Upper bounds on the signed (k,k)-domatic number of digraphs

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Abstract

Let D be a simple digraph with vertex set $V(D)$, and let $f : V(D) \to \{-1, 1\}$ be a two-valued function. If $k \geq 1$ is an integer and $\sum_{x \in N^-[v]} f(x) \geq k$ for each $v \in V(D)$, where $N^-[v]$ consists of v and all vertices of D from which arcs go into v, then f is a signed k-dominating function on D. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed k-dominating functions on D with the property that $\sum_{i=1}^{d} f_i(x) \leq k$ for each $x \in V(D)$, is called a signed (k,k)-dominating family (of functions) on D. The maximum number of functions in a signed (k,k)-dominating family on D is the signed (k,k)-domatic number of D.

In this article we mainly present upper bounds on the signed (k,k)-domatic number, in particular for regular digraphs.

2010 Mathematics Subject Classification: 05C20, 05C69

Keywords and phrases: Signed (k,k)-domatic number, Signed k-dominating function, Signed k-domination number, Regular digraphs

1. Terminology and introduction

We consider finite and simple digraphs D with vertex set $V(D)$ and arc set $A(D)$. The *order* $n = n(D)$ of a digraph D is the number of its vertices. If v is a vertex of the digraph D, then $N^+(v) = N_D^+(v) = \{x|(v,x) \in A(D)\}$ and $N^-(v) = N_D^-(v) = \{x|(x,v) \in A(D)\}$ are the *out-neigbourhood* and *in-neighbourhood* of the vertex v. We call the vertices in $N^+(v)$ and $N^-(v)$ the *out-neighbours* and *in-neighbours* of v. Likewise, $N^+[v] = N_D^+[v] = N^+(v) \cup \{v\}$ and $N^-[v] = N_D^-[v] = N^-(v) \cup \{v\}$. The numbers $d_D^+(v) = d^+(v) = |N^+(v)|$ and $d_D^-(v) = d^-(v) = |N^-(v)|$ are the *out-degree* and *in-degree* of v, respectively. The *minimum* and *maximum out-degree* and *minimum* and *maximum in-degree* of a digraph D are denoted by $\delta^+(D)$, $\Delta^+(D)$, $\delta^-(D)$ and $\Delta^-(D)$, respectively. A digraph D is *regular* or *r-regular* if $\delta^+(D) = \Delta^+(D) = \delta^-(D) = \Delta^-(D) = r$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. If X and Y are disjoint subsets of $V(D)$, then (X, Y) is the set of arcs from X to Y. The *complete digraph* of order n is denoted by K_n^*. Consult [4, 5] for notation and terminology which are not defined here. Further information on domination and related topics are contained in [2, 6, 10, 11].

If $k \geq 1$ is an integer, then the *signed k-dominating function* is defined as a two-valued function $f : V(D) \to \{-1, 1\}$ such that $\sum_{x \in N^-[v]} f(x) \geq k$ for each $v \in V(D)$. The sum $\sum_{x \in V(D)} f(x) = f(V(D))$ is called the weight $w(f)$ of f. The minimum of weights $w(f)$, taken over all signed k-dominating functions f on D, is called the *signed k-domination number* of D, denoted by $\gamma_{kS}(D)$. The signed k-domination number of digraphs was introduced by Atapou, Hajypory, Sheikholeslami and Volkmann [3]. If $k = 1$, then the signed k-domination number $\gamma_{kS}(D)$ is the usual *signed domination number* $\gamma_S(D)$, which was introduced by Zelinka in [12] and has been studied by several authors (see for example [7, 9]).

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed k-dominating functions on D with the property that $\sum_{i=1}^{d} f_i(x) \leq k$ for each vertex $x \in V(D)$, is called in [8] a *signed (k, k)-dominating family* on D. The maximum number of functions in a signed (k, k)-dominating family on D is the *signed (k, k)-domatic number* of D, denoted by $d_S^k(D)$. As the assumption $\delta^-(D) \geq k - 1$ is necessary, we always assume that when we discuss $\gamma_{kS}(D)$ or $d_S^k(D)$, all digraphs involved satisfy $\delta^-(D) \geq k - 1$. The special case $k = 1$ was also discussed in [1, 9].

In this paper we continue the studies of the signed (k, k)-domatic number of digraphs. First we present upper bounds on $d_S^k(D)$ for regular digraphs in terms of order. Finally, we show that $d_S^k(D) \leq n - 1$ with exception of the case that the digraph D of order n is isomorphic to the complete digraph K_n^* and n and k are of the same parity with $k \leq n - 2$. For regular digraphs we can improve this result by the upper bound $n - 2$.

The following basic results are useful for our investigations.

Proposition 1.1 (Sheikholeslami, Volkmann [8] 2012) If D is a digraph with $\delta^-(D) \geq k - 1$, then $d_S^k(D) \leq \delta^-(D) + 1$.

Proposition 1.2 (Sheikholeslami, Volkmann [8] 2012) If D is a digraph such that $\delta^-(D)$ and k are both odd or $\delta^-(D)$ and k are both even, then

$$d_S^k(D) \leq \frac{k}{k+1}(\delta^-(D) + 1).$$

Proposition 1.3 (Sheikholeslami, Volkmann [8] 2012) Let $k \geq 2$ be an integer, and let D be a digraph with $\delta^-(D) \geq k-1$. Then $d_S^k(D) = 1$ if and only if for every vertex $v \in V(D)$, the set $N^+[v]$ contains a vertex x such that $d^-(x) \leq k$.

Proposition 1.4 (Sheikholeslami, Volkmann [8] 2012) If D is a digraph of order n with $\delta^-(D) \geq k-1$, then $\gamma_{kS}(D) \cdot d_S^k(D) \leq k \cdot n$.

2. Regular digraphs

Throughout this section, if f is a signed k-dominating function on a digraph D, then we let P and M denote the sets of those vertices in D which are assigned under f the values 1 and -1, respectively. Thus $|P| + |M| = n(D)$.

Theorem 2.1 If $k \geq 1$ is an odd integer, and D is a $2r$-regular digraph of odd order $n = 2q+1 \geq 3$ with $r \leq q-1$, then

$$d_S^k(D) \leq \left\lfloor \frac{kn}{k+2} \right\rfloor .$$

Proof. If f is an arbitrary signed k-dominating function on D, then we firstly show that

$$|P| \geq q + \frac{k+3}{2}. \tag{1}$$

Because of $\sum_{x \in N^-[y]} f(x) \geq k$ for each vertex $y \in V(D)$, we observe that each vertex $u \in P$ has at most $(2r+1-k)/2$ in-neighbors in M and thus

$$|(M,P)| \leq |P| \cdot \frac{2r+1-k}{2}. \tag{2}$$

In addition, each vertex $v \in M$ has at most $(2r-k-1)/2$ in-neighbors in M and so

$$|A(D[M])| \leq |M| \cdot \frac{2r-k-1}{2}.$$

Since $d_D^+(x) = 2r$ for each vertex $x \in V(D)$, we deduce from the last inequality that

$$|(M,P)| \geq 2r \cdot |M| - |M| \cdot \frac{2r-k-1}{2} = |M| \cdot \frac{2r+k+1}{2}. \tag{3}$$

Using (2) and (3), we obtain

$$|P| \cdot \frac{2r+1-k}{2} \geq (2q+1-|P|)\frac{2r+1+k}{2}$$

and thus

$$|P| \geq \frac{(2r+k+1)(2q+1)}{4r+2}.$$

If we suppose that $|P| \leq q + \frac{k+1}{2}$, then the last inequality leads to

$$q + \frac{k+1}{2} \geq |P| \geq \frac{(2r+k+1)(2q+1)}{4r+2}.$$

It follows that $r \geq q$. This is a contradiction to the hypothesis $r \leq q-1$ and thus (1) is proved.

Now let $\{f_1, f_2, \ldots, f_d\}$ be a signed (k,k)-dominating family on D with $d = d_S^k(D)$. Since $\sum_{i=1}^{d} f_i(u) \leq k$ for every $u \in V(D)$, each of these sums contains at least $\lceil (d-k)/2 \rceil$ summands of value -1. Using this and inequality (1), we see that the sum

$$\sum_{x \in V(D)} \sum_{i=1}^{d} f_i(x) = \sum_{i=1}^{d} \sum_{x \in V(D)} f_i(x) \tag{4}$$

contains at least $(2q+1)\lceil (d-k)/2 \rceil$ summands of value -1 and at least $d(q+(k+3)/2)$ summands of value 1. As the sum (4) consists of exactly $d(2q+1)$ summands, it follows that

$$(2q+1)\frac{d-k}{2} + d\left(q + \frac{k+3}{2}\right) \leq (2q+1)\left\lceil \frac{d-k}{2} \right\rceil + d\left(q + \frac{k+3}{2}\right) \leq d(2q+1).$$

We deduce that

$$(2q+1)(d-k) + d(2q+k+3) \leq 2d(2q+1),$$

and thus $d(k+2) \leq k(2q+1)$. This yields to the desired bound immediately. □

Example 2.2 Let D be the complete digraph K_n^* of odd order $n = 2q+1 \geq 3$, and let $\{1, 2, \ldots, n\}$ be the vertex set of D. Since D is complete, we observe that $N^-[v] = V(D)$ for each vertex $v \in V(D)$. Let $k \geq 1$ be an odd integer with $k \leq 2q-1$. Define the signed k-dominating functions $f_1, f_2, \ldots, f_n$ by

$$f_i(i) = f_i(i+1) = \cdots = f_i\left(i - 1 + \frac{2q+k+1}{2}\right) = 1$$

and $f_i(j) = -1$ for the remaining vertices $j \in V(D)$ for $i = 1, 2, \ldots, n$, where all numbers are taken modulo n. It is easy to see that

$$\sum_{x \in V(D)} f_i(x) = \frac{2q+k+1}{2} - \left(2q + 1 - \frac{2q+k+1}{2}\right) = k$$

for $1 \leq i \leq n$ and $\sum_{i=1}^{n} f_i(x) = k$ for each $x \in V(D)$. Hence $\{f_1, f_2, \ldots, f_n\}$ is a signed (k,k)-dominating family on D, and we conclude that $d_S^k(D) \geq n$. In view of

Proposition 1.1, it holds $d_S^k(D) \leq \delta^-(D) + 1 = n$, and so $d_S^k(K_n^*) = n$ when n and k are odd and $k \leq n-2$.

Example 2.2 demonstrates that the condition $r \leq q-1$ in Theorem 2.1 is necessary, since that theorem is not valid for $r = q$. If k is even in Theorem 2.1, then we can improve the bound on the signed (k,k)-domatic number.

Theorem 2.3 If $k \geq 2$ is an even integer, and D is a $2r$-regular digraph of odd order $n = 2q+1 \geq 3$ with $r \leq q-1$, then

$$d_S^k(D) \leq \left\lfloor \frac{kn}{k+3} \right\rfloor.$$

Proof. If f is a signed k-dominating function on D, then we show that

$$|P| \geq q + \frac{k+4}{2}. \tag{5}$$

As D is a $2r$-regular digraph and k is even, the condition $\sum_{x\in N^-[y]} f(x) \geq k$ leads to $\sum_{x\in N^-[y]} f(x) \geq k+1$ for each vertex $y \in V(D)$. This implies that each vertex $u \in P$ has at most $(2r-k)/2$ in-neighbors in M and thus

$$|(M,P)| \leq |P| \cdot \frac{2r-k}{2}. \tag{6}$$

In addition, each vertex $v \in M$ has at most $(2r-k-2)/2$ in-neighbors in M and so

$$|A(D[M])| \leq |M| \cdot \frac{2r-k-2}{2}.$$

Since $d_D^+(x) = 2r$ for each vertex $x \in V(D)$, we deduce from the last inequality that

$$|(M,P)| \geq 2r \cdot |M| - |M| \cdot \frac{2r-k-2}{2} = |M| \cdot \frac{2r+k+2}{2}. \tag{7}$$

Applying (6) and (7), we obtain

$$|P| \cdot \frac{2r-k}{2} \geq (2q+1-|P|)\frac{2r+2+k}{2}$$

and thus

$$|P| \geq \frac{(2r+k+2)(2q+1)}{4r+2}.$$

If we suppose that $|P| \leq q + \frac{k+2}{2}$, then the last inequality leads to

$$q + \frac{k+2}{2} \geq |P| \geq \frac{(2r+k+2)(2q+1)}{4r+2}.$$

It follows that $r \geq q$. This is a contradiction to the hypothesis $r \leq q-1$ and thus (5) is proved.

Now let $\{f_1, f_2, \ldots, f_d\}$ be a signed (k,k)-dominating family on D with $d = d_S^k(D)$. Since $\sum_{i=1}^{d} f_i(u) \leq k$ for every $u \in V(D)$, each of these sums contains at least $\lceil (d-k)/2 \rceil$ summands of value -1. Using this and inequality (5), we see that the sum

$$\sum_{x \in V(D)} \sum_{i=1}^{d} f_i(x) = \sum_{i=1}^{d} \sum_{x \in V(D)} f_i(x) \tag{8}$$

contains at least $(2q+1)\lceil (d-k)/2 \rceil$ summands of value -1 and at least $d(q+(k+4)/2)$ summands of value 1. As the sum (8) consists of exactly $d(2q+1)$ summands, it follows that

$$(2q+1)\frac{d-k}{2} + d\left(q + \frac{k+4}{2}\right) \leq (2q+1)\left\lceil \frac{d-k}{2} \right\rceil + d\left(q + \frac{k+4}{2}\right) \leq d(2q+1).$$

We deduce that

$$(2q+1)(d-k) + d(2q+k+4) \leq 2d(2q+1),$$

and thus $d(k+3) \leq k(2q+1)$. This yields to the desired bound immediately. □

The proofs of the next upper bounds on regular digraphs are analogously to that of Theorem 2.1.

Theorem 2.4 If $k \geq 1$ is an odd integer, and D is a $2r$-regular digraph of even order $n = 2q \geq 4$, then

$$d_S^k(D) \leq \left\lfloor \frac{kn}{k+1} \right\rfloor.$$

Theorem 2.5 If $k \geq 2$ is an even integer, and D is a $(2r+1)$-regular digraph of odd order $n = 2q+1 \geq 3$, then

$$d_S^k(D) \leq \left\lfloor \frac{kn}{k+1} \right\rfloor.$$

Theorem 2.6 If $k \geq 2$ is an even integer, and D is a $(2r+1)$-regular digraph of even order $n = 2q \geq 4$ with $r \leq q-2$, then

$$d_S^k(D) \leq \left\lfloor \frac{kn}{k+2} \right\rfloor.$$

If k is even in Theorem 2.4 or k odd in Theorems 2.5 and 2.6, then we can improve the upper bounds on the signed (k,k)-domatic number analogously to the proof of Theorem 2.3.

Theorem 2.7 If $k \geq 2$ is an even integer, and D is a $2r$-regular digraph of even order $n = 2q \geq 4$, then

$$d_S^k(D) \leq \left\lfloor \frac{kn}{k+2} \right\rfloor.$$

Theorem 2.8 If $k \geq 1$ is an odd integer, and D is a $(2r+1)$-regular digraph of odd order $n = 2q+1 \geq 3$, then

$$d_S^k(D) \leq \left\lfloor \frac{kn}{k+2} \right\rfloor.$$

Theorem 2.9 If $k \geq 1$ is an odd integer, and D is a $(2r+1)$-regular digraph of even order $n = 2q \geq 4$ with $r \leq q-2$, then

$$d_S^k(D) \leq \left\lfloor \frac{kn}{k+3} \right\rfloor.$$

Example 2.10 Let D be the complete digraph K_n^* of even order $n = 2q \geq 4$, and let $\{1, 2, \ldots, n\}$ be the vertex set of D. Let $k \geq 2$ be an even integer with $k \leq 2q-2$. Define the signed k-dominating functions $f_1, f_2, \ldots, f_n$ by

$$f_i(i) = f_i(i+1) = \cdots = f_i\left(i - 1 + \frac{2q+k}{2}\right) = 1$$

and $f_i(j) = -1$ for the remaining vertices $j \in V(D)$ for $i = 1, 2, \ldots, n$, where all numbers are taken modulo n. It is easy to see that

$$\sum_{x \in V(D)} f_i(x) = \frac{2q+k}{2} - \left(2q - \frac{2q+k}{2}\right) = k$$

for $1 \leq i \leq n$ and $\sum_{i=1}^{n} f_i(x) = k$ for each $x \in V(D)$. Hence $\{f_1, f_2, \ldots, f_n\}$ is a signed (k,k)-dominating family on D, and we conclude that $d_S^k(D) \geq n$. In view of Proposition 1.2, it holds $d_S^k(D) \leq \delta(G) + 1 = n$ and so $d_S^k(K_n^*) = n$ when n and k are even and $k \leq n-2$.

Example 2.10 shows that Theorem 2.6 is not valid for $r = q-1$.

3. Further upper bounds

Theorem 3.1 Let $k \geq 1$ be an integer, and let K_n^* be the complete digraph of order $n \geq 3$.

(a) If $n-1 \leq k \leq n$, then $d_S^k(K_n^*) = 1$.

(b) If $k \leq n-2$ and k and n are odd or k and n are even, then $d_S^k(K_n^*) = n$.

(c) If $k \leq n-2$ and k and n are of different parity, then $d_S^k(K_n^*) \leq n-2$.

Proof. (a) Proposition 1.3 and the condition $n \geq 3$ lead to $d_S^k(K_n^*) = 1$ when $n-1 \leq k \leq n$.

(b) Examples 2.2 and 2.10 show that (b) is valid.

(c) If n is odd and k is even, then $\delta^-(K_n^*) = n-1$ is even. According to Proposition 1.2, we have

$$d_S^k(K_n^*) \leq \frac{k}{k+1}(\delta^-(K_n^*)+1) = \frac{kn}{k+1} < n-1$$

when $k < n-1$ and so $d_S^k(K_n^*) \leq n-2$ when $k \leq n-2$. If n is even and k is odd, then $\delta^-(K_n^*) = n-1$ is odd. Again Proposition 1.2 yields to the desired bound. □

Corollary 3.2 If $k \geq 1$ is an integer and D a digraph of order $n \geq 3$, then $d_S^k(D) \leq n-1$ with exception of the case that D is isomorphic to the complete digraph K_n^* and k and n are odd or k and n are even with $k \leq n-2$.

Proof. If $\delta^-(D) \leq n-2$, then Proposition 1.1 implies that $d_S^k(D) \leq \delta^-(D)+1 \leq n-1$. If $\delta^-(D) = n-1$, then Theorem 3.1 leads to the desired result. □

For regular digraphs we can improve the bound in Corollary 3.2.

Theorem 3.3 If $k \geq 1$ is an integer and D a δ-regular digraph of order $n \geq 3$, then $d_S^k(D) \leq n-2$ with exception of the case that D is isomorphic to the complete digraph K_n^* and k and n are odd or k and n are even with $k \leq n-2$.

Proof. If $\delta = n-1$, then Theorem 3.1 leads to the desired result. If $\delta \leq n-3$, then we deduce from Proposition 1.1 that $d_S^k(D) \leq \delta+1 \leq n-2$.

Now assume that $\delta = n-2$. If δ and k are odd or δ and k are even, then Proposition 1.2 implies that

$$d_S^k(D) \leq \frac{k}{k+1}(\delta+1) = \frac{k}{k+1}(n-1) < n-1$$

and so $d_S^k(D) \leq n-2$.

If δ is even and k is odd, then $n = \delta+2$ is even. Applying Theorem 2.4, we obtain

$$d_S^k(D) \leq \frac{kn}{k+1} < n-1$$

when $k < n-1$ and so $d_S^k(D) \leq n-2$ when $k < n-1$. In the remaining case that $n-1 \leq k \leq n$, we deduce from Proposition 1.3 that $d_S^k(D) = 1 \leq n-2$.

Finally, assume that δ is odd and k is even. Then $n = \delta+2$ is odd, and Theorem 2.5 leads to

$$d_S^k(D) \leq \frac{kn}{k+1} < n-1$$

when $k < n-1$ and so $d_S^k(D) \leq n-2$ when $k < n-1$. In the remaining case that $n-1 \leq k \leq n$, we deduce from Proposition 1.3 that $d_S^k(D) = 1 \leq n-2$. Since we have discussed all possible cases, the proof is complete. □

If D is a digraph of order n with $\delta^-(D) \geq k+1$, then we have proved in [8] that

$$\gamma_{kS}(D) + d_S^k(D) \leq n + k. \tag{9}$$

Next we will show that this inequality remains valid for all digraphs with $\delta^-(D) \geq k-1$.

Theorem 3.4 Let $k \geq 1$ be an integer, and let D be a digraph of order n. If $\delta^-(D) \geq k-1$, then

$$\gamma_{kS}(D) + d_S^k(D) \leq n + k.$$

Proof. Assume first that $k-1 \leq \delta^-(D) \leq k$. If $\gamma_{kS}(D) = n$, then $d_S^k(D) = 1$ and therefore $\gamma_{kS}(D) + d_S^k(D) = n+1 \leq n+k$, as desired. If $\gamma_{kS}(D) \leq n-1$, then Proposition 1.1 and the condition $\delta^-(D) \leq k$ imply

$$\gamma_{kS}(D) + d_S^k(D) \leq n - 1 + \delta^-(D) + 1 \leq n - 1 + k + 1 = n + k.$$

In the case $\delta^-(D) \geq k+1$, the desired bound follows from inequality (9), and we are done. □

For $k \geq 2$ we will improve Theorem 3.4.

Theorem 3.5 Let $k \geq 2$ be an integer, and let D be a digraph of order $n \geq 3$ such that $\delta^-(D) \geq k-1$. Then

$$\gamma_{kS}(D) + d_S^k(D) \leq n + k - 1$$

with exception of the case that D is isomorphic to the complete digraph K_n^* and k and n are odd or k and n are even with $k \leq n-2$.

Proof. Assume first that $k-1 \leq \delta^-(D) \leq k$. If $\gamma_{kS}(D) = n$, then $d_S^k(D) = 1$ and therefore $\gamma_{kS}(D) + d_S^k(D) = n+1 \leq n+k-1$ since $k \geq 2$. If $\gamma_{kS}(D) \leq n-1$, then $\gamma_{kS}(D) \leq n-2$ (see also [3]). Therefore we deduce from Proposition 1.1 and the condition $\delta^-(D) \leq k$ that

$$\gamma_{kS}(D) + d_S^k(D) \leq n - 2 + \delta^-(D) + 1 \leq n - 2 + k + 1 = n + k - 1.$$

Assume next that $\delta^-(D) \geq k+1$. If $\gamma_{kS}(D) \leq k$, then Corollary 3.2 implies that $\gamma_{kS}(D) + d_S^k(D) \leq k + n - 1$ with exception of the case that D is isomorphic to the complete digraph K_n^* and k and n are odd or k and n are even with $k \leq n-2$.

Now assume that $\gamma_{kS}(D) \geq k+1$. If $\gamma_{kS}(D) = n$, then $d_S^k(D) = 1$ and so $\gamma_{kS}(D) + d_S^k(D) = n+1 \leq n+k-1$, as desired. If $\gamma_{kS}(D) \leq n-1$, then $k+1 \leq \gamma_{kS}(D) \leq n-2$.

Using Proposition 1.4 and the fact that the function $g(x) = x + (kn)/x$ is decreasing for $k+1 \le x \le \sqrt{kn}$ and increasing for $\sqrt{kn} \le x \le n-2$, we obtain

$$\begin{aligned} \gamma_{kS}(D) + d_S^k(D) &\le \gamma_{kS}(D) + \frac{kn}{\gamma_{kS}(D)} \\ &\le \max\left\{k+1+\frac{kn}{k+1}, n-2+\frac{kn}{n-2}\right\} < n+k \end{aligned}$$

and thus $\gamma_{kS}(D) + d_S^k(D) \le n+k-1$ with exception of the case that $\delta^-(D) = k+1 = n-1$ and so D is isomorphic to the complete digraph K_n^* with $k = n-2$. This completes the proof. □

It is easy to see that $\gamma_{kS}(K_n^*) = k$ when $n+k$ is even (see also [3]). Using Examples 2.2 and 2.10, we see that $\gamma_{kS}(K_n^*) + d_S^k(K_n^*) = k+n$ when $n+k$ is even and $k \le n-2$. Therefore the given upper bound in Theorem 3.4 is sharp and the bound in Theorem 3.5 is not valid in these cases. If C_n is an oriented cycle of length n, then $\gamma_{1S}(D) + d_S^1(D) = n+1$. This example shows that Theorem 3.5 is not valid for $k=1$ in general.

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