The Natural Partial Order on Regular Γ-Semigroups

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Abstract : In this paper, we introduce partial orders on the set of all idempotents of a Γ-semigroup and introduce a natural partial order on a regular Γ-semigroup. We determine when the partial orders are compatible with respect to the multiplication. Finally, we find the least primitive congruence on a regular Γ-semigroup.

Keywords the natural order, regular Γ-semigroups, primitive congruence
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1. Introduction

The various relations are used in the study of the structure and properties of semigroups. Green's relations and the natural partial orders are important relations which are most notable and useful tool in semigroup theory. We know that Green's relations are the equivalence relations. In 2012, Xiang-zhi and Kar-ping [21] studied a Green generalized relation on a semigroup S which is a combination of the well known Green-* relation and $\widetilde{\mathcal{R}}$ Green relation on S. The well-known natural partial order for a regular semigroup play an important role in the structure of regular semigroup. Many authors studied partial orders on semigroups and special class of semigroups. In 1952, Vegner [20] introduced a natural partial order on an inverse semigroup S as follows:

$$a \leqslant b \text{ if and only if } a = eb \text{ for some } e \in E(S), \tag{1}$$

where $E(S)$ denotes the set of idempotents of S. Later, Mitsch [6] defined the natural order on an inverse semigroup S by

$$a \leqslant b \text{ if and only if } ab' = aa' \tag{2}$$

where a', b' denote the unique inverse of a and b respectively and showed that the partial order (1) and (2) are equivalent. Moreover, an inverse semigroup S is totally ordered with respect to its natural ordering if and only if $ab = ba = a$ or $ab = ba = b$ for all $a, b \in S$. Furthermore, Nambooripad [8] defined a partial order $\leqslant$ on a regular semigroup S by

$$a \leqslant b \text{ if and only if } R_a \leqslant R_b \text{ and } a = fb \text{ for some } f \in E(R_a) \tag{3}$$

where $R_a \leqslant R_b$ if and only if $S^1a \subseteq S^1b$ and $E(R_a)$ denotes the set of idempotents in R_a, that coincides with the relation defined above on inverse semigroups. Such a relation $\leqslant$ is called the natural partial order on S. We see that the relation (3) is generalization of the relation (1). And Nambooripad proved that the natural partial order on a regular subsemigroup T of a regular semigroup S is the restriction of the natural partial order on S to T. Mitsch [5] used properties of the natural partial order on a regular semigroup to define the natural partial order on a semigroup. The relation $\leqslant$ on a semigroup S is defined by

$$a \leqslant b \text{ if and only if } a = xb = by \text{ and } xa = a \text{ for some } x, y \in S^1. \tag{4}$$

Then the relation $\leqslant$ is a partial order on a semigroup. We know that the relation (4) generalized the relations (1) and (3). In 1994, Mitsch [7] studied certain properties of the natural partial order with respect to the structure of a semigroup. For example, if S^2 is regular on a semigroup S then the natural partial order on S is compatible on the right with multiplication if and only if S satisfies $\mathcal{L}$-majorization. If the natural partial order on a semigroup S is compatible with multiplication then

$$\omega := \{(a, b) \in S \times S \mid c \leqslant a \text{ and } c \leqslant b \text{ for some } c \in S\}$$

is a congruence on S. Petrich [11] used a definition of the partial order on a regular semigroup S as follows:

$$a \leqslant b \text{ if and only if } a = eb = bf \text{ for some } e, f \in E(S) \tag{5}$$

and showed that the natural partial order is compatible if and only if S is locally inverse if and only if S satisfies $\mathcal{L}$- and $\mathcal{R}$-majorization. Next, Srinivas [19] proved that $E(S)$ is a normal band if and only if the natural partial order (5) and μ coincide on a subsemigroup $Reg(S)$ where

$$\mu := \{(a, b) \in S \times S \mid sa = sb = a = at = bt \text{ for some } s, t \in S^1\}.$$

The concept of Γ-semigroups has been introduced by Sen [13] in 1981. Sen and Saha [12] changed the definition, which is more general definition and gave the definition of the Γ-semigroup via a mapping as follows: A nonempty set S is called a Γ-*semigroup* if there exists a mapping from $S \times \Gamma \times S$ to S written as $(a, \alpha, b) \mapsto a\alpha b$ satisfying the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. For example, if S is the set of all $m \times n$ matrices and Γ is the set of all $n \times m$ matrices over a field then for $A, B \in S$ the product AB can not be defined i.e., S is not a semigroup under the usual matrix multiplication. But for all $A, B, C \in S$ and $\alpha, \beta \in \Gamma$ we have $A\alpha B \in S$ and since the matrix multiplication is associative, we have $(A\alpha B)\beta C = A\alpha(B\beta C)$. Hence S is a Γ-semigroup. Let X, Y be nonempty sets. If S is the set of all functions from X to Y and Γ is the set of all functions from Y to X then for $f, g \in S$ the composition $f \circ g$ can not defined. Thus S is not a semigroup under the usual composite function but S is a Γ-semigroup i.e. for all $f, g, h \in S, \alpha, \beta \in \Gamma$ we have $f \circ \alpha \circ g \in S$ and $(f \circ \alpha \circ g) \circ \beta \circ h = f \circ \alpha \circ (g \circ \beta \circ h)$. The notions of regular Γ-semigroups have been studied and developed in [14]. Many authors tried to transfer results of semigroups to Γ-semigroup and some of them used the definition of a Γ-semigroup introduced by Sen in 1981 and 1986 (see [14], [16], [17], [15], [18]). Thus we will be interested to study the natural partial orders in Γ-semigroups and special class of Γ-semigroups.

In this paper, we shall construct a natural partial order on a regular Γ-semigroup and construct a partial order on the set of all idempotents of a Γ-semigroup. The first aim of this paper is to extend the properties of semigroups to the properties of regular Γ-semigroups by using the partial order on Γ-semigroups. We determine when the partial orders are (left, right) compatible with respect to the semigroup multiplication. Finally, we find the congruence $\omega := \{(a, b) \in S \times S \mid c \leqslant a \text{ and } c \leqslant b \text{ for some } c \in S\}$ such that ω is the least primitive congruence on a regular Γ-semigroup.

2. Preliminaries

An element x in a Γ-semigroup S is *regular* if there exist $y \in S$, $\alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x$. A Γ-semigroup S is said to be a *regular Γ-semigroup* [14] if every element in a S is regular. For any Γ-semigroup S, $a, x \in S$, and $\alpha, \beta \in \Gamma$, an element x is called an *(α, β)-inverse* of a if $a = a\alpha x\beta a$ and $x = x\beta a\alpha x$. The set of all (α, β)-inverses of an element a in a Γ-semigroup S is denoted by $V_\alpha^\beta(a)$. That is,

$$V_\alpha^\beta(a) := \{x \in S \mid a = a\alpha x\beta a \text{ and } x = x\beta a\alpha x\}.$$

Seth [17] showed that if a is a regular element then $V_\alpha^\beta(a) \neq \emptyset$ for some $\alpha, \beta \in \Gamma$. An element e of a regular Γ-semigroup S is called an *α-idempotent* [17], where $\alpha \in \Gamma$, if $e\alpha e = e$. The set of all α-idempotents is denoted by $E_\alpha(S)$ and we denote $\bigcup_{\alpha \in \Gamma} E_\alpha(S)$ by $E(S)$. Every element of $E(S)$ is called an *idempotent element* of S. In a regular Γ-semigroup S, we have that $E(S)$ is a non-empty set.

Green's equivalence relations [1] $\mathcal{R}, \mathcal{L}$ and $\mathcal{H}$ on a Γ-semigroup S are defined by the following rules :

(1) $a\mathcal{R}b$ if and only if $a\Gamma S \cup \{a\} = b\Gamma S \cup \{b\}$.
(2) $a\mathcal{L}b$ if and only if $S\Gamma a \cup \{a\} = S\Gamma b \cup \{b\}$.
(3) $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

The $\mathcal{R}$-class (resp. $\mathcal{L}$-class, $\mathcal{H}$-class) containing the element a will be written R_a (resp. L_a, H_a).

Lemma 2.1. [1] *Let S be a Γ-semigroup. Then for all $a, b \in S$, we have*

(1) $a\mathcal{R}b$ if and only if $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = b\alpha x$ and $b = a\beta y$.

(2) $a\mathcal{L}b$ if and only if $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = x\alpha b$ and $b = y\beta a$.

(3) $a\mathcal{H}b$ if and only if $a\mathcal{L}b$ and $a\mathcal{R}b$.

Lemma 2.2. [1] *Let S be a Γ-semigroup, $\alpha \in \Gamma$ and let e be an α-idempotent. Then*

(1) $e\alpha a = a$ for all $a \in R_e$.

(2) $a\alpha e = a$ for all $a \in L_e$.

(3) $a\alpha e = a = e\alpha a$ for all $a \in H_e$.

Proposition 2.3. [16] *Let S be a regular Γ-semigroup. Then for all $a \in S$, there exist $\alpha, \beta \in \Gamma$ and $a' \in V_\alpha^\beta(a)$ such that $a'\beta a \in E_\alpha(S)$ and $a\alpha a' \in E_\beta(S)$.*

3. The Natural Partial Order

In this section, we construct a relation on a regular Γ-semigroup S by extending the partial order in [8]. Let a, b be elements of a regular Γ-semigroup S. Define

$$R_a \leqslant R_b \text{ if and only if } a\Gamma S \cup \{a\} \subseteq b\Gamma S \cup \{b\},$$

and define $a \leqslant_n b$ if

$$R_a \leqslant R_b \text{ and } a = f\beta b \text{ for some } f \in E_\beta(R_a), \beta \in \Gamma.$$

It is easy to show that $\leqslant_n$ is a partial order on a regular Γ-semigroup S and we call the *natural partial order* on a regular Γ-semigroup. For convenience, we write a symbol $\leqslant$ for the natural partial order $\leqslant_n$.

We start with elementary properties of idempotent elements of Green's relations.

Lemma 3.4. *Let S be a regular Γ-semigroup and let $a \in S$. Then the following statements hold.*

(1) For all $\alpha \in \Gamma, e \in E_\alpha(R_a)$ if and only if there exist $\gamma \in \Gamma, a' \in V_\gamma^\alpha(a)$ such that $e = a\gamma a'$.

(2) For all $\alpha \in \Gamma, e \in E_\alpha(L_a)$ if and only if there exist $\gamma \in \Gamma, a' \in V_\alpha^\gamma(a)$ such that $e = a'\gamma a$.

(3) For all $\alpha, \beta \in \Gamma, e \in E_\alpha(L_a), f \in E_\beta(R_a)$ if and only if there exist $a' \in V_\alpha^\beta(a)$ such that $e = a'\beta a$ and $f = a\alpha a'$.

Proof. (1) Let $\alpha \in \Gamma$ and $e \in E_\alpha(R_a)$. By Lemma 2.1(1), we get $e = a$ or $e = a\gamma x$ for some $x \in S, \gamma \in \Gamma$. This is obvious when $a = e$. Assume that $e = a\gamma x$. By Lemma 2.2(1), we have

$$a = e\alpha a = a\gamma(x\alpha e)\alpha a$$

and

$$x\alpha e = x\alpha a\gamma x\alpha e = (x\alpha e)\alpha a\gamma(x\alpha e),$$

which implies that $x\alpha e \in V_{\gamma}^{\alpha}(a)$. Set $a' = x\alpha e$. We obtain that $a\gamma a' = (a\gamma x)\alpha e = e\alpha e = e$. The converse part is obvious.

(2) The proof is similar to the proof of (1).

(3) Let $\alpha, \beta \in \Gamma, e \in E_{\alpha}(L_a), f \in E_{\beta}(R_a)$ be such that $e\mathcal{L}a$ and $f\mathcal{R}a$. By Lemma 2.1, we have that $e = a$ or there exist $\gamma \in \Gamma, x \in S$, $e = x\gamma a$ and $f = a$ or there exist $\theta \in \Gamma, y \in S$, $f = a\theta y$.

Case 1: $e = a = f$. Then we set $a' = a$.

Case 2: $a = e$ and $f = a\theta y$. Then, we can set $a' = f$. Thus $e = a'\beta a$ and $f = e\theta y = a\alpha f = a\alpha a'$.

Case 3: $f = a$ and $e = x\gamma a$. Then this proof is similar to the second case and set $a' = e$.

Case 4: $e = x\gamma a$ and $f = a\theta y$. Then we choose $a' = e\theta y\beta f$. By Lemma 2.2, we have that

$$a'\beta a\alpha a' = e\theta y\beta a\theta y\beta f = a',$$

and

$$a\alpha a'\beta a = a\theta y\beta a = a.$$

Thus $a' \in V_{\alpha}^{\beta}(a)$ and we obtain that $a'\beta a = e\theta y\beta a = x\gamma f\beta a = e$ and $a\alpha a' = a\theta y\beta f = f$. The converse part is obvious. □

In the proof of Lemma 3.4, we see that any two elements in $\mathcal{L}$-class [$\mathcal{R}$-class, $\mathcal{H}$-class] may be alike and the proof of them is obvious.

Next, we show that the natural partial order has an alternative characterization:

Theorem 3.5. *Let a, b be elements of a regular Γ-semigroup S. Then the following statements are equivalent.*

(1) $a \leqslant b$.

(2) $a \in b\Gamma S$ and there exist $\alpha, \beta \in \Gamma, a' \in V_{\alpha}^{\beta}(a)$ such that $a = a\alpha a'\beta b$.

(3) There exist $\beta, \gamma \in \Gamma, f \in E_{\beta}(S), g \in E_{\gamma}(S)$ such that $a = f\beta b = b\gamma g$.

(4) $H_a \leqslant H_b$ and for all $\alpha, \delta \in \Gamma, b' \in V_{\alpha}^{\delta}(b), a = a\alpha b'\delta a$.

(5) $H_a \leqslant H_b$ and there exist $\alpha, \delta \in \Gamma, b' \in V_{\alpha}^{\delta}(b), a = a\alpha b'\delta a$.

Proof. (1) ⇒ (2) Let $a \leqslant b$. Then $R_a \leqslant R_b$ and $a = f\beta b$ for some $f \in E_{\beta}(R_a), \beta \in \Gamma$. By Lemma 3.4(1), there exist $\alpha \in \Gamma, a' \in V_{\alpha}^{\beta}(a)$ such that $a\alpha a' = f$. Thus $a = a\alpha a'\beta b$. Since $R_a \leqslant R_b$, we have $a\Gamma S \cup \{a\} \subseteq b\Gamma S \cup \{b\}$ which implies that $a \in b\Gamma S$.

(2) ⇒ (3) By assumption, $a = b\gamma u$ for some $\gamma \in \Gamma, u \in S$. Set $f = a\alpha a' \in E_{\beta}(S)$ and $g = u\alpha a'\beta b$, So we have $a = f\beta b$. Thus $b\gamma g = b\gamma u\alpha a'\beta b = a\alpha a'\beta b = a$ with $g \in E_{\gamma}(S)$.

(3) ⇒ (4) By assumption, $a \in b\Gamma S$ and $a\Gamma S \subseteq b\Gamma S$ which implies that $a\Gamma S \cup \{a\} \subseteq b\Gamma S \cup \{b\}$, so $R_a \leqslant R_b$. Similarly, we can show that $S\Gamma a \cup \{a\} \subseteq S\Gamma b \cup \{b\}$, so that $L_a \leqslant L_b$. Thus $H_a \leqslant H_b$. Let $\alpha, \delta \in \Gamma, b' \in V_{\alpha}^{\delta}(b)$, we have immediately that $a\alpha b'\delta a = a$.

(4) $\Rightarrow$ (5) This part is obvious.

(5) $\Rightarrow$ (1) By assumption, $R_a \leqslant R_b$ and $L_a \leqslant L_b$. Let $a' \in V_\gamma^\beta(a)$ for some $\beta, \gamma \in \Gamma$. Set $f = a\gamma a'\beta a\alpha b'$. Then $a = a\gamma a'\beta a\alpha b'\delta a = f\delta a$ and $f \in E_\delta(S)$, which prove that $f \in E_\delta(R_a)$. Since $L_a \leqslant L_b$, we get that $a = u\theta b$ for some $u \in S, \theta \in \Gamma$. Thus $f\delta b = a\alpha b'\delta b = u\theta b = a$. Therefore $a \leqslant b$. $\square$

Let S be a Γ-semigroup. We define relations on $E(S)$ as follows : For $e, f \in E(S)$, define

$$\begin{aligned}
&(1)\ \ e \preccurlyeq^l f \ \Leftrightarrow\ e = e\alpha f && \text{if } e \in E_\alpha(S) \text{ for some } \alpha \in \Gamma,\\
&(2)\ \ e \preccurlyeq^r f \ \Leftrightarrow\ e = f\beta e && \text{if } f \in E_\beta(S) \text{ for some} \beta \in \Gamma,\\
&(3)\ \ e \preccurlyeq f \ \ \Leftrightarrow\ e \preccurlyeq^l f \text{ and } e \preccurlyeq^r f, &&\\
&\qquad\qquad\quad\ \Leftrightarrow\ e = e\alpha f = f\beta e && \text{if } e \in E_\alpha(S), f \in E_\beta(S) \text{ for some } \alpha, \beta \in \Gamma.
\end{aligned}$$

It is easy to show that $\preccurlyeq$ is a partial order on $E(S)$.

The next result give a relationship between the natural partial order and the partial order on $E(S)$.

Proposition 3.6. *Let S be a regular Γ-semigroup and $a, b \in S$. Then the following statements are equivalent.*

(1) $a \leqslant b$.

(2) For every $f \in E(R_b)$, there exist $\alpha \in \Gamma, e \in E_\alpha(R_a)$ such that $e \preccurlyeq f$ and $a = e\alpha b$.

(3) For every $f' \in E(L_b)$, there exist $\alpha \in \Gamma, e' \in E_\alpha(L_a)$ such that $e' \preccurlyeq f'$ and $a = b\alpha e'$.

Proof. (1) $\Rightarrow$ (2) Let $f \in E(R_b)$. Then there exits $\beta \in \Gamma$ such that $f \in E_\beta(S)$. By assumption, $a = h\gamma b$ for some $h \in E_\gamma(R_a), \gamma \in \Gamma$ and $R_a \leqslant R_b$ which implies that $R_h = R_a \leqslant R_b = R_f$. By Lemma 2.2(1), we have $h = f\beta h$ and $h\gamma f \in E_\beta(S)$. Choose $e = h\gamma f$. Then $h = e\beta h$ which implies $e\mathcal{R}h$, and so $e\mathcal{R}a$. Thus $a = h\gamma b = h\gamma f\beta b = e\beta b$, and $e = e\beta f,\ e = h\gamma f = f\beta e$. Therefore $e \preccurlyeq f$.

(2) $\Rightarrow$ (3) Let $f' \in E(L_b)$. The $f' \in E(S) \cap L_b$ which implies that $f' \in E_\beta(S)$ for some $\beta \in \Gamma$. By Lemma 3.4(1), there exist $\gamma \in \Gamma, b' \in V_\beta^\gamma(b)$ such that $f' = b'\gamma b$. Clearly, $b\beta b' \in E_\gamma(R_b)$. Set $f = b\beta b'$. By assumption, there exist $\alpha \in \Gamma, e \in E_\alpha(R_a)$ such that $e \preccurlyeq f$ and $a = e\alpha b$. Set $e' = b'\gamma e\alpha b$. Clearly, $e' \in E_\beta(S), e' = b'\gamma a$ and $a = b\beta e'$. Thus $e' \in E_\beta(L_a)$. Consider, $e' = b'\gamma b\beta b'\gamma e\alpha b = f'\beta e'$, and $e' = b'\gamma e\alpha b\beta b'\gamma b = e'\beta f'$. Therefore $e' \preccurlyeq f'$.

(3) $\Rightarrow$ (1) Let $b' \in V_\gamma^\delta(b)$ for some $\gamma, \delta \in \Gamma$. By Lemma 3.4, $b'\delta b \in E_\gamma(L_b)$. By assumption, there exist $\alpha \in \Gamma, e' \in E_\alpha(L_a)$ such that $e' \preccurlyeq b'\delta b$ and $a = b\alpha e'$. Set $f = b\alpha e'\alpha b'$. Clearly, $f \in E_\delta(S)$. Thus $f\delta b = b\alpha e' = a$. By Theorem 3.5, we have $a \leqslant b$. $\square$

The following remark follows immediately from the above propositions.

Remark. *Let S be a regular Γ-semigroup and $a, b \in S$. Then the following statements are equivalent.*

(1) $a \leqslant b$.

(2) If $f \in E_\beta(R_b)$ for some $\beta \in \Gamma$, then there exist $e \in E_\beta(R_a)$ such that $e \preccurlyeq f$ and $a = e\beta b$.

(3) If $f' \in E_\beta(L_b)$ for some $\beta \in \Gamma$, then there exist $e' \in E_\beta(L_a)$ such that $e' \preccurlyeq f'$ and $a = b\beta e'$.

Lemma 3.7. *Let S be a regular Γ-semigroup. Then the following conditions hold :*

(1) $\preccurlyeq \circ \mathcal{L} \;=\; \mathcal{L} \circ \preccurlyeq$.

(2) $\preccurlyeq \circ \mathcal{R} \;=\; \mathcal{R} \circ \preccurlyeq$.

Proof. (1) Let $(e, f) \in \preccurlyeq \circ \mathcal{L}$ where $e, f \in E(S)$. Then there exists $h \in E(S)$ such that $e \preccurlyeq h$ and $h\mathcal{L}f$.

Case 1. $e, f, h \in E_\alpha(S)$ for some $\alpha \in \Gamma$. Then $e = e\alpha h = h\alpha e$, $h = h\alpha f$ and $f = f\alpha h$. Consider $e = e\alpha f$ and $f = f\alpha e$, so $e\mathcal{L}f$. Clearly, $f \preccurlyeq f$. Thus $(e, f) \in \mathcal{L} \circ \preccurlyeq$.

Case 2. $e, h \in E_\alpha(S), f \in E_\beta(S)$ for some $\alpha, \beta \in \Gamma$. By Lemma 2.2(2), $h = h\beta f$ and $f = f\alpha h$. Then $e = e\beta f$ and $f = f\alpha e$, which implies that $e\mathcal{L}f$. Therefore $(e, f) \in \mathcal{L} \circ \preccurlyeq$.

Case 3. $e, f \in E_\alpha(S), h \in E_\beta(S)$ for some $\alpha, \beta \in \Gamma$. Similar to proof of case 2.

Case 4. $e \in E_\alpha(S), f \in E_\beta(S)$ and $h \in E_\gamma(S)$ for some $\alpha, \beta, \gamma \in \Gamma$. Then $e = e\alpha h = h\gamma e$. By Lemma 2.2, we get that $e = e\beta f$ and $f = f\alpha h\gamma e = f\gamma e$, which prove that $e\mathcal{L}f$. By cases (1)-(4), we get $(e, f) \in \mathcal{L} \circ \preccurlyeq$ which implies that $\preccurlyeq \circ \mathcal{L} \;\subseteq\; \mathcal{L} \circ \preccurlyeq$.

Similarly, we can show that $\mathcal{L} \circ \preccurlyeq \subseteq \preccurlyeq \circ \mathcal{L}$.

(2) Similar to proof of (1). $\square$

Proposition 3.8. *Let S be a regular Γ-semigroup and $a, b \in S$. Then $H_a \leqslant H_b$ if and only if $a \in b\Gamma S\Gamma b$.*

Proof. Assume that $H_a \leqslant H_b$. Then $L_a \leqslant L_b$ and $R_a \leqslant R_b$. Since $a \in S$, we have $a = a\alpha c\beta a$ for some $\alpha, \beta \in \Gamma, c \in S$. If $a = b$, it is obvious. If $a = x\gamma b$ and $a = b\delta y$ for some $\gamma, \delta \in \Gamma, x, y \in S$, we get that $a = a\alpha c\beta a = b\delta y\alpha c\beta x\gamma b \in b\Gamma S\Gamma b$. The converse part is clear. $\square$

Proposition 3.9. *Let S be a regular Γ-semigroup. Then the following statements hold.*

(1) If $e \in E(S), a \in S$ and $a \leqslant e$ then $a \in E(S)$.

(2) For any $a, b \in S, a\mathcal{R}b$ and $a \leqslant b$ implies $a = b$.

(3) If $a \leqslant c, b \leqslant c$ and $H_a \leqslant H_b$ then $a \leqslant b$.

Proof. (1) Let $e \in E(S)$. Then $e \in E_\alpha(S)$ for some $\alpha \in \Gamma$. By assumption, there exist $\beta, \gamma \in \Gamma, f \in E_\beta(S), g \in E_\gamma(S)$ such that $a = e\beta f = g\gamma e$. Thus $a\alpha a = g\gamma e\beta f = a\beta f = e\beta f = a$, so $a \in E(S)$.

(2) Let $a\mathcal{R}b$. Then there exist $x \in S, \theta \in \Gamma$ such that $b = a\theta x$. Since $a \leqslant b$, we get that $a = f\beta b$ for some $\beta \in \Gamma, f \in E_\beta(S)$. Thus $a = f\beta a\theta x = f\beta b\theta x = b$.

(3) Assume that $a \leqslant c, b \leqslant c$ and $H_a \leqslant H_b$. Let $c' \in V_\alpha^\beta(c)$ for some $\alpha, \beta \in \Gamma$. Then $c\alpha c' \in E_\beta(R_c)$. By Remark (2), there exist $e \in E_\beta(R_a), f \in E_\beta(R_b)$ such that $e \preccurlyeq c\alpha c', f \preccurlyeq c\alpha c'$ and $a = e\beta c, b = f\beta c$. By assumption and Proposition 3.8, we have $a \in b\Gamma S\Gamma b$. Then $a = b\delta x\theta b$ for some $\delta, \theta \in \Gamma, x \in S$. Thus $(c'\beta f)\beta b\alpha(c'\beta f) = c'\beta f$ and $b\alpha(c'\beta f)\beta b = f\beta c = b$, from which get that $c'\beta f \in V_\alpha^\beta(b)$. Set $b' = c'\beta f$. Since $e \preccurlyeq c\alpha c'$ and by Theorem 3.5, we obtain that

$$e = e\beta c\alpha c' = a\alpha c' = b\delta x\theta b\alpha c' = f\beta c\delta x\theta b\alpha c' = f\beta b\delta x\theta b\alpha c' = f\beta a\alpha c' = f\beta e\beta c\alpha c' = f\beta e.$$

Therefore, $a\alpha b'\beta a = e\beta f\beta e\beta c = e\beta c = a$. Again, by Theorem 3.5, we have that $a \leqslant b$. $\square$

Note that by Proposition 3.9(1), if $e \in E_\alpha(S)$ and $a \leqslant e$ then $a \in E_\alpha(S)$.

Proposition 3.10. *Let e be an α-idempotent and f an β-idempotent of a regular Γ-semigroup S. Then the following statements hold.*

(1) If $e \preccurlyeq f$, then $e \in E_\beta(S)$.

(2) $V_\alpha^\beta(f\beta e) \neq \emptyset$.

Proof. (1) This follows directly from the definition of the relation $\preccurlyeq$.

(2) Since $f\beta e$ is a regular element, we can choose $x \in S, \gamma, \delta \in \Gamma$ such that $f\beta e = (f\beta e)\gamma x\delta(f\beta e)$. It follows that

$$(e\gamma x\delta f\beta e\gamma x\delta f)\beta(f\beta e)\alpha(e\gamma x\delta f\beta e\gamma x\delta f) = e\gamma x\delta f\beta e\gamma x\delta f,$$

and

$$(f\beta e)\alpha(e\gamma x\delta f\beta e\gamma x\delta f)\beta(f\beta e) = f\beta e$$

which proves that $e\gamma x\delta f\beta e\gamma x\delta f \in V_\alpha^\beta(f\beta e)$. Therefore $V_\alpha^\beta(f\beta e) \neq \emptyset$. $\square$

A regular Γ-semigroup S is called an *$\mathcal{L}$-unipotent*[*$\mathcal{R}$-unipotent*] if every $\mathcal{L}$-class [$\mathcal{R}$-class] of S contains only one idempotent.

Proposition 3.11. *Let S be a regular Γ-semigroup. If S is $\mathcal{L}$-unipotent then $e\alpha f\beta e = f\beta e$ for all $e \in E_\alpha(S), f \in E_\beta(S)$ for some $\alpha, \beta \in \Gamma$.*

Proof. Let $e \in E_\alpha(S), f \in E_\beta(S)$ for some $\alpha, \beta \in \Gamma$. By Proposition 3.10(2), we can choose $x \in V_\alpha^\beta(f\beta e)$. Then

$$(x\beta f\beta e)\alpha(x\beta f\beta e) = x\beta f\beta e \text{ and } (e\alpha x\beta f\beta e)\alpha(e\alpha x\beta f\beta e) = e\alpha x\beta f\beta e,$$

so $x\beta f\beta e, e\alpha x\beta f\beta e \in E_\alpha(S)$, and immediately it follows that $(x\beta f\beta e)\mathcal{L}(e\alpha x\beta f\beta e)$. The hypothesis implies that

$$x\beta f\beta e = e\alpha x\beta f\beta e. \tag{6}$$

Now, $x = e\alpha x\beta f\beta e\alpha x = e\alpha x$. It follows that $x = x\beta f\beta x$, that is $x\beta f \in E_\beta(S)$. Thus $(f\beta e\alpha x\beta f)\beta(f\beta e\alpha x\beta f) = f\beta e\alpha x\beta f$, we obtain that $f\beta e\alpha x\beta f \in E_\beta(S)$ and $(x\beta f)\mathcal{L}(f\beta e\alpha x\beta f)$. Again, the hypothesis implies that $x\beta f = f\beta e\alpha x\beta f$. Then

$$x\beta f\beta e = f\beta e. \tag{7}$$

By (6) and (7), we get that $f\beta e = e\alpha x\beta f\beta e$. Again by (7), $f\beta e = f\beta e\alpha f\beta e$. Thus $e\alpha f\beta e = e\alpha x\beta f\beta e = f\beta e$. □

Proposition 3.12. *Let S be a regular Γ-semigroup. If e, f are α-idempotent with $e\alpha f\alpha e = f\alpha e$ then S is an $\mathcal{L}$-unipotent.*

Proof. Let $\alpha \in \Gamma$ and $e, f \in E_\alpha(S)$ be such that $e\mathcal{L}f$. By Lemma 2.2(1), we get that $e = e\alpha f$ and $f = f\alpha e$. By the hypothesis we get that $e = e\alpha f = e\alpha f\alpha e = f\alpha e = f$. □

By Proposition 3.11 and 3.12, we get that S is an $\mathcal{L}$-unipotent if and only if $e\alpha f\alpha e = f\alpha e$ for some $e, f \in E_\alpha(S), \alpha \in \Gamma$.

Proposition 3.13. *Let S be a regular Γ-semigroup and $e \in E(S)$. If $a \in e\Gamma S\Gamma e$, then the following statements hold.*

(1) There exist $\gamma, \delta \in \Gamma, a' \in V_\gamma^\delta(a) \cap e\Gamma S\Gamma e$ such that $a'\delta a \preccurlyeq e$.

(2) There exist $\gamma, \delta \in \Gamma, a'' \in V_\gamma^\delta(a) \cap e\Gamma S\Gamma e$ such that $a\gamma a'' \preccurlyeq e$.

(3) If $a', a'' \in V_\gamma^\delta(a) \cap e\Gamma S\Gamma e$ then $a'\delta a\mathcal{L}a''\delta a$ and $a\gamma a'\mathcal{R}a\gamma a''$.

Proof. (1) Let $e \in E_\alpha(S)$ for some $\alpha \in \Gamma$ and let $a \in e\Gamma S\Gamma e$. Then there exist $\beta, \gamma \in \Gamma, x \in S$ such that $a = e\beta x\gamma e$. Since a is a regular element of S, we get that $a = a\delta y\theta a$ for some $y \in S, \delta, \theta \in \Gamma$. Set $a' = e\delta y\theta a\delta y\theta e$. Then

$$\begin{aligned} a'\alpha a\alpha a' &= e\delta y\theta a\delta y\theta e\alpha e\beta x\gamma e\alpha e\delta y\theta a\delta y\theta e \\ &= e\delta y\theta a\delta y\theta a\delta y\theta a\delta y\theta e \\ &= e\delta y\theta a\delta y\theta e \\ &= a', \end{aligned}$$

and

$$a\alpha a'\alpha a = e\beta x\gamma e\alpha e\delta y\theta a\delta y\theta e\alpha e\beta x\gamma e = a\delta y\theta a\delta y\theta a = a,$$

which then implies that $a' \in V_\alpha^\alpha(a) \cap e\Gamma S\Gamma e$. This immediately implies that $a'\alpha a \preccurlyeq e$.

(2) This part of proof is similar to (1).

(3) This is trivial. □

Proposition 3.14. *The partial order on $E(S)$ of a regular semigroup S is the restriction of the natural partial order on S to $E(S)$.*

Proof. Let $e, f \in E(S)$ be such that $e \leqslant f$. Then there exists $\beta \in \Gamma$ such that $f \in E_\beta(S)$. By the proof of Proposition 3.9(1), we get that $e \in E_\beta(S)$. Since $e \leqslant f$, there exist $g \in E_\gamma(S), h \in E_\delta(S)$ for some $\gamma, \delta \in \Gamma$ such that $e = f\gamma g = h\delta f$. Thus $e = f\beta f\gamma g = f\beta e$ and $e = h\delta f\beta f = e\beta f$. Therefore $e \preccurlyeq f$. The converse is obvious. □

A regular Γ-semigroup S satisfies *$\mathcal{L}$-majorization* [*$\mathcal{R}$-majorization*] if for any $a, b, c \in S, a \leqslant c, b \leqslant c$ and $a\mathcal{L}b$ [$a\mathcal{R}b$] imply that $a = b$.

Theorem 3.15. *Let S be a regular Γ-semigroup. Then the following statements are equivalent.*

(1) $\leqslant$ is right compatible.

(2) S satisfies $\mathcal{L}$-majorization for idempotents.

(3) S satisfies $\mathcal{L}$-majorization.

Proof. (1) ⇒ (2) Let $e, f, g \in E(S)$ be such that $f \preccurlyeq e, g \preccurlyeq e$ and $f\mathcal{L}g$. Then there exist $\alpha \in \Gamma$ such that $e \in E_\alpha(S)$. By Proposition 3.10, we have that $f, g \in E_\alpha(S)$. Thus $f = f\alpha g$ and $g = g\alpha f$. By hypothesis, we get that

$$f = f\alpha g \preccurlyeq e\alpha g = g \text{ and } g = g\alpha f \preccurlyeq e\alpha f = f.$$

Therefore $f = g$.

(2) ⇒ (3) Let $a, b, c \in S$ be such that $a \leqslant c,\ b \leqslant c$ and $a\mathcal{L}b$. Then $a = e\alpha c = c\beta f$ for some $e \in E_\alpha(S), f \in E_\beta(S), \alpha, \beta \in \Gamma$. Let $c' \in V_\gamma^\delta(c)$ for some $\gamma, \delta \in \Gamma$. It follows that

$$c'\delta a = (c'\delta c)\gamma(c'\delta a),\ \ c'\delta a = c'\delta e\alpha c\gamma c'\delta c = (c'\delta a)\gamma(c'\delta c)$$

and

$$c'\delta a = c'\delta e\alpha c\beta f = (c'\delta a)\gamma(c'\delta a)$$

which proves that $c'\delta a \leqslant c'\delta c$. Thus $a = c\gamma c'\delta c\beta f = c\gamma c'\delta a$, we have that $a\mathcal{L}c'\delta a$. Similarly, since $b \leqslant c$ we have $a = e_1\alpha_1 c = c\beta_1 f_1$ for some $e_1 \in E_{\alpha_1}(S), f_1 \in E_{\beta_1}(S), \alpha_1, \beta_1 \in \Gamma$. Then $c'\delta b = (c'\delta c)\gamma(c'\delta b),\ c'\delta b = c'\delta e_1\alpha_1 c\gamma c'\delta c = (c'\delta b)\gamma(c'\delta c)$ and $c'\delta b = c'\delta e_1\alpha_1 c\beta_1 f = (c'\delta b)\gamma(c'\delta b)$ which proves that $c'\delta b \leqslant c'\delta c, b = c\gamma c'\delta b$ and $b\mathcal{L}c'\delta b$ with $c'\delta b \in E_\gamma(S)$ which implies that $c'\delta a\mathcal{L}c'\delta b$. By the hypothesis, we obtain $c'\delta a = c'\delta b$. Therefore $a = b$.

(3) ⇒ (1) Let $a \leqslant b$. By Theorem 3.5(3), $a = e\alpha b = b\beta f$ for some $e \in E_\alpha(S), f \in E_\beta(S), \alpha, \beta \in \Gamma$. Also, let $c \in S, \theta \in \Gamma$ and $x \in V_\gamma^\delta(a\theta c)$ for some $\gamma, \delta \in \Gamma$. Then

$$b\theta(c\gamma x\delta a) = (b\theta c\gamma x\delta e)\alpha b,\quad (c\gamma x\delta a)\theta(c\gamma x\delta a) = c\gamma x\delta a,$$

and

$$(b\theta c\gamma x\delta e)\alpha(b\theta c\gamma x\delta e) = b\theta c\gamma x\delta a\theta c\gamma x\delta e = b\theta c\gamma x\delta e$$

which proves that $b\theta c\gamma x\delta a \leqslant b$. Again, we have that

$$a\theta c\gamma x\delta a = b\beta(f\theta c\gamma x\delta e)\alpha b = (a\theta c\gamma x\delta e)\alpha b,$$

$$(f\theta c\gamma x\delta a)\beta(f\theta c\gamma x\delta a) = f\theta c\gamma x\delta a,$$

and

$$(a\theta c\gamma x\delta e)\alpha(a\theta c\gamma x\delta e) = a\theta c\gamma x\delta e$$

which give $a\theta c\gamma x\delta a \leqslant b$. It is easy to show that $(b\theta c\gamma x\delta a)\mathcal{L}(a\theta c\gamma x\delta a)$. By the hypothesis, we get that $b\theta c\gamma x\delta a = a\theta c\gamma x\delta a$. Since $x \in V_\gamma^\delta(a\theta c)$, we get that

$$a\theta c = a\theta c\gamma x\delta a\theta c = b\theta c\gamma x\delta a\theta c \text{ and } a\theta c = e\alpha b\theta c$$

with $x\delta a\theta c \in E_\gamma(S)$. We conclude that $a\theta c \leqslant b\theta c$. □

Dually, we get the following statement.

Corollary 3.16. *Let S be a regular Γ-semigroup. Then the following statements are equivalent.*

(1) $\leqslant$ is left compatible.

(2) S satisfies $\mathcal{R}$-majorization for idempotents.

(3) S satisfies $\mathcal{R}$-majorization.

Proof. The proof is similar to that of Theorem 3.15. □

Corollary 3.17. *Let S be a regular Γ-semigroup. Then the following statements are equivalent.*

(1) $\leqslant$ is compatible.

(2) S satisfies $\mathcal{L}$- and $\mathcal{R}$-majorization for idempotents.

(3) S satisfies $\mathcal{L}$- and $\mathcal{R}$-majorization.

Proof. It follows from Theorem 3.15 and Corollary 3.16. □

Finally, we find a relation on a regular Γ-semigroup S and show that this relation is a congruence on S.

Theorem 3.18. *Let S be a regular Γ-semigroup and the natural partial order on S be compatible with multiplication. Then*

$$\omega := \{(a, b) \in S \times S \mid c \leqslant a \text{ and } c \leqslant b \text{ for some } c \in S\}$$

is a congruence on S.

Proof. Note that ω is a reflexive and symmetric. Next, we will show that ω is a transitive. Let $(a,b),(b,c) \in \omega$. Then there exist $x, y \in S$ such that $x \leqslant a, x \leqslant b$ and $y \leqslant b, y \leqslant c$. It implies that $x = f\beta b$ and $y = b\alpha e$ for some $f \in E_\beta(S), e \in E_\alpha(S), \beta, \alpha \in \Gamma$. Indeed,

$$x\alpha e = f\beta b\alpha e = f\beta y.$$

Set $z = x\alpha e = f\beta y$. By hypothesis and $x \leqslant b$ we get that $z = x\alpha e \leqslant b\alpha e = y$ and $y \leqslant b$ implies that $z = f\beta y \leqslant f\beta b = x$, so $z \leqslant x \leqslant a$ and $z \leqslant y \leqslant c$. It implies that $(a,c) \in \omega$. By hypothesis, ω is compatible. Therefore ω is a congruence on S. □

A non-zero element of a regular Γ-semigroup S is *primitive* if it is minimal among the non-zero elements of S. A regular Γ-semigroup S is said to be *primitive* if each of its non-zero idempotents is primitive. A congruence ρ on a regular Γ-semigroup S is called *primitive* if S/ρ is primitive. Clearly, if S is trivially ordered then S is primitive.

A mapping $\phi : X \to Y$ of a quasi-ordered set $(X, \leqslant_X)$ into a quasi-ordered set $(Y, \leqslant_Y)$ *reflecting*[8] if for all $y, y' \in X\phi$ such that $y' \leqslant_Y y$ and $x \in X$ with $x\phi = y$ there is some $x' \in X$ such that $x' \leqslant_X x$ and $x'\phi = y'$.

Theorem 3.19. *Let S be a regular Γ-semigroup such that ω is a congruence and the natural homomorphism for ω is reflecting the natural partial order. Then ω is the least primitive congruence on S.*

Proof. Define the natural homomorphism $\varphi : S \to S/\omega$ by $s\varphi = s\omega$ for all $s \in S$. We will show that S/ω is trivially ordered. Let $y, z \in S/\omega$ be such that $y \leq z$. Since φ is reflecting the natural partial order, there exist $s, t \in S$ such that $s \leqslant t$ and $s\varphi = y, t\varphi = z$. Since $s \leqslant t$, we now get that $s\omega t$. Thus

$$y = s\varphi = s\omega = t\omega = t\varphi = z.$$

Therefore S/ω is trivially ordered.

Let ρ be any congruence on S/ω such that $(S/\rho, \leq)$ is trivially ordered and let ψ denotes the natural homomorphism corresponding ρ. Suppose that $s\omega t$. There exists $w \in S$ such that $w \leqslant s$ and $w \leqslant t$, giving $w\psi \leq s\psi$ and $w\psi \leq t\psi$ in S/ρ. Since S/ρ is trivially ordered, we obtain that $s\psi = w\psi = t\psi$, so $s\rho = t\rho$. Thus $s\rho t$ immediately implies that $\omega \subseteq \rho$. Therefore ω is the least primitive congruence on S. □

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