

Rainbow C_4 's and Directed C_4 's: the Bipartite Case Study

Bo Ning^{a*} and Jun Ge^{b†}

^aDepartment of Applied Mathematics, School of Science,
Northwestern Polytechnical University, Xi'an, Shaanxi 710072, P.R. China

^bSchool of Mathematical Sciences,
Xiamen University, Xiamen, Fujian 361005, P.R. China

Abstract

In this paper we obtain a new sufficient condition for the existence of directed cycles of length 4 in oriented bipartite graphs. As a corollary, a conjecture of H. Li is confirmed. As an application, a sufficient condition for the existence of rainbow cycles of length 4 in bipartite edge-colored graphs is obtained.

Keywords: Rainbow cycle; Edge-colored graph; Directed cycle; Oriented bipartite graph

Mathematics Subject Classification (2010): 05C38, 05C15, 05C20

1 Introduction

For terminology and notation not defined here, we refer to [2]. Let $G = (V, E)$ be a simple graph. An *edge-coloring* of G is a mapping $C : E \rightarrow \mathbb{N}^+$, where \mathbb{N}^+ is the set of positive integers. We call G^c an *edge-colored graph* (or briefly, a *colored graph*) if G is assigned an edge-coloring C . Let v be a vertex of G^c . The *color degree* of v in G^c , denoted by $d_G^c(v)$ (or briefly, $d^c(v)$), is the number of colors of the edges incident to v . A *color neighborhood set* $N^c(v)$ of v is a subset of $N(v)$, the neighborhood of v , such that the colors of edges between v and $N^c(v)$ are pairwise distinct. Let H be a subgraph of G , then $C(H) = \{C(e) : e \in E(H)\}$ is called the *color set* of H .

A subgraph of a colored graph is called *rainbow* (sometimes called *heterochromatic*, or *colorful*) if all edges of it have distinct colors. The existence of rainbow subgraphs has been studied for a long time. A problem of rainbow Hamilton cycles in colored complete graphs was mentioned by Erdős, Nešetřil and Rödl [6], and later studied by Hahn and Thomassen

*E-mail address: ningbo_math84@mail.nwpu.edu.cn (BO NING).

†Corresponding author. E-mail address: mathsgjun@163.com (JUN GE).

[8], Frieze and Reed [7] and Albert, Frieze and Reed [1], respectively. Rainbow matchings were studied by Wang and H. Li [16], Lesaulnier et al. [11], and Kostochka and Yancey [10]. Chen and X. Li [4, 5] studied the existence of long rainbow paths. A recent article on strong rainbow connection can be found in [15]. For a survey on the study of rainbow subgraphs in colored graphs, we refer to [9].

In particular, rainbow short cycles have received much attention. Broersma et al. [3] studied the existence of rainbow C_3 's and C_4 's under color neighborhood union condition. Later, H. Li and Wang [14] obtained two results on the existence of rainbow C_3 's and C_4 's under colored degree condition.

Theorem 1 (Li and Wang [14]). *Let G^c be a colored graph of order $n \geq 3$. If $d^c(v) \geq (4\sqrt{7}/7 - 1)n + 3 - 4\sqrt{7}/7$ for each $v \in V(G)$, then G^c has either a rainbow C_3 or a rainbow C_4 .*

Theorem 2 (Li and Wang [14]). *Let G^c be a colored graph of order $n \geq 3$. If $d^c(v) \geq (\sqrt{7} + 1)n/6$ for each $v \in V(G)$, then G^c has a rainbow C_3 .*

H. Li and Wang [14] conjectured that every colored graph G^c of order $n \geq 3$ has a rainbow C_3 if $d^c(v) \geq (n + 1)/2$ for each $v \in V(G)$. This conjecture was proved by H. Li [13] and stronger results were proved by B. Li et al. [12] with different methods as follows.

Theorem 3 (Li [13]). *Let G^c be a colored graph of order $n \geq 3$. If $d^c(v) \geq (n + 1)/2$ for each $v \in V(G)$, then G^c has a rainbow C_3 .*

Theorem 4 (Li, Ning, Xu and Zhang [12]). *Let G^c be a colored graph of order $n \geq 3$. If $\sum_{v \in V(G)} d^c(v) \geq n(n + 1)/2$, then G^c has a rainbow C_3 .*

Theorem 5 (Li, Ning, Xu and Zhang [12]). *Let G^c be a colored graph of order $n \geq 3$. If $d^c(v) \geq n/2$ for each $v \in V(G)$, then G^c has a rainbow C_3 or $G \in \{K_{n/2, n/2}, K_4 - e, K_4\}$.*

The existence of rainbow C_4 's in special colored graphs has also been studied. Wang et al. [17] obtained a result on the existence of rainbow C_4 's in triangle-free colored graphs. Recently, H. Li [13] got a result on the existence of rainbow C_4 's in balanced bipartite colored graphs.

Theorem 6 (Wang, Li, Zhu and Liu [17]). *Let G^c be a triangle-free colored graph of order $n \geq 9$. If $d^c(v) \geq (3 - \sqrt{5})n/2 + 1$ for each $v \in V(G)$, then G^c has a rainbow C_4 .*

Theorem 7 (Li [13]). *Let G^c be a balanced bipartite colored graph of order $2n$ with bipartition (A, B) . If $d^c(v) > 3n/5 + 1$ for each $v \in A \cup B$, then G^c has a rainbow C_4 .*

While in [13], H. Li made a tiny error in the proof of Theorem 7. Notice that $K_{3,3}$ is 3-edge-colorable, and a proper 3-edge-coloring of $K_{3,3}$ satisfies the condition of Theorem 7, but it has no rainbow C_4 since there are only 3 colors. We point out that, in order to correct it, the condition $d^c(u) > 3n/5 + 1$ should be changed into $d^c(u) > (3n + 8)/5$.

Now we turn to finite simple oriented graphs, i.e., finite graphs without multiple edges and loops in which each edge is replaced by exactly one arc. Let $D[A, B]$ be an oriented bipartite graph with bipartition (A, B) . When there is no ambiguity, we use D instead of $D[A, B]$. For $A_1 \subseteq A$ and $B_1 \subseteq B$, we denote by $A_D(A_1, B_1)$ the set of arcs from A_1 to B_1 in $D[A, B]$.

The study of rainbow cycles in colored graphs is largely related to the study of oriented cycles in digraphs. For a wonderful example, see the introduction of [13]. In particular, motivated by the study of short rainbow cycles in colored graphs, H. Li [13] proposed the following nice conjecture and proved for balanced oriented bipartite graphs.

Conjecture 1 (Li [13]). Let D be an oriented bipartite graph with bipartition (A, B) . If $d^+(u) > |B|/3$ for each $u \in A$ and $d^+(v) > |A|/3$ for each $v \in B$, then D has a directed C_4 .

Theorem 8 (Li [13]). Let D be a balanced oriented bipartite graph with bipartition (A, B) , where $|A| = |B| = n$. If $d^+(v) > n/3$ for each $v \in A \cup B$, then D has a directed C_4 .

We state a construction from [17] to show that if Conjecture 1 holds, then it would be almost the best possible. Let m and n be two positive integers divisible by 3. Let $|M_0| = |M_1| = |M_2| = m/3$ and $|N_0| = |N_1| = |N_2| = n/3$. We construct an oriented bipartite graph with bipartition (M, N) , where $M = M_0 \cup M_1 \cup M_2$ and $N = N_0 \cup N_1 \cup N_2$, by creating all possible arcs from M_i to N_i , and from N_i to M_{i+1} , $i = 0, 1, 2$ (modulo 3). In the rest parts, we use $D^*(m, n)$ to denote the construction above.

The first purpose of this paper is to confirm Conjecture 1. In fact, we prove a stronger result as follows.

Theorem 9. Let D be an oriented bipartite graph with bipartition (A, B) , where $|A| = m \geq 2$ and $|B| = n \geq 2$. If $d^+(u) \geq n/3$ for each $u \in A$ and $d^+(v) \geq m/3$ for each $v \in B$, then either D has a directed C_4 or $D = D^*(m, n)$.

By using Theorem 9, we can extend Theorem 7 as follows.

Theorem 10. Let G^c be a bipartite colored graph with bipartition (A, B) . If $d^c(u) \geq (3|B| + 8)/5$ for each $u \in A$ and $d^c(v) \geq (3|A| + 8)/5$ for each $v \in B$, then G^c has a rainbow C_4 .

2 Proofs

Proof of Theorem 9

Let $\mathcal{D}(m, n)$ be the family of digraphs consisting of those oriented bipartite graphs with bipartition (A, B) which satisfies the condition of Theorem 9, where $m = |A|$ and $n = |B|$.

First we claim that it is sufficient to prove for those m and n which are both multiples of 3. Suppose Theorem 9 holds for $\mathcal{D}(m, n)$ with $m \equiv n \equiv 0 \pmod{3}$. For any $D \in \mathcal{D}(m', n')$, where m' and n' are not both multiples of 3, let $s_1 = 3\lceil \frac{m'}{3} \rceil - m'$, $A^* = \{u_1, \dots, u_{s_1}\}$ and $A' = A \cup A^*$. Let $s_2 = 3\lceil \frac{n'}{3} \rceil - n'$, $B^* = \{v_1, \dots, v_{s_2}\}$ and $B' = B \cup B^*$. Now we construct a new oriented bipartite graph D' with bipartition (A', B') , where $A(D') = A(D) \cup \{u'v, v'u : u \in A, u' \in A^*, v \in B, v' \in B^*\}$. Notice that $d_{D'}^+(u) \geq \lceil \frac{|B|}{3} \rceil = \frac{|B|}{3}$ for each $u \in A$ and $d_{D'}^+(u') = n' > \frac{|B'|}{3}$ for each $u' \in A^*$. Similarly, $d_{D'}^+(v) \geq \lceil \frac{|A|}{3} \rceil = \frac{|A|}{3}$ for each $v \in B$ and $d_{D'}^+(v') = m' > \frac{|A'|}{3}$ for each $v' \in B^*$. It follows that $D' \in \mathcal{D}(3\lceil \frac{m'}{3} \rceil, 3\lceil \frac{n'}{3} \rceil)$. Hence D' has a directed C_4 or $D' = D^*(3\lceil \frac{m'}{3} \rceil, 3\lceil \frac{n'}{3} \rceil)$. Since A^* and B^* are not both empty sets and the vertices in A^* and B^* only have outdegrees, $D' \neq D^*(3\lceil \frac{m'}{3} \rceil, 3\lceil \frac{n'}{3} \rceil)$, and moreover, the directed C_4 in D' is also in D . The proof of our claim is complete.

Now assume $D \in \mathcal{D}(m, n)$, where $m = 3m_1$, $n = 3n_1$ and m_1, n_1 are two positive integers. Let D_1 be a spanning subdigraph of D satisfying $d_{D_1}^+(u) = n_1$ for each $u \in A$ and $d_{D_1}^+(v) = m_1$ for each $v \in B$. Suppose D has no directed C_4 , obviously, D_1 also has no directed C_4 . Let u_0 be a vertex with maximum indegree k_1 among A , and v_0 be a vertex with maximum indegree k_2 among B . Let $B_1 = N_{D_1}^-(u_0)$, $B_2 = N_{D_1}^+(u_0)$, $A_3 = N_{D_1}^+(B_2)$ and $B_3 = N_{D_1}^+(A_3) - B_2$, where $|B_1| = k_1$, $|B_2| = n_1$. Since D_1 has no directed C_4 , we have $N_{D_1}^+(A_3) \cap B_1 = \emptyset$. Since k_2 is the maximum indegree of all vertices in B , we get

$$|B_3|k_2 \geq |N_{D_1}^-(B_3)| = |A_{D_1}(A_3, B_3)|. \quad (1)$$

Since D_1 has no directed C_4 , there is no arc from A_3 to B_1 , which implies all arcs starting from A_3 have heads in $B_2 \cup B_3$. Hence $|A_{D_1}(A_3, B_3)| = |A_3|n_1 - |A_{D_1}(A_3, B_2)|$. Since

$$|A_{D_1}(A_3, B_2)| \leq |A_3||B_2| - |A_{D_1}(B_2, A_3)|, \quad (2)$$

we obtain

$$|A_{D_1}(A_3, B_3)| \geq |A_3|n_1 - (|A_3||B_2| - |A_{D_1}(B_2, A_3)|) = |A_{D_1}(B_2, A_3)| = m_1n_1. \quad (3)$$

Together with (1) and (3), we obtain $|B_3| \geq \frac{m_1n_1}{k_2}$. Therefore

$$3n_1 = |B| \geq |B_1| + |B_2| + |B_3| \geq k_1 + n_1 + \frac{m_1n_1}{k_2} \geq 3\sqrt[3]{\frac{k_1m_1n_1^2}{k_2}}. \quad (4)$$

It follows that $k_2n_1 \geq k_1m_1$. By symmetry, we also have $k_1m_1 \geq k_2n_1$. Thus, $k_1m_1 = k_2n_1$ and all the inequalities (1)-(4) are actually equalities. These facts imply $|B_1| = |B_2| = |B_3| = n_1 = k_1$, $m_1 = k_2$, $|A_{D_1}(B_2, A_3)| = |A_{D_1}(A_3, B_3)| = m_1n_1$, and all vertices in A have indegree n_1 in D_1 , all vertices in B have indegree m_1 in D_1 .

Next we show $|A_3| = m_1$. First, choose a vertex $v' \in B_2$, $d^+(v') = m_1$, so $|A_3| \geq m_1$ by the definition of A_3 . Assume that $|A_3| > m_1$. Since the inequality (2) becomes equality, the underlying graph of $D_1[A_3, B_2]$ is a complete bipartite graph. Hence there exists a vertex, say $u' \in A_3$, such that $u'v' \in D_1$. Since $u' \in A_3$, there exist a vertex $v'' \in B_2$, such that $v''u' \in D_1$ by the choice of A_3 . Note that $d_{D_1[B_2, A_3]}^+(v') = d_{D_1[B_2, A_3]}^+(v'') = m_1$ and $u' \notin N^+(v')$. It follows that there exists a vertex, say $u'' \in A_3$, such that $v'u'' \in D_1$ and $v''u'' \notin D_1$. Since the underlying graph of $D_1[B_2, A_3]$ is a complete bipartite graph, $u''v'' \in D_1$. Now $C = u'v'u''v''u'$ is a directed C_4 in D_1 , a contradiction. Hence $|A_3| = m_1$.

Now let $A_1 = N_{D_1}^+(B_3)$ and $A_2 = N_{D_1}^-(B_2)$. Note that $|\{vu : v \in B_2, u \in A_3\}| = m_1n_1 = |A_{D_1}(B_2, A_3)|$. It follows that $A_{D_1}(B_2, A_3) = \{vu : v \in B_2, u \in A_3\}$. Similarly, $A_{D_1}(A_3, B_3) = \{uv : u \in A_3, v \in B_3\}$. So there is no arc with tail in A_3 and head in B_2 , or arc with tail in B_3 and head in A_3 , follows $A_1 \cap A_3 = A_2 \cap A_3 = \emptyset$. Since D_1 has no directed C_4 , $A_1 \cap A_2 = \emptyset$.

Since $d_{D_1}^+(u) = d_{D_1}^-(u) = n_1$ for each $u \in A$ and $d_{D_1}^+(v) = d_{D_1}^-(v) = m_1$ for each $v \in B$, we obtain $|A_1| \geq m_1$ by its definition, and $n_1|A_2| \geq |A_{D_1}(A_2, B_2)| = \sum_{v \in B_2} d_{D_1}^-(v) = m_1n_1$, follows that $|A_2| \geq m_1$. Since $|A_1| + |A_2| + |A_3| = 3m_1$ and A_1, A_2, A_3 are pairwise disjoint, $|A_1| = |A_2| = |A_3| = m_1$. Now apparently, $A_{D_1}(A_2, B_2) = \{uv : u \in A_2, v \in B_2\}$ and $A_{D_1}(B_3, A_1) = \{vu : v \in B_3, u \in A_1\}$. Since A_1 and A_2 are disjoint, $A_{D_1}(A_1, B_2) = \emptyset$. Furthermore, $A_{D_1}(A_1, B_3) = \emptyset$, hence $N^+(A_1) \subset B_1$. $\sum_{v \in A_1} d^+(v) = m_1n_1$ implies $A_{D_1}(A_1, B_1) = \{uv : u \in A_1, v \in B_1\}$. Similarly, $A_{D_1}(B_1, A_2) = \{vu : v \in B_1, u \in A_2\}$. Therefore $D_1 = D^*(m, n)$. If there is any arc in D but not in D_1 , then obviously, there would be a directed C_4 in D , a contradiction. Thus $D = D_1 = D^*(m, n)$. The proof is complete. \square

Proof of Theorem 10

First note that the color degree condition implies that $|A| \geq 4$ and $|B| \geq 4$.

Suppose not. Let G^c be a colored graph which satisfies the condition of Theorem 10 but has no rainbow C_4 . Set $|A| = n_1$ and $|B| = n_2$.

Choose an edge $e = xy \in E(G^c)$ such that $C(xy) = c_0$. Let $N^c(x) = \{y, y_1, y_2, \dots, y_{r-1}\}$ and $N^c(y) = \{x, x_1, x_2, \dots, x_{s-1}\}$. Since $d^c(x) \geq \frac{3n_2+8}{5}$ and $d^c(y) \geq \frac{3n_1+8}{5}$, we can set $r = \lceil \frac{3n_2+8}{5} \rceil$ and $s = \lceil \frac{3n_1+8}{5} \rceil$. Let $A_1 = \{x_1, x_2, \dots, x_{s-1}\}$ and $B_1 = \{y_1, y_2, \dots, y_{r-1}\}$. Note that $G^c[A_1, B_1]$ is also a bipartite colored graph.

The following claim can be deduced immediately from the definition of color neighbor-

hood set and the assumption that G^c has no rainbow C_4 .

Claim 1. For any edge $x_i y_j$ and $C(xy_j) \neq C(yx_i)$, where $1 \leq i \leq s-1$ and $1 \leq j \leq r-1$, we have $C(x_i y_j) \in \{C(xy), C(xy_j), C(yx_i)\}$.

Now we construct an oriented bipartite graph $D = D[A_1, B_1]$ as follows. For any edge $x_i y_j \in E(G[A_1, B_1])$, such that $C(x_i y_j) \neq C(xy)$ and $C(xy_j) \neq C(yx_i)$, then $C(x_i y_j) = C(xy_j)$ or $C(x_i y_j) = C(yx_i)$ by Claim 1. If $C(x_i y_j) = C(xy_j)$, we define an arc $x_i y_j$ in D , and if $C(x_i y_j) = C(yx_i)$, we define an arc $y_j x_i$ in D . Let $G' = G[D]$ be the underlying graph of D .

In the following, for convenience, when we mention the color of an arc in D , we mean the color of the corresponding edge in $E(G')$.

Claim 2. There is no directed C_4 in D .

Proof. Suppose $Q = x_i y_j x_p y_q x_i$ is a directed C_4 in D . By the definition of D , we have $C(x_i y_j) = C(xy_j)$, $C(y_j x_p) = C(yx_p)$, $C(x_p y_q) = C(xy_q)$ and $C(y_q x_i) = C(yx_i)$. The existence of arcs $x_i y_j$ and $y_j x_p$ implies $C(xy_j) \neq C(yx_i)$ and $C(xy_j) \neq C(yx_p)$, hence $C(x_i y_j) \neq C(y_q x_i)$ and $C(x_i y_j) \neq C(y_j x_p)$. We have $C(xy_j) \neq C(xy_q)$ from the definition of B_1 , hence $C(x_i y_j) \neq C(x_p y_q)$. So $C(x_i y_j)$ is different from the colors of all other three edges in the cycle Q in G^c . Similarly we can prove that all edges in Q receive distinct colors in G^c , and therefore, Q is a rainbow C_4 in G^c , a contradiction. \square

Claim 3. $D \neq D^*(|A_1|, |B_1|)$.

Proof. Assume that $D = D^*(|A_1|, |B_1|)$. Without loss of generality, set $A_1 = X_1 \cup X_2 \cup X_3$ and $B_1 = Y_1 \cup Y_2 \cup Y_3$, where $|X_1| = |X_2| = |X_3|$ and $|Y_1| = |Y_2| = |Y_3|$. Since $s-1 \equiv 0 \pmod{3}$ and $r-1 \equiv 0 \pmod{3}$, we have $\lceil \frac{3n_i+3}{5} \rceil \equiv 0 \pmod{3}$, $i = 1, 2$. It follows that $n_i \equiv 3$ or $4 \pmod{5}$, $i = 1, 2$.

First, we claim that $D \neq D^*(3, 3)$. Suppose not. Then $\lceil \frac{3n_1+3}{5} \rceil = \lceil \frac{3n_2+3}{5} \rceil = 3$. Hence $n_1 = n_2 = 4$. In this case, we may suppose that $X_i = \{x_i\}$ and $Y_i = \{y_i\}$ for $i = 1, 2, 3$. The existence of arc $x_1 y_1$ in D implies that $C(x_1 y_1) = C(xy_1)$. Hence $d^c(y_1) \leq 3 < \frac{3|A|+8}{5}$, a contradiction.

Let v_0 be an arbitrary vertex of D . Without loss of generality, assume $v_0 \in Y_1$. If $n_1 = 5k + \lambda$, $\lambda = 3$ or 4 , then $|X_1| = |X_2| = |X_3| = k+1$. From the definition of D , we know that each edge in $\{uv_0 : u \in X_1\}$ has the same color $C(v_0 x)$, and there are $k+1$ different colors in $\{v_0 u : u \in X_2\}$. Since $|A \setminus (A_1 \cup \{x\})| = 5k + \lambda - (3k + 3) - 1 = 2k + \lambda - 4$, there are at most $2k + \lambda - 4$ different colors in the edge set $\{v_0 u : u \in A \setminus (A_1 \cup \{x\})\}$. These facts mean that there are at most $3k + \lambda - 2$ different colors in the edge set $\{v_0 u : u \in A \setminus X_3\}$.

Since $d^c(v) \geq 3k + \lambda$, there exists an edge $v_0u_0 \in \{v_0u : u \in X_3\}$, such that v_0u_0 has a new color and v_0u_0 is not in $E(G')$.

Next we show that any directed path of length 3 in $D = D^*(|A_1|, |B_1|)$ is rainbow. Without loss of generality, we choose a directed path $uv'u'v$, where $u' \in X_1$, $v \in Y_1$, $u \in X_3$ and $v' \in Y_3$. By the construction of D , $C(u'v) = C(xv)$, $C(v'u') = C(yu')$ and $C(uv') = C(xv')$. Since $u'v$ exists in D , $C(xv) \neq C(yu')$. It follows that $C(u'v) \neq C(v'u')$. Similarly, we have $C(v'u') \neq C(uv')$. Since $C(u'v) = C(xv)$ and $C(uv') = C(xv')$ and $C(xv) \neq C(xv')$ by the choice of $N^c(x)$, we have $C(u'v) \neq C(uv')$.

Now we fix a vertex $v_0 \in Y_1$. By the analysis above, there exists an edge $v_0u_0 \in \{v_0u \in E(G) : u \in X_3\}$, which is not in $E(G')$ and satisfies $C(v_0u_0) \notin \{C(uv_0) : u \in X_1\}$. Since $D = D^*(|A_1|, |B_1|) \neq D^*(3, 3)$, there are at least two arcs in $A_D(Y_3, X_1)$ with distinct colors, and we can choose one of them, say $v'_0u'_0 \in A_D(Y_3, X_1)$, such that $C(v'_0u'_0) \neq C(v_0u_0)$. By the analysis before, there is an edge $u_0v''_0 \in E(G^c)$, where $v''_0 \in Y_1$, such that it is not in $E(G')$ and satisfies $C(u_0v''_0) \neq C(u_0v'_0)$. Now we will show that $v''_0 = v_0$. First, the deletion of u_0v_0 means $C(u_0v_0) = C(xy)$ or $C(xv_0) = C(yu_0)$. If $C(u_0v_0) = C(xy)$, then the existence of arcs u'_0v_0 , $u'_0v'_0$ and $u_0v'_0$ in D implies $C(xy) \neq C(u'_0v_0)$, $C(xy) \neq C(u'_0v'_0)$ and $C(xy) \neq C(u_0v'_0)$. Since colors of arcs in $\{u'_0v_0, v'_0u'_0, u_0v'_0\}$ are pairwise distinct, $u'_0v'_0u_0v_0u'_0$ is a rainbow C_4 in G^c , a contradiction. Hence $C(xv_0) = C(yu_0)$. Similarly, we may obtain $C(xv''_0) = C(yu_0)$ from the deletion of $u_0v''_0$ when constructing D . It follows that $C(xv_0) = C(xv''_0)$. Now we can get $v_0 = v''_0$ directly from the definition of $N^c(x)$. From the analysis above, colors of arcs in $\{u'_0v_0, v'_0u'_0, u_0v'_0\}$ are pairwise distinct, and $C(v_0u_0)$ is different from all of the three. Therefore $u'_0v_0u_0v'_0u'_0$ is also a rainbow C_4 in G^c , a contradiction. \square

By Claims 2 and 3, D has no directed C_4 and $D \neq D^*(|A_1|, |B_1|)$. By Theorem 9, there exists a vertex, without loss of generality, say $y_j \in B_1$, such that $d_D^+(y_j) < \frac{|A_1|}{3}$. By the construction of D , we know there are less than $\frac{|A_1|}{3} + 1$ different colors in $C(E_{G'}(y_j, A_1))$. For any edge e adjacent to y_j which is in $E(G[A_1, B_1]) \setminus E(G')$, $C(e) = C(xy)$ or there exists an edge yx_i such that $C(yx_i) = C(xy_j)$ in G^c , and in this case, $e = x_iy_j$ and $C(e)$ maybe missed in $C(E_{G'}(y_j, A_1))$. This implies that there are at most three colors in $C(xy_j) \cup C(E_G(y_j, A_1)) \setminus C(E_{G'}(y_j, A_1))$. However, $C(xy_j)$ is also in $C(E_{G'}(y_j, A_1))$. Hence there are less than $\frac{|A_1|}{3} + 3$ different colors in $C(E_G(y_j, \{x\} \cup A_1))$. It follows that there are more than $d^c(y_j) - 3 - \frac{|A_1|}{3} \geq s - 3 - \frac{s-1}{3}$ different colors in the color set $\{y_jx' : x' \in A \setminus (A_1 \cup \{x\})\}$. Hence y_j has more than $s - 3 - \frac{s-1}{3}$ different neighbors in $A \setminus (A_1 \cup \{x\})$. Now we have

$$n_1 = |A| \geq |A_1| + |\{x\}| + |N(y_j) \setminus (A_1 \cup \{x\})| > (s-1) + 1 + s - 3 - \frac{s-1}{3} = \frac{5s-8}{3} \geq n_1,$$

a contradiction.

The proof is complete. □

Acknowledgement

Bo Ning is supported by NSFC (No. 11271300) and the Doctorate Foundation of Northwestern Polytechnical University (cx201326). Jun Ge is supported by NSFC (No. 11171279 and No. 11271307). The authors are grateful to editors and anonymous referees for helpful comments on an earlier version of this article.

References

- [1] M. Albert, A. Frieze, B. Reed, Multicoloured Hamilton cycles, *Electron. J. Combin.* **2** (1995), #R10.
- [2] J. A. Bondy, U. S. R. Murty, Graph theory, Graduate Texts in Mathematics 244, Springer, New York, (2008).
- [3] H. J. Broersma, X. Li, G. Woeginger, S. Zhang, Paths and cycles in colored graphs, *Australas. J. Combin.* **31** (2005), 299-311.
- [4] H. Chen, X. Li, Long heterochromatic paths in edge-colored graphs, *Electron. J. Combin.* **12** (2005), #R33.
- [5] H. Chen, X. Li, Color neighborhood union conditions for long heterochromatic paths in edge-colored graphs, *Electron. J. Combin.* **14** (2007), #R77.
- [6] P. Erdős, J. Nešetřil, V. Rödl, On some problems related to partitions of edges of a graph, in: Graphs and Other Combinatorial Topics, Teubner, Leipzig (1983), 54-63.
- [7] A. Frieze, B. Reed, Polychromatic Hamilton cycles, *Discrete Math.* **118** (1993), 69-74.
- [8] G. Hahn, C. Thomassen, Path and cycle sub-Ramsey numbers and an edge-colouring conjecture, *Discrete Math.* **62** (1) (1986), 29-33.
- [9] M. Kano, X. Li, Monochromatic and heterochromatic subgraphs in edge-colored graphs-a survey, *Graphs Combin.* **24** (4) (2008), 237-263.
- [10] A. Kostochka, M. Yancey, Large rainbow matchings in edge-coloured graphs, *Combin. Probab. Comput.* **21** (2012), 255-263.

- [11] T. D. LeSaulnier, C. Stocker, P. S. Wenger, D. B. West, Rainbow matching in edge-colored graphs, *Electron. J. Combin.* **17** (2010), #N26.
- [12] B. Li, B. Ning, C. Xu, S. Zhang, Heterochromatic triangles in edge-colored graphs, Submitted, arxiv:1212.6348 (2012).
- [13] H. Li, Rainbow C_3 's and C_4 's in edge-colored graphs, *Discrete Math.* **313** (19) (2013), 1893-1896.
- [14] H. Li, G. Wang, Color degree and heterochromatic cycles in edge-colored graphs, *European J. Combin.* **33** (8) (2012), 1958-1964.
- [15] X. Li, Y. Sun, On the strong rainbow connection of a graph, *Bull. Malays. Math. Sci. Soc. (2)* **36** (2) (2013), 299-311.
- [16] G. Wang, H. Li, Heterochromatic matchings in edge-colored graphs, *Electron. J. Combin.* **15** (2008), #R138.
- [17] G. Wang, H. Li, Y. Zhu, G. Liu, A note on heterochromatic C_4 in edge-colored triangle-free graphs, *Graphs Combin.* **28** (6) (2012), 901-905.