_3)$. We obtain an *nsd*-6-coloring of G, which is a contradiction. Therefore, $d_G(v_1) \neq 5$. Similarly, $d_G(v_2) \neq 5$.

If $d_G(v_1) = 3$ and x_1, y_1 are the other two neighbors of v_1. Consider $G' = G\backslash\{uu_1, uu_2, u_1v_1\}$, then G' has an *nsd*-6-coloring c. Let $S_1 = [6]\backslash(\{c(v_1x_1) + c(v_1y_1)\}\bigcup\{c(uu_3)\})$, $S_2 = [6]\backslash(\{c(uu_3)\}\bigcup\{c(u_2v_2)\}\bigcup\{w(v_2)-w(u_2)\})$, $S_3 = [6]\backslash(\{c(v_1x_1), c(v_1y_1)\}\bigcup\{w(x_1)-w(v_1)\}\bigcup\{w(y_1) - w(v_1)\})$, then $|S_1| \geq 4$, $|S_2| \geq 3$, $|S_3| \geq 2$. We can choose $\alpha_i \in S_i, i = 1, 2, 3$ such that $\alpha_1 \neq \alpha_2, \alpha_1 \neq \alpha_3$, u can be distinguished from u_1, u_2, u_3, and v_1 can be distinguished from x_1, y_1. We obtain an *nsd*-6-coloring of G, a contradiction. Therefore, $d_G(v_1) \neq 3$. Similarly, $d_G(v_2) \neq 3$.

Now we assume that $d_G(v_1) = d_G(v_2) = 4$, x_i, y_i, z_i are the other three neighbors of v_i, $i = 1, 2$. Consider $G' = G\backslash\{uu_1, uu_2\}$, then G' has an *nsd*-6-coloring c. If $c(v_1x_1) + c(v_1y_1) + c(v_1z_1) > 6$, then u_1 and v_1 can be distinguished. Let $S_1 = [6]\backslash(\{c(uu_3)\}\bigcup\{c(u_1v_1)\}\bigcup\{w(u_2)-c(uu_3)\})$, $S_2 = [6]\backslash(\{c(uu_3)\}\bigcup\{c(u_2v_2)\}\bigcup\{w(v_2) - w(u_2)\}\bigcup\{w(u_1) - c(uu_3)\})$, then $|S_1| \geq 3, |S_2| \geq 2$. We can choose $\alpha_i \in S_i, i = 1, 2$ such that $\alpha_1 \neq \alpha_2$, $\alpha_1 + \alpha_2 + c(uu_3) \neq w(u_3)$.

We obtain an nsd-6-coloring of G, which is a contradiction. Therefore, $c(v_1x_1) + c(v_1y_1) + c(v_1z_1) = 6$. Similarly, $c(v_2x_2)+c(v_2y_2)+c(v_2z_2) = 6$. Without loss of generality we assume that $c(v_ix_i) = 1, c(v_iy_i) = 2, c(v_iz_i) = 3, i = 1, 2$. Suppose that $c(u_1v_1) \neq c(u_2v_2)$ or $c(uu_3) = 6$, then we can obtain an nsd-6-coloring of G as previously. Hence, $c(u_1v_1) = c(u_2v_2)$, $c(uu_3) \neq 6$. From the above discussion, $d_G(u_3) \geq 3$. If $d_G(u_3) = 3$, let x_3, y_3 be the neighbors of u_3 distinct from u. Consider the graph $G' = G\backslash\{uu_1, uu_2, uu_3\}$, then G' has an nsd-6-coloring c. Let $S_1 = [5]\backslash\{c(u_1v_1)\}$, $S_2 = [5]\backslash\{c(u_2v_2)\}$, $S_3 = [6]\backslash(\{c(v_3x_3), c(v_3y_3)\} \bigcup\{w(x_3)-w(u_3)\} \bigcup\{w(y_3) - w(u_3)\})$, then $|S_1| \geq 4$, $|S_2| \geq 4$, $|S_3| \geq 2$. From the above discussion we know that $S_1 = S_2 = \{1, 2, 3, 4\}$ or $\{1, 2, 3, 5\}$, so we can choose $\alpha_i \in S_i, i = 1, 2, 3$ such that $\alpha_1, \alpha_2, \alpha_3$ are pairwise distinct and u can be distinguished from u_1, u_2, u_3. We obtain an nsd-6-coloring of G, which is a contradiction. Therefore, $d_G(u_3) \geq 4$. If $c(uu_3) \in \{1, 2, 3\}$, color uu_1, uu_2 properly with $\{1, 2, 3\}\backslash\{c(uu_3)\}$. Otherwise, properly color uu_1, uu_2 with colors in $\{1, 2, 3\}$. In both cases, we obtain an nsd-6-coloring of G, a contradiction.

(iii) Suppose to the contrary that $d_H(u_1) = d_H(u_2) = 2$, $u_iv_i \in E(H), d_H(v_i) = 2, i = 1, 2$, x_i is the other neighbor of $v_i, i = 1, 2$. By Claim 2.3 (iii), $d_G(u) = d_H(u) = 4$. Consider the graph $G' = G\backslash\{uu_1, uu_2\}$, then G' has an nsd-k-coloring c. Let $S_i = [k]\backslash(\{c(uu_3), c(uu_4)\} \bigcup\{c(v_ix_i)\}), i = 1, 2$, then $|S_i| \geq 3$. By Lemma 2.1 (i), we can choose $\alpha_i \in S_i, i = 1, 2$ such that $\alpha_1 \neq \alpha_2$ and u can be distinguished from u_3, u_4. Recolor u_iv_i with a color distinct from $\alpha_i, c(v_ix_i), w(x_i)-c(v_ix_i), \alpha_1+\alpha_2+c(uu_3)+c(uu_4)-\alpha_i$, $i = 1, 2$, then u can be distinguished from u_1, u_2. We obtain an nsd-k-coloring of G, a contradiction.

Now assume that u is adjacent to a bad 2-vertex u_1 with $u_1v_1 \in E(H), d_H(v_1) = 2, v_1x_1 \in E(H), x_1 \neq v_1$. Suppose to the contrary that $d_H(u_i) = 2, u_iv_i \in E(H), v_i \neq u, i = 2, 3, 4$. By Claim 2.3 (iii), $d_G(u) = d_H(u) = 4$. Consider $G' = G\backslash\{uu_1\}$, then G' has an nsd-k-coloring c. Let $S = [k]\backslash(\{c(uu_2), c(uu_3), c(uu_4)\} \bigcup\{c(v_1x_1)\})$. When $k \leq 7$, if $1 \in S$, then color uu_1 with $\alpha \in S\setminus\{1\}$, otherwise color uu_1 with $\alpha \in S\setminus\{2\}$. In both cases $w(u)+\alpha > w(u_i), i = 2, 3, 4$. Then recolor u_1v_1 with a color distinct from $\alpha, c(v_1x_1), w(u), w(x_1) - c(v_1x_1)$. We obtain an nsd-k-coloring of G, a contradiction. When $k \geq 8$, $|S| \geq 4$, we can choose $\alpha \in S$ such that $\alpha + w(u) \neq w(u_i), i = 2, 3, 4$. Then recolor u_1v_1 with a color distinct from $\alpha, c(v_1x_1), w(u), w(x_1) - c(v_1x_1)$. We obtain an nsd-k-coloring of G, a contradiction.

(iv) Suppose to the contrary that $d_H(u_i) = 2, i = 1, 2, \cdots, l$, $u_iv_i \in E(H), v_i \neq u, i = 1, 2, \cdots, l$ and $d_H(v_j) = 2, v_jx_j \in E(H), x_j \neq v_j, j = 1, 2, \cdots, l-1$. By Claim 2.3 (iii), $d_G(u) = d_H(u) = l$. Let $G' = G\backslash\{uu_1\}$, then G' has an nsd-k-coloring c. If $l < \Delta = k-1$, color uu_1 with $\alpha \in [k]\setminus(\{c(uu_2), \cdots, c(uu_l)\} \bigcup\{c(v_1x_1)\} \bigcup\{\alpha+\sum_{i=2}^{l-1} c(uu_i)\})$. Otherwise color uu_1 with $\alpha \in [k] \setminus (\{c(uu_2), \cdots, c(uu_l)\} \bigcup\{c(v_1x_1)\})$. In both cases, u can be distinguished from u_l. Properly recolor u_iv_i such that u can be distinguished from u_i and v_i can be distinguished from $x_i, i = 1, 2, \cdots, l-1$. We obtain an nsd-k-coloring of G, a contradiction.

□

Claim 2.5. *Let* $u \in V(H), d_H(u) = 5, uu_i \in E(H), i = 1, 2, 3, 4, 5.$

(i) *If* $\Delta(G) \geq 6$*, then* u *is adjacent to at most two bad 2-vertices. If* $\Delta(G) = 5$ *and* u *is adjacent to three bad 2-vertices, then* u *is adjacent to at most one good 2-vertex.*

Furthermore, if $\Delta(G) = 5$ *and* u *is a bad 5-vertex, then by Claim 2.3 (iii),* $d_G(u) = d_H(u) = 5$*. Let* $d_H(u_i) = 2, u_iv_i \in E(H)$*,* $d_H(v_i) = 2$*,* x_i *be the other neighbor of* $v_i, i = 1, 2, 3, 4$*, we have*

(ii) $d_H(u_5) \geq 4$.

(iii) *If* $d_H(u_5) = 4$*, then* u_5 *is adjacent to no bad 2-vertex.*

(iv) *If* $d_H(u_5) = 5$*, then* u_5 *is adjacent to at most two bad 2-vertices.*

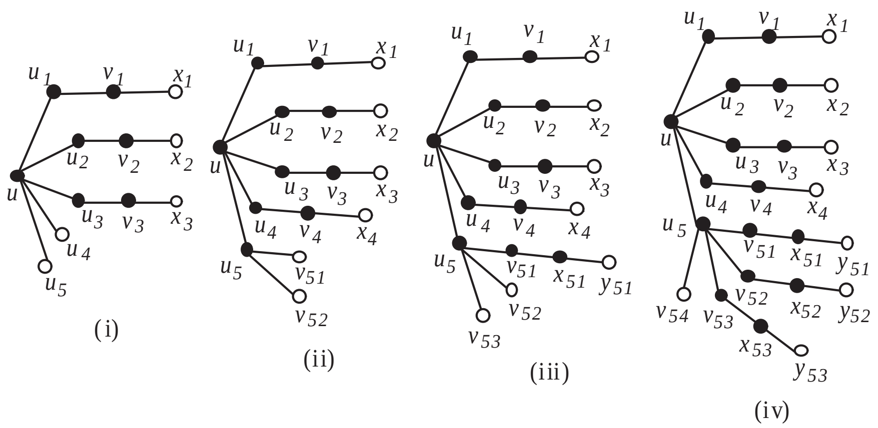

Figure 2.3: Illustration of Claim 2.5

Proof. (i) Assume $\Delta(G) \geq 6$. Suppose to the contrary that $d_H(u_1) = d_H(u_2) = d_H(u_3) = 2$, v_i is the other neighbor of u_i with $d_H(v_i) = 2$, x_i is the other neighbor of $v_i, i = 1, 2, 3$. By Claim 2.3 (iii), $d_G(u) = d_H(u) = 5$. Consider the graph $G' = G \backslash \{uu_1, uu_2, uu_3\}$, then G' has an *nsd*-k-coloring c. Let $S_i = [k] \backslash (\{c(uu_4), c(uu_5)\} \bigcup \{c(v_ix_i)\}), i = 1, 2, 3$, then $|S_i| \geq 4, i = 1, 2, 3$. We can choose $\alpha_i \in S_i, i = 1, 2, 3$ such that $\alpha_1, \alpha_2, \alpha_3$ are pairwise distinct and u can be distinguished by u_4, u_5. Recolor u_iv_i with a color distinct from $\alpha_i, c(v_ix_i), w(x_i) - c(v_ix_i), \sum_{i=1}^{3} \alpha_i + w(u) - \alpha_i, i = 1, 2, 3$, then u can be distinguished from u_1, u_2, u_3. We obtain an *nsd*-k-coloring of G, a contradiction.

Assume that $\Delta(G) = 5$. Suppose to the contrary that $d_H(u_i) = 2$, $u_iv_i \in E(H), v_i \neq u, i = 1, 2, \cdots, 5$ and $d_H(v_j) = 2, v_jx_j \in E(H), j = 1, 2, 3$. By Claim 2.3 (iii), $d_G(u) = d_H(u) = 5$. Consider the graph $G' = G \backslash \{uu_1\}$, then G' has an *nsd*-6-coloring c. Color uu_1 with $\alpha \in [6] \backslash (\{c(uu_2), \cdots, c(uu_5)\} \bigcup \{c(v_1x_1)\})$. Then recolor u_1v_1 with a color distinct from $\alpha, c(v_1x_1), w(x_i) - c(v_1x_1), \sum_{i=2}^{5} c(uu_i)$. It can be seen that $w(u) + \alpha > w(u_i), i = 2, \cdots, 5$, so we obtain an *nsd*-6-coloring of G, a contradiction.

(ii) By Claim 2.4 (iv), $d_H(u_5) \geq 3$. Suppose to the contrary that $d_H(u_5) = 3$, v_{51}, v_{52} are the other two neighbors of u_5. Consider the graph $G' = G \backslash \{uu_1, uu_2, uu_3, uu_4, uu_5\}$, then G' has an *nsd*-6-coloring c. Let $S_i = [6] \backslash \{c(v_i x_i)\}, i = 1, 2, 3, 4$, $S_5 = [6] \backslash (\{c(u_5 v_{51}), c(u_5 v_{52})\} \bigcup \{w(v_{51}) - w(u_5)\} \bigcup \{w(v_{52}) - w(u_5)\})$, then $|S_i| \geq 5, i = 1, 2, 3, 4$, $|S_5| \geq 2$. We can choose $\alpha_i \in S_i, i = 1, 2, 3, 4, 5$ such that $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are pairwise distinct and u can be distinguished by u_5. Obviously, u can be distinguished by u_1, u_2, u_3, u_4. Recolor $u_i v_i$ with a color distinct from $\alpha_i, c(v_i x_i), w(x_i) - c(v_i x_i), i = 1, 2, 3, 4$ and we obtain an *nsd*-6-coloring of G, a contradiction.

(iii) Let v_{51}, v_{52}, v_{53} be the other three neighbors of u_5. Suppose to the contrary that $d_H(v_{51}) = d_H(x_{51}) = 2, v_{51}x_{51} \in E(H)$, $x_{51}y_{51} \in E(H), y_{51} \neq v_{51}$. By Claim 2.3 (iii), $d_G(u_5) = d_H(u_5) = 4$. Consider the graph $G' = G \backslash \{uu_1, uu_2, uu_3, uu_4\}$, then G' has an *nsd*-6-coloring c. Assume that $c(uu_5) = \alpha_5$, $c(u_5 v_{5j}) = \beta_j, j = 1, 2, 3$, $c(v_i x_i) = \gamma_i, i = 1, 2, 3, 4$, $c(x_{51} y_{51}) = \eta$. Let $S_i = [6] \backslash (\{\gamma_i\} \bigcup \{\alpha_5\}), i = 1, 2, 3, 4$, then $|S_i| \geq 4, i = 1, 2, 3, 4$.

If there exists some $\gamma_i = \alpha_5$ $(i \in \{1, 2, 3, 4\})$ or $\gamma_i \neq \gamma_j$ $(i \neq j, i, j \in \{1, 2, 3, 4\})$, then we can choose $\alpha_i \in S_i, i = 1, 2, 3, 4$ such that u can be distinguished by u_5. Recolor $u_i v_i$ with a color distinct from $\gamma_i, \alpha_i, w(x_i) - c(v_i x_i), i = 1, 2, 3, 4$, we can obtain an *nsd*-6-coloring of G, a contradiction. Therefore, we assume that $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma, \alpha_5 \neq \gamma$. Then $S_1 = S_2 = S_3 = S_4 = [6] \backslash \{\alpha_5, \gamma\}$. Assume that $S_i = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, i = 1, 2, 3, 4$. Color uu_i with $\alpha_i, i = 1, 2, 3, 4$.

If $|\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \bigcap \{\beta_1, \beta_2, \beta_3\}| = 3$, then u can be distinguished by u_5, recolor $u_i v_i, i = 1, 2, 3, 4$ as previously and we obtain an *nsd*-6-coloring of G, a contradiction. Furthermore, $|\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \bigcap \{\beta_1, \beta_2, \beta_3\}| \geq 2$ because there are six colors in total. Therefore, $|\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \bigcap \{\beta_1, \beta_2, \beta_3\}| = 2$. Without loss of generality we assume that $\{\alpha_1, \alpha_2\} \subseteq \{\beta_1, \beta_2, \beta_3\}$.

If $\beta_1 = \gamma$, then $\{\alpha_1, \alpha_2\} = \{\beta_2, \beta_3\}$. If $\alpha_5 \neq \eta$, suppose u can not be distinguished by u_5, then $\gamma = \alpha_3 + \alpha_4$. Recolor uu_5 with γ and recolor $u_5 v_{51}$ with α_5, we obtain an *nsd*-6-coloring of G, a contradiction. Therefore, $\alpha_5 = \eta$. Recolor $u_5 v_{51}$ with one of α_3, α_4, or exchange the colors of uu_3 and uu_5, or exchange the colors of uu_4 and uu_5 such that u_5 can be distinguished by u, v_{52}, v_{53}. It is easy to see that u can be distinguished by u_1, u_2, u_3, u_4. Recolor $u_i v_i, i = 1, 2, 3, 4$ and $v_{51} x_{51}$ as previously and we can obtain an *nsd*-6-coloring of G, a contradiction.

If $\beta_1 \neq \gamma$. Without loss of generality we assume that $\beta_1 = \alpha_1, \beta_2 = \alpha_2$. Recolor $u_5 v_{51}$ with one of α_3, α_4, or exchange the colors of uu_3 and uu_5, or exchange the colors of uu_4 and uu_5 such that u_5 can be distinguished by u, v_{52}, v_{53}. It is easy to see that u can be distinguished by u_1, u_2, u_3, u_4. Recolor $u_i v_i, i = 1, 2, 3, 4$ and $v_{51} x_{51}$ as previously and we can obtain an *nsd*-6-coloring of G, a contradiction.

(iv) Let $v_{51}, v_{52}, v_{53}, v_{54}$ be the other four neighbors of u_5. Suppose to the contrary that

$d_H(v_{5i}) = d_H(x_{5i}) = 2, v_{5i}x_{5i} \in E(H)$, $x_{5i}y_{5i} \in E(H), y_{5i} \neq v_{5i}, i = 1,2,3$. By Claim 2.3 (iii), $d_G(u_5) = d_H(u_5) = 5$. Consider the graph $G' = G\backslash\{uu_1, uu_2, uu_3, uu_4\}$, then G' has an *nsd*-6-coloring c. Assume that $c(uu_5) = \alpha_5$, $c(u_5v_{5j}) = \beta_j, j = 1,2,3,4$, $c(v_ix_i) = \gamma_i, i = 1,2,3,4$. Let $S_i = [6]\backslash(\{\gamma_i\} \bigcup \{\alpha_5\}), i = 1,2,3,4$, then $|S_i| \geq 4, i = 1,2,3,4$.

If there exists some $\gamma_i = \alpha_5$ ($i \in \{1,2,3,4\}$) or $\gamma_i \neq \gamma_j$ ($i \neq j, i,j \in \{1,2,3,4\}$), then we can choose $\alpha_i \in S_i, i = 1,2,3,4$ such that u can be distinguished by u_5. Recolor u_iv_i with a color distinct from $\alpha_i, \gamma_i, w(x_i) - c(v_ix_i), i = 1,2,3,4$. We obtain an *nsd*-6-coloring of G, a contradiction. Therefore, we assume that $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma, \alpha_5 \neq \gamma$. Then $S_1 = S_2 = S_3 = S_4 = [6]\backslash\{\alpha_5, \gamma\}$. Assume that $S_i = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Color uu_i with α_i and recolor u_iv_i with a color distinct from $\gamma, \alpha_i, w(x_i) - c_(v_ix_i), i = 1,2,3,4$.

If $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \neq \{\beta_1, \beta_2, \beta_3, \beta_4\}$, then u can be distinguished by u_5, it is easy to see that u can be distinguished from u_1, u_2, u_3, u_4. We obtain an *nsd*-6-coloring of G, a contradiction. Therefore, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$. Without loss of generality we assume that $\alpha_i = \beta_i, i = 1,2,3,4$. If $c(x_{51}y_{51}) \neq \gamma$, then recolor u_5v_{51} with γ. We can see that u can be distinguished from u_5. If u_5 can not be distinguished from v_{54}, then exchange the colors of uu_1 and uu_5. Recolor $v_{51}x_{51}$ with a color distinct from $\gamma_1, c(x_{51}y_{51}), w(y_{51}) - c(x_{51}y_{51})$. We obtain an *nsd*-6-coloring of G, a contradiction. Similarly, $c(x_{52}y_{52}) = c(x_{53}y_{53}) = \gamma$. Recolor uu_5 with γ. Then recolor u_5v_{51} or recolor u_5u_{52} with α_5 such that u_5 can be distinguished by v_{54}. Recolor $v_{5i}x_{5i}$ with a color distinct from $\gamma, \alpha_5, \alpha_i, w(y_{5i}) - c(x_{5i}y_{5i}), i = 1,2$. We obtain an *nsd*-6-coloring of G, a contradiction. □

In order to complete the proof, we use a discharging procedure. For every $v \in V(H)$, we define the original charge of v to be $ch(v) = d_H(v) = l$. We then redistribute the charges according to the rules R1, R2 and R3 (below). To complete the proof, our aim is to prove that, for every vertex v, the new charge $ch^*(v)$ is at least 8/3.

The discharging rules are defined as follows:

(R1) Every 4^+-vertex gives $\frac{2}{3}$ to each adjacent bad 2-vertex.

(R2) Every 3^+-vertex gives $\frac{1}{3}$ to each adjacent good 2-vertex.

(R3) If u is a bad 5-vertex, $u_i, i = 1,2,3,4,5$ are the neighbors of u, u_1, u_2, u_3, u_4 are bad 2-vertices, then u_5 gives $\frac{1}{3}$ to u.

Case $l = 2$**.** Observe that $ch(v) = 2$. Suppose v is a good 2-vertex. Hence, by (R2), $ch^*(v) \geq 2+2\times\frac{1}{3} = \frac{8}{3}$. Suppose v is bad, By Claim 2.4 (i) and Claim 2.4 (ii), v is adjacent to at most one 2-vertex and is adjacent to a 4^+-vertex. Hence, by (R1), $ch^*(v) = 2+1\times\frac{2}{3} = \frac{8}{3}$.

Case $l = 3$**.** Observe that $ch(v) = 3$. By Claim 2.4 (ii), v is adjacent to no bad 2-vertex and is adjacent to at most one good 2-vertex. By Claim 2.5 (ii), v is adjacent to no bad 5-vertex. By (R2) and (R3), $ch^*(v) \geq 3 - 1 \times \frac{1}{3} = \frac{8}{3}$.

Case $l = 4$**.** Observe that $ch(v) = 4$. Suppose v is not adjacent to a bad 2-vertex. Then, by (R2) and (R3), $ch^*(v) \geq 4 - 4 \times \frac{1}{3} = \frac{8}{3}$. Assume now, v is adjacent to a bad 2-vertex. By Claim 2.4 (iii), v is adjacent to at most two good 2-vertices. By Claim 2.5 (iii), v is adjacent to no bad 5-vertex. Hence by (R1), (R2) and (R3), $ch^*(v) \geq 4 - 1 \times \frac{2}{3} - 2 \times \frac{1}{3} = \frac{8}{3}$.

Case $l = 5$**.** Observe that $ch(v) = 5$. Suppose v is adjacent to at most two bad 2-vertices, then by (R1), (R2) and (R3), $ch^*(v) \geq 5 - 2 \times \frac{2}{3} - 3 \times \frac{1}{3} = \frac{8}{3}$. If v is adjacent to three bad 2-vertices, then by Claim 2.5 (i) and (iv), v is adjacent to at most one good 2-vertex and v is adjacent to no bad 5-vertex. Hence by (R1), (R2) and (R3), $ch^*(v) \geq 5 - 3 \times \frac{2}{3} - 1 \times \frac{1}{3} = \frac{8}{3}$. Assume now, v is a bad 5-vertex. By (R1), (R2) and (R3), $ch^*(v) \geq 5 - 4 \times \frac{2}{3} + 1 \times \frac{1}{3} = \frac{8}{3}$.

Case $l \geq 6$**.** Observe that $ch(v) = l$. By Claim 2.5 (i), any 5-vertex in H is good. By Claim 2.4 (iv), v is adjacent to at most $(l - 1)$ bad 2-vertices. Moreover if v is adjacent to $(l - 1)$ bad 2-vertices, then its last neighbor has degree at least 3. It follows by (R1), $ch^*(v) \geq l - (l - 1) \times \frac{2}{3} \geq \frac{8}{3}$.

This completes the proof of Theorem 1.3.

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