Maximal thermo-geometric parameter in a nonlinear heat conduction equation

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Abstract. A nonlinear heat conduction equation is studied, and the maximal thermo-geometric parameter in the equation is analytically determined, above which thermal instability occurs. The first-order result yields an acceptable error, and the variational iteration method is recommended for a higher accurate prediction.

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1. Introduction

We consider a rectangular longitudinal one-dimensional fin, which is attached to a fixed base surface of temperature T_b and extends into a fluid of temperature T_b. The dimensionless governing equation is[1]

(1.1) $$\frac{d^2\theta}{dx^2}+\beta\theta\frac{d^2\theta}{dx^2}+\beta(\frac{d\theta}{dx})^2-M^2(1+\beta\theta)^n=0\,,\quad \theta>0$$

with boundary conditions

(1.2) $\theta'(0)=0$ and $\theta(1)=1$

where θ is the dimensionless temperature, $\beta=\lambda(T_b-T_a)$ is the gradient of thermal conductivity, M is the thermo-geometric parameter. The exponent, n, represents laminar film boiling or condensation when $n=-1/4$, laminar natural convection when $n=1/4$, turbulent natural convection when $n=1/3$, nucleate boiling when n=2, radiation when n=3 and n vanishes for a constant heat transfer coefficient.

It is very important to study the effect of M on the heat transfer. It is obvious that the increase of M might result in negative θ at x=0, contradicting the assumption. It is, therefore, important to identify the maximal value for the thermo-geometric parameter.

The detailed derivation of Eq.(1) was given in [1], and the thermal characteristics was elucidated in [2]. Harley and Moitsheki [1] gave a numerical investigation, and obtained the maximal values for various n. Some effective analytical methods were successfully applied to the problem [3,4,5], there are alternative numerical/analytical methods, such as the three-point implicit block multistep method [6], the variational iteration method [7-12], reproducing kernel method [13,14], a complete review on various analytical methods is available in [15]. In this paper we will suggest a simple analytical approach to identification of the maximal value of the thermo-geometric parameter.

2. Maximal thermo-geometric parameter

In this study, neither an exact solution nor an approximate solution is searched for, only the maximal M in Eq.(1.1) is considered. For this end, we choose a very simple trial function in the form

(2.1) $$\theta(x) = a_0 + a_1 x + a_2 x^2$$

By the boundary conditions, Eq.(1.2), we have

(2.2) $$a_1 = 0$$

(2.3) $$a_0 + a_1 + a_2 = 1$$

By Eq.(1.1) and Eq.(1.2), we obtain

(2.4) $$\theta''(0) = M^2 (1 + \beta\theta(0))^n = M^2 (1 + \beta a_0)^n$$

Eq.(2.4) means

(2.5) $$2a_2 = M^2 (1 + \beta a_0)^n$$

Submitting Eq.(2.2) and (2.5) into Eq.(2.3) results in

(2.6) $$a_0 + \frac{M^2}{2}(1 + \beta a_0)^n = 1$$

Setting $a_0 = \theta_{\min}(0) = 0$, we obtain maximal value for *M*, which is

(2.7) $$M_{\max} = \sqrt{2} = 1.414$$

Comparison of Eq.(2.7) with the numerical results given in [1] reveals that the maximal error is 16.5% for $-4 < n < 3$. The accuracy is 3% and 5.4% for *n*=2 and *n*=3, respectively.

When the thermo-geometric parameter reaches its maximal value, thermal instability occurs [2], so in practical applications we should follow $M << M_{\max}$, and the 16.5% error is acceptable.

If a higher accurate prediction is needed, the variational iteration algorithm [7,15] is recommended.

According to the variational iteration method [7,15], the following iteration formulation (variational iteration algorithm-II [15]) can be constructed

$$(2.8)\ \theta_{p+1}(x)=\theta_0(x)+\int_0^x (x-s)\left\{\beta\theta_p(s)\frac{d^2\theta_p(s)}{ds^2}+\beta\left(\frac{d\theta_p(s)}{ds}\right)^2-M^2\left[1+\beta\theta_p(s)\right]^n\right\}ds$$

We begin with $\theta_0(x)=\theta(0)=a_0$, by Eq.(2.8), we have

$$(2.9)\qquad \theta_1(x)=a_0+\int_0^x (x-s)\left\{-M^2(1+\beta a_0^n)\right\}ds=a_0-\frac{1}{2}M^2(1+\beta a_0^n)x^2$$

If the first-order approximate solution is enough, then by the boundary condition, $\theta(1)=1$, the following result is obtained.

$$(2.10)\qquad \theta_1(1)=a_0-\frac{1}{2}M^2(1+\beta a_0^n)=1$$

which is exactly same with Eq.(2.6).

The solution process can continue without any difficulty by using some mathematical software, and a higher accurate result can be obtained.

3. Conclusion

In practical applications, we need neither an exact solution nor an approximate solution, but a criterion for some parameters in the studied equation, for example, the condition of resonance for a nonlinear oscillator. In this paper we suggest a simple but effective approach to identification of the maximal thermo-geometric parameter in Eq.(1), the 16.5% accuracy of the first-order prediction is acceptable considering it should follow $M \ll M_{\max}$ though a higher accuracy can be obtained by the variational iteration method.

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