Properties of the wave curves in the shallow water equations with discontinuous topography

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Abstract

We first establish the monotonicity of the curves of composite waves for shallow water equations with discontinuous topography. Second, a critical investigation of the Riemann problem yields deterministic results for large data on the existence of Riemann solutions made of Lax shocks, rarefaction waves, and admissible stationary contacts. Although multiple solutions can be constructed for certain Riemann data, we can determine relatively large neighborhoods of Riemann data in which the Riemann problem admits a unique solution.

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1. Introduction

The curves of composite waves in the shallow water equations with discontinuous topography play an essential role in solving the Riemann problem. However, the monotonicity of this kind of curves has not been proved, partly due to its complicatedness of composing waves in one characteristic family along a curve of waves in another characteristic family. This gives us a motivation for this study. Furthermore, we provide in this paper a critical investigation the Riemann problem for shallow water equations with discontinuous topography. Precisely, the model is given by

$$\begin{aligned}
&\partial_t h + \partial_x(hu) = 0,\\
&\partial_t(hu) + \partial_x\left(h\left(u^2 + \frac{gh}{2}\right)\right) = -gh\partial_x a,\\
&\partial_t a = 0,
\end{aligned} \tag{1.1}$$

where the height of the water from the bottom to the surface, denoted by h, and the fluid velocity u are the main unknowns. Here, g is the gravity constant, and $a = a(x)$ (with $x \in \mathbb{R}$) is the height of the bottom from a given level. Observe that the third equation in (1.1) is a trivial equation.

The Riemann problem for (1.1) is the Cauchy problem with the initial data, called the Riemann data, of the form

$$(h,u,a)(x,0) = \begin{cases} (h_L,u_L,a_L), & \text{for} \quad x<0, \\ (h_R,u_R,a_R), & \text{for} \quad x>0. \end{cases} \tag{1.2}$$

It has been known that the system of balance laws in nonconservative form (1.1) is hyperbolic whose characteristic fields may coincide, see [15] for example. Building Riemann solutions of this kind of systems would involve the construction of curves of composite waves, which include waves in a genuinely nonlinear and a linearly degenerate characteristic fields. For example, in the general case a (local) existence result was established by [5]. The Riemann problem for (1.1) was studied in [15, 16], where the existence for large data was obtained. Furthermore, it was shown in [16] that up to three solutions can be constructed for certain Riemann data. However, the monotonicity property of the curves of composite waves has not been proved before. That leaves an open question for the completeness of the theory of this kind of systems. Furthermore, when does a Riemann solution existence, precisely? Hence, the current paper has two goals: the first goal is to establish the monotonicity property of the curves of composite waves - and so the domain of uniqueness could be found, and the second goal is to seek for a deterministic version of the existence of Riemann solutions - where *explicit* large domains of existence could be found.

Systems of balance laws in nonconservative form have attracted many authors. A general framework for systems of balance laws in nonconservative form was introduced by Dal Maso-LeFloch-Murat [4], see also LeFloch [13]. The standard admissibility criterion for shock waves for hyperbolic systems of conservation laws was addressed in the pioneering work by Lax [12]. Shock waves and the related traveling waves in scalar conservation laws with a nonzero right-hand side were studied by Isaacson-Temple [7, 8] and Thanh [24]. As mentioned above, a local existence of Riemann solutions for general systems of balance laws with resonance was established by Goatin-LeFloch[5]. The Riemann problem for fluid flows in a nozzle with discontinuous cross-section were considered by MarchesinPaes-Leme [17], Andrianov-Warnecke [1], LeFloch-Thanh [14], Kroener-LeFloch-Thanh [11], and Thanh [20]. Recently, the Riemann problem and exact solutions for two-phase flow models are considered by Andrianov-Warnecke [2], Schwendeman-Wahle-Kapila [19], and Thanh [23, 22]. Numerical schemes for shallow water equations were studied by Chinnayya-LeRoux-Seguin [3], Thanh-Fazlul-Ismail [21, 16], Jin-Wen [9, 10], Rosatti-Begnudelli [18], and Gallardo-Parés-Castro [6]. See also the references therein.

The organization of this paper is as follows. In Section 2 we recall basic concepts and properties of the system (1.1). In Section 3 we study the monotonicity of the curves of composite waves. Finally, in Section 4 we present deterministic version of existence results, and uniqueness of Riemann solutions of the problem (1.1)-(1.2).

2. Preliminaries

2.1. Wave curves

The system (1.1) can be re-written as a nonconservative system as

$$\partial_t U + A(U)\partial_x U = 0, \tag{2.1}$$

where

$$U = \begin{pmatrix} h \\ u \\ a \end{pmatrix}, \quad A(U) = \begin{pmatrix} u & h & 0 \\ g & u & g \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix $A = A(U)$ has three real eigenvalues

$$\lambda_1(U) := u - \sqrt{gh} < \lambda_2(U) := u + \sqrt{gh}, \quad \lambda_3(U) := 0, \tag{2.2}$$

together with the corresponding eigenvectors which can be chosen as

$$r_1(U) := \begin{pmatrix} h \\ -\sqrt{gh} \\ 0 \end{pmatrix}, \quad r_2(U) := \begin{pmatrix} h \\ \sqrt{gh} \\ 0 \end{pmatrix}, \quad r_3(U) := \begin{pmatrix} gh \\ -gu \\ u^2 - gh \end{pmatrix}. \tag{2.3}$$

Thus, the system (2.1) is hyperbolic. Moreover, the first and the third characteristic speeds can coincide, i.e.,

$$\lambda_1(U) = \lambda_3(U) = 0,$$

on the surface

$$\mathcal{C}_+ := \Big\{(h, u, a)| \quad u = \sqrt{gh}\Big\}, \tag{2.4}$$

and the second and the third characteristic fields can coincide, i.e.,

$$\lambda_2(U) = \lambda_3(U) = 0,$$

on the surface

$$\mathcal{C}_- := \Big\{(h, u, a)| \quad u = -\sqrt{gh}\Big\}. \tag{2.5}$$

The above argument means that the system (1.1)-(1.2) is hyperbolic, but is *not strictly hyperbolic.*

Besides, it is easy to see that the first and second characteristic fields (λ_1, r_1), (λ_2, r_2) are genuinely nonlinear, that is

$$\nabla\lambda_1 \cdot r_1 \neq 0, \quad \nabla\lambda_2 \cdot r_2 \neq 0,$$

and that the third characteristic field (λ_3, r_3) is linearly degenerate, that is

$$\nabla\lambda_3 \cdot r_3 = 0.$$

Set

$$\begin{aligned}
&\mathcal{C} := \mathcal{C}_+ \cup \mathcal{C}_-,\\
&G_1 := \Big\{U| \quad \lambda_1(U) > \lambda_3(U)\Big\} = \Big\{U| \quad u > \sqrt{gh}\Big\},\\
&G_2 := \Big\{U| \quad \lambda_2(U) > \lambda_3(U) > \lambda_1(U)\Big\} = \Big\{U| \quad |u| < \sqrt{gh}\Big\},\\
&G_2^+ := \{U \in G_2| \quad u \geq 0\} = \Big\{U| \quad 0 \leq u < \sqrt{gh}\Big\},\\
&G_2^- := \{U \in G_2| \quad u < 0\} = \Big\{U| \quad 0 > u > -\sqrt{gh}\Big\},\\
&G_3 := \Big\{U| \quad \lambda_3(U) > \lambda_2(U)\Big\} = \Big\{U| \quad u < -\sqrt{gh}\Big\}.
\end{aligned} \tag{2.6}$$

As discussed in [15], across a discontinuity there are two possibilities:

(i) either the bottom height a remains constant,

(ii) or the discontinuity is stationary (i.e. propagates with zero speed).

Let us consider the first the case (i), where the system (1.1) is reduced to the usual shallow water equations with flat bottom. Then, we can determine the Rankine-Hugoniot relations and the admissibility criterion for shock waves as usual. Let us recall that a *shock wave* of (1.1) is a weak solution of the form

$$U(x,t) = \begin{cases} U_-, & x < st, \\ U_+, & x > st, \end{cases} \tag{2.7}$$

where U_-, U_+ are the left-hand and right-hand states, respectively, and $s = s(U_-, U_+)$ is the shock speed. A shock wave (2.7) is *admissible*, called an i-Lax shock, if it satisfies the Lax shock inequalities, see [12],

$$\lambda_i(U_+) < s(U_-, U_+) < \lambda_i(U_-), \quad i = 1, 2. \tag{2.8}$$

From now on, we consider admissible shock waves, only.

Given a left-hand state U_0, the set of all right-hand states that can be connected to U_0 by an i-Lax shock forms a curve, denoted by $\mathcal{S}_i(U_0), i = 1, 2$. In a backward way, given a right-hand state U_0, the set of all left-hand states that can be connected to U_0 by an i-Lax shock forms a curve, denoted by $\mathcal{S}_i^B(U_0), i = 1, 2$. These curves are defined by

$$\begin{aligned}
\mathcal{S}_1(U_0): &\quad u = u_0 - \sqrt{\frac{g}{2}}(h - h_0)\sqrt{\frac{1}{h} + \frac{1}{h_0}}, \quad h > h_0, \\
\mathcal{S}_2(U_0): &\quad u = u_0 + \sqrt{\frac{g}{2}}(h - h_0)\sqrt{\frac{1}{h} + \frac{1}{h_0}}, \quad h < h_0, \\
\mathcal{S}_1^B(U_0): &\quad u = u_0 - \sqrt{\frac{g}{2}}(h - h_0)\sqrt{\frac{1}{h} + \frac{1}{h_0}}, \quad h < h_0, \\
\mathcal{S}_2^B(U_0): &\quad u = u_0 + \sqrt{\frac{g}{2}}(h - h_0)\sqrt{\frac{1}{h} + \frac{1}{h_0}}, \quad h > h_0,
\end{aligned} \tag{2.9}$$

see [15].

It is interesting that the shock speeds in the nonlinear characteristic fields may coincide with the characteristic speed of the linearly degenerate field as stated in the following lemma.

Lemma 2.1 (Lem. 2.1, [16])**.** *Consider the projection on the (h, u)-plan. To every $U_L = (h_L, u_L) \in G_1$ there exists exactly one point $U_L^{\#} \in S_1(U_L) \cap G_2^+$ such that the 1-shock speed $\overline{\lambda}_1(U_L, U_L^{\#}) = 0$. The state $U_L^{\#} = (h_L^{\#}, u_L^{\#})$ is defined by*

$$h_L^{\#} = \frac{-h_L + \sqrt{h_L^2 + 8h_L u_L^2/g}}{2}, \quad u_L^{\#} = \frac{u_L h_L}{h_L^{\#}}.$$

Moreover, for any $U \in S_1(U_L)$, the shock speed $\overline{\lambda}_1(U_L, U) > 0$ if and only if U is located above $U_L^{\#}$ on $S_1(U_L)$.

Next, let us consider rarefaction waves, which are piecewise smooth self-similar solutions of (2.1). It was shown by [**?**] that the bottom height a remains constant through any rarefaction fan. Given a left-hand state U_0, the set of all right-hand states that can be connected to U_0 by an i-rarefaction waves of (2.1) forms a curve, denoted by $\mathcal{R}_i(U_0), i = 1, 2$. In a backward way, given a right-hand state U_0, the set of all left-hand states that can be connected to U_0 by an i-rarefaction wave forms a curve, denoted by $\mathcal{R}_i^B(U_0), i = 1, 2$. These curves are given by

$$\begin{array}{lll} \mathcal{R}_1(U_0): & u = u_0 - 2\sqrt{g}(\sqrt{h} - \sqrt{h_0}), & h \le h_0, \\ \mathcal{R}_2(U_0): & u = u_0 + 2\sqrt{g}(\sqrt{h} - \sqrt{h_0}), & h \ge h_0, \\ \mathcal{R}_1^B(U_0): & u = u_0 - 2\sqrt{g}(\sqrt{h} - \sqrt{h_0}), & h \ge h_0, \\ \mathcal{R}_2^B(U_0): & u = u_0 + 2\sqrt{g}(\sqrt{h} - \sqrt{h_0}), & h \le h_0, \end{array} \tag{2.10}$$

see [15]. We can therefore define the forward and backward wave wave curves in the nonlinear characteristic fields as follows

$$\begin{array}{l} \mathcal{W}_i(U_0) = \mathcal{R}_i(U_0) \cup \mathcal{S}_i(U_0), \\ \mathcal{W}_i^B(U_0) = \mathcal{R}_i^B(U_0) \cup \mathcal{S}_i^B(U_0), \quad i = 1, 2. \end{array} \tag{2.11}$$

As seen above, the curves $\mathcal{W}_i(U_0)$ can be parameterized as a function $u = w_i(U_0; h)$ of $h \ge 0$, and the curves $\mathcal{W}_i^B(U_0)$ can be parameterized as a function $u = w_i^B(U_0; h)$ of $h \ge 0$, $i = 1, 2$. Precisely,

$$\begin{array}{ll} \mathcal{W}_1(U_0): & u = w_1(U_0; h) := \begin{cases} u_0 - 2\sqrt{g}\left(\sqrt{h} - \sqrt{h_0}\right), & h \le h_0, \\ u_0 - \sqrt{\frac{g}{2}}\,(h - h_0)\sqrt{\frac{1}{h} + \frac{1}{h_0}}, & h > h_0, \end{cases} \\ \mathcal{W}_2(U_0): & u = w_2(U_0; h) := \begin{cases} u_0 + 2\sqrt{g}\left(\sqrt{h} - \sqrt{h_0}\right), & h \ge h_0, \\ u_0 + \sqrt{\frac{g}{2}}\,(h - h_0)\sqrt{\frac{1}{h} + \frac{1}{h_0}}, & h < h_0, \end{cases} \\ \mathcal{W}_1^B(U_0): & u = w_1^B(U_0; h) =: \begin{cases} u_0 - 2\sqrt{g}\left(\sqrt{h} - \sqrt{h_0}\right), & h \ge h_0, \\ u_0 - \sqrt{\frac{g}{2}}\,(h - h_0)\sqrt{\frac{1}{h} + \frac{1}{h_0}}, & h < h_0, \end{cases} \\ \mathcal{W}_2^B(U_0): & u = w_2^B(U_0; h) := \begin{cases} u_0 + 2\sqrt{g}\left(\sqrt{h} - \sqrt{h_0}\right), & h \le h_0, \\ u_0 + \sqrt{\frac{g}{2}}\,(h - h_0)\sqrt{\frac{1}{h} + \frac{1}{h_0}}, & h > h_0. \end{cases} \end{array} \tag{2.12}$$

It was shown in [15] that $w_1(U_0; h)$ *and* $w_1^B(U_0; h)$ *are strictly convex and strictly decreasing functions of* h*, while* $w_2(U_0; h)$ *and* $w_2^B(U_0; h)$ *are strictly concave and strictly increasing functions of* $h \ge 0$.

Let us now consider the case (ii), where the discontinuity satisfies the jump relations

$$\begin{array}{l} [hu] = 0, \\ \left[\dfrac{u^2}{2} + g(h + a)\right] = 0. \end{array}$$

The last jump relations determine the stationary-wave curve (parameterized with h) as follows

$$\mathcal{W}_3(U_0): \qquad \begin{aligned} u &= w_3(U_0;h) := \frac{u_0 h_0}{h}, \quad h \geq 0, \\ a &= a_0 + \frac{u_0^2 - u^2}{2g} + h_0 - h. \end{aligned} \tag{2.13}$$

It is easy to check that *the function $w_3(U_0;h), h \geq 0$, is strictly convex and strictly decreasing for $u_0 > 0$, and strictly concave and strictly increasing for $u_0 < 0$.*

2.2. Properties of stationary contacts

Given a state $U_0 = (h_0, u_0, a_0)$ and another bottom level $a \neq a_0$, we let $U = (h, u, a)$ be the corresponding right-hand state of the stationary contact issuing from the given left-hand state U_0. We now determine h, u in terms of U_0, a, as follows. Substituting $u = h_0 u_0 / h$ from the first equation of (2.13) to the second equation of (2.13) , we obtain

$$a_0 + \frac{1}{2g}\left(u_0^2 - \left(\frac{h_0 u_0}{h}\right)^2\right) + h_0 - h = a. \tag{2.14}$$

Multiplying both sides of (2.14) by $2gh^2$, and then re-arranging terms, we get

$$F(h) = F(U_0, a; h) := 2gh^3 + [2g(a - a_0 - h_0) - u_0^2]h^2 + h_0^2 u_0^2 = 0. \tag{2.15}$$

We easily check

$$\begin{aligned} F(0) &= h_0^2 u_0^2 \geq 0, \\ F'(h) &= 6gh^2 + 2[2g(a - a_0 - h_0) - u_0^2]h, \\ F''(h) &= 12gh + 2[2g(a - a_0 - h_0) - u_0^2], \end{aligned}$$

so that

$$F'(h) = 0 \text{ iff } h = 0 \text{ or } h = h_* = h_*(U_0, a) := \frac{u_0^2 + 2g(a_0 + h_0 - a)}{3g}. \tag{2.16}$$

If $a > a_0 + h_0 + \dfrac{u_0^2}{2g}$, then $h_* < 0$ and $F'(h) > 0, h > 0$. Since $F(0) \geq 0$, there is no root for (2.15). Otherwise, if $a \leq a_0 + h_0 + \dfrac{u_0^2}{2g}$, then $F'(h) > 0, h > h_*$ and $F'(h) < 0, 0 < h < h_*$. In this case, $F(h)$ has two zeros h if and only if

$$F_{min} := F(h_*) = -gh_*^3 + h_0^2 u_0^2 \leq 0,$$

or

$$h_* \geq h_{min}(U_0) := (h_0^2 u_0^2 / g)^{1/3}.$$

It is easy to check that $h_* \geq h_{min}(U_0)$ if and only if

$$a \leq a_{max}(U_0) := a_0 + h_0 + \frac{u_0^2}{2g} - \frac{3}{2g^{1/3}}(h_0 u_0)^{2/3}.$$

Lemma 2.2 (Lem. 2.2, [16]). *Given a state* $U_0 = (h_0, u_0, a_0)$ *and a bottom level* $a \neq a_0$. *The following conclusions holds.*

(i) $a_{max}(U_0) \geq a_0$, $a_{max}(U_0) = a_0$ *if and only if* $(h_0, u_0) \in \mathcal{C}_\pm$.

(ii) The nonlinear equation (2.14) admits a root if and only if $a \leq a_{max}(U_0)$, *and in this case it has two roots* $\varphi_1(a) \leq h_*(U_0, a) \leq \varphi_2(a)$. *Moreover, if* $a < a_{max}(U_0)$, *these two roots are distinct.*

(iii) According to the part (ii), whenever $a \leq a_{max}(U_0)$, *there are two states* $U_i(a) = (\varphi_i(a), u_i(a), a)$, *where* $u_i(a) = h_0u_0/\varphi_i(a)$, $i = 1, 2$ *to which a stationary contact from* U_0 *is possible. Moreover, the locations of these states can be determined as follows*

$$\begin{array}{ll} U_1(a) \in G_1 & \text{if} \quad u_0 > 0, \\ U_1(a) \in G_3 & \text{if} \quad u_0 < 0, \\ U_2(a) \in G_2. & \end{array}$$

We next prove the following result, which has been stated in [15] without a proof.

Lemma 2.3. *We have the following comparisons*

(i) If $a < a_0$, *then*

$$\varphi_1(a) < h_0 < \varphi_2(a).$$

(ii) If $a_0 < a < a_{max}(U_0)$, *then*

$$\begin{aligned} h_0 &< \varphi_1(a) \text{ for } U_0 \in G_1 \cup G_3, \\ h_0 &> \varphi_2(a) \text{ for } U_0 \in G_2. \end{aligned}$$

Proof. We have

$$F(h_0) = 2g(a - a_0)h_0.$$

If $a < a_0$, then $a < a_0 \leq a_{max}(U_0)$ and $F(h_0) < 0$. It implies that $F(h)$ has two zeros $\varphi_{1,2}(a)$ such that

$$\varphi_1(a) < h_0 < \varphi_2(a).$$

If $a_0 < a < a_{max}(U_0)$, then $F(h_0) > 0$ and $F(h)$ has two distinct zeros $\varphi_{1,2}(a)$ such that

$$h_0 < \varphi_1(a) \text{ or } h_0 > \varphi_2(a).$$

In the case $U_0 \in G_1 \cup G_3$, $h_0 < \varphi_1(a)$ since $h_* = \left(\frac{u_0^2h_0^2}{g}\right)^{1/3} > h_0$. In the case $U_0 \in G_2$, $h_0 > \varphi_2(a)$ since $F'(h_0) = 2h_0[gh_0 - u_0^2 + 2g(a - a_0)] > 0$. □

From Lemma 2.2 , we can construct two-parameter wave sets. The Riemann problem may therefore admit up to a one-parameter family of solutions. To select a unique solution, we impose an admissibility condition for stationary contacts, referred to as the Monotonicity Criterion and defined as follows

(MC) Along any stationary curve $\mathcal{W}_3(U_0)$, the bottom level a is monotone as a function of h. The total variation of the bottom level component of any Riemann solution must not exceed $|a_L - a_R|$, where a_L, a_R are left-hand and right-hand bottom levels.

A similar criterion was used in [15].

Lemma 2.4. *The Monotonicity Criterion implies that any stationary shock does not cross the boundary of strict hyperbolicity, in other words*

(i) If $U_0 \in G_1 \cup G_3$, then only the stationary contact based on the value $\varphi_1(a)$ is allowed.

(ii) If $U_0 \in G_2$, then only the stationary contact using $\varphi_2(a)$ is allowed.

3. Monotone property of curves of composite waves

Observe that by the transformation $x \mapsto -x$, $u \mapsto -u$, a left-hand (right-hand) state $U = (h, u, a)$ in G_2^- (in $G_3 \cup \mathcal{C}_-$) will be transformed to the right-hand (left-hand, respectively) state $V = (h, -u, a)$ in G_2^+ (in $G_1 \cup \mathcal{C}_+$, respectively). Thus, the construction of wave curves and therefore the Riemann solutions for Riemann data around $\mathcal{C}_-$ can be obtained from the one for Riemann data around $\mathcal{C}_+$. Thus, without loss of generality, in the sequel we consider only the case where Riemann data are in $G_1 \cup \mathcal{C}_+ \cup G_2^+$. Moreover, the construction will be relied on the left-hand state U_L (and hence the region of the right-hand states will follow) if $a_L > a_R$, and the construction will be relied on the right-hand state U_R (and hence the region of the left-hand states will follow), otherwise.

Notations

(i) $W_k(U_i, U_j)$ $(S_k(U_i, U_j), R_k(U_i, U_j))$ denotes the kth-wave (kth-shock, kth-rarefaction wave, respectively) connecting the left-hand state U_i to the right-hand state U_j, $k = 1, 2, 3$.

(ii) $W_m(U_i, U_j) \oplus W_n(U_j, U_k)$ indicates that there is an mth-wave from the left-hand state U_i to the right-hand state U_j, followed by an nth-wave from the left-hand state U_j to the right-hand state U_k, $m, n \in \{1, 2, 3\}$.

(iii) We will sometimes write for simplicity in this section the curves defined by (2.1) as $u = w_i(h)$ instead of $u = w_i(U_0; h)$, and $u = w_i^B(h)$ instead of $u = w_i^B(U_0; h), i = 1, 2$, when U_0 is clear, if this does not any confusion.

(iv) $U^{\#}$ denotes the state resulted by a shock wave from U with zero speed; U^0 denotes the state resulted by an admissible stationary contact from U.

3.1. Case : $U_L \in G_1 \cup \mathcal{C}_+$ and $a_L > a_R$

Let $U_\pm = (h_\pm, u_\pm)$ stand for the states at which the wave $\mathcal{W}_1(U_L)$ intersects with the curves $\mathcal{C}_\pm$, respectively. From $U_L(a_L)$ the Riemann solution can begin with a stationary contact wave to some state $U_L^0(a_R) \in G_1$ using $\varphi_1(U_L, a_R)$. There is one state $U_L^{0\#}(a_R) \in \mathcal{W}_1(U_L^0) \cap G_2^+$ such that $\overline{\lambda}_1(U_L^0, U_L^{0\#}) = 0$ and $\overline{\lambda}_1(U_L^0, U) > 0$ for $h_L^0 < h < h_L^{0\#}$, $\overline{\lambda}_1(U_L^0, U) < 0$ for $h > h_L^{0\#}$. So, the solution can continue by a 1-wave from U_L^0 to state U such that $0 \leq h \leq h_L^{0\#}$. The set of these states U form the curve composite $\mathcal{W}_{3\to1}(U_L)$. The composite curve $\mathcal{W}_{3\to1}(U_L)$ is a part of the curve $\mathcal{W}_1(U_L^0)$. The curve $\mathcal{W}_1(U_L^0)$ intersects axis $h = 0$ at the point $I = \left(0, u_{up} := u_L^0 + 2\sqrt{gh_L^0}\right)$. I and $U_L^{0\#}$ are two endpoints of the composite curve $\mathcal{W}_{3\to1}(U_L)$.

From U_L the Riemann solution can begin with 1-shock to state $A \in \mathcal{S}_1(U_L) \cap G_2$ such that $\overline{\lambda}_1(U_L, A) \leq 0$. So, A is between $U_L^{\#}$ and U_-. The solution continue with a stationary contact wave from state $A(a_L)$ to state $A^0(a_R) \in G_2$ using $\varphi_2(A, a_R)$. The set of such states A^0 forms the curve composite $\mathcal{W}_{1\to 3}(U_L)$. $U_L^{\#0}$ and U_-^0 are two endpoints of this curve, where

$$U_-^0 = (h_-^0, u_-^0) = (\varphi_2(U_-, a_R), w_3(U_-, \varphi_2(U_-, a_R)))\,.$$

Of course, the curve of composite waves can be constructed beyond $h > h_-^0$. However, in the region G_3, the characteristic speed $\lambda_2(U) < 0 = \lambda_3(U), U \in G_3$. In this case, the construction of the Riemann solution(as seen below) may not be well-defined. That is the reason why we stop the composite curve at U_-^0.

Besides, at each level $a \in [a_R, a_L]$, from $U_L(a_L)$ the Riemann solution can begin with the stationary contact wave to some state $B(a) \in G_1 \cup \mathcal{C}_+$ using $\varphi_1(U_L, a)$. The solution is continued with 1-shock from state B to state $B^{\#} \in G_2^+ \cup \mathcal{C}_+$ such that $\overline{\lambda}_1(B, B^{\#}) = 0$. Then, the solution is continued with the stationary contact wave from state $B^{\#}(a)$ to state $B^{\#0}(a_R) \in G_2$ using $\varphi_2(B^{\#}, a_R)$. The set of such states $B^{\#0}$ forms the curve composite $\mathcal{W}_{3\to 1\to 3}(U_L)$. If $a = a_R$, then $B = U_L^0$ and $B^{\#} = U_L^{0\#} = B^{\#0}$. If $a = a_L$, then $B = U_L$, $B^{\#} = U_L^{\#}$ and $B^{\#0} = U_L^{\#0}$. So, $U_L^{\#0}$ and $U_L^{0\#}$ are two endpoints of this curve. The *curve of composite waves* $\Gamma(U_L)$ is defined as follows

$$\Gamma(U_L) := \mathcal{W}_{3\to 1}(U_L) \cup \mathcal{W}_{1\to 3}(U_L) \cup \mathcal{W}_{3\to 1\to 3}(U_L). \tag{3.1}$$

Thus, $\Gamma(U_L)$ has two endpoints I and $U_-^0 = (h_-^0, u_-^0) \in G_2^-$.

As mentioned above, the first part of $\Gamma(U_L)$, which is $\mathcal{W}_{3\to 1}(U_L)$ is an arc of the curve $\mathcal{W}_1(U_L^0)$, and so it is strictly decreasing as written in the form $u = u(h)$. The third part of $\Gamma(U_L)$, which is $\mathcal{W}_{3\to 1\to 3}(U_L)$ is an arc of the hyperbola $u/h =$ positive constant, so it is also strictly decreasing as written in the form $u = u(h)$ as well. Now, we consider the monotone property of the composite curve $\mathcal{W}_{1\to 3}(U_L)$. As seen in Section 2, given a state $U_0 = (h_0, u_0)$ at topography level a_0, the state $U = (h, u)$ at topography level $a \neq a_0$ that can be connected to U_0 by a stationary wave satisfies the equations

$$F(h; U_0) = 2gh^3 + (2g(a - a_0 - h_0) - u_0^2)h^2 + h_0^2 u_0^2 = 0, \qquad u = \frac{u_0 h_0}{h}. \tag{3.2}$$

In view by Lemmas 2.2 and 2.3, the equation $F(h; U_0) = 0$ could admit two roots denoted by $\varphi_1(U_0, a)$ and $\varphi_2(U_0, a)$ such that

$$\varphi_1(U_0, a) < h_0 < \varphi_2(U_0, a) \quad \text{if} \quad a < a_0.$$

Moreover, as seen in Section 2, the Monotone Criterion selects $\varphi_2(U_0, a)$ when $U_0 \in G_2$, and the state resulting from the stationary wave $U = (h = \varphi_2(U_0, a), u = u_0 h_0/\varphi_2(U_0, a)) \in G_2$.

Now, let U_0 vary on the curve $\mathcal{W}_1(U_L) \cap G_2$, so we replace U_0 by $U_1(h) = (h, u = w_1(U_L; h)) \in \mathcal{W}_1(U_L) \cap G_2$, at the topography level a_L. In the sequel for simplicity we write

$$w_1(h) = w_1(U_L; h), \quad \varphi_2(h) = \varphi_2(U_1(h), a_R) \quad \text{where} \quad U_1(h) = (h, u = w_1(U_L; h)).$$

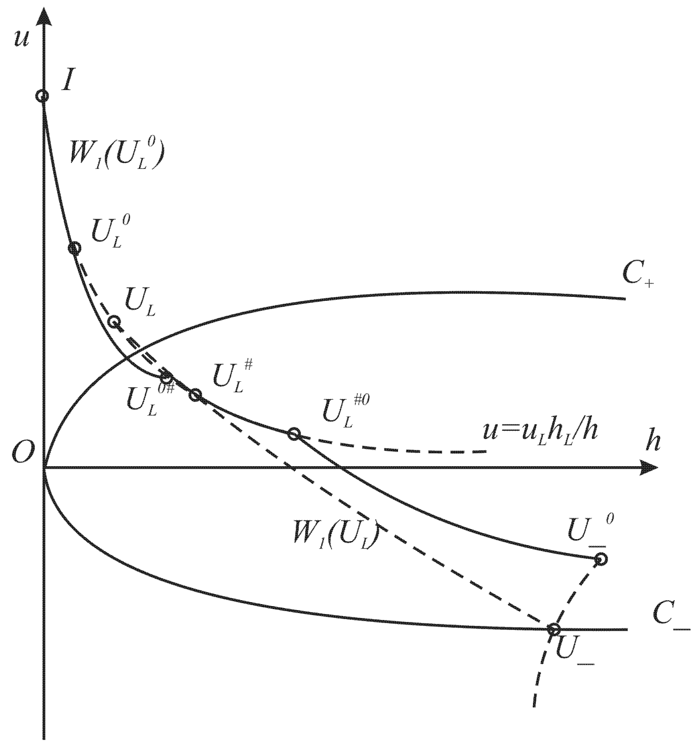

Figure 1: The composite wave curve $\Gamma(U_L)$.

Then, the state on the other side of the admissible stationary wave from $U_1(h)$ is determined by

$$U_1^0(h) = (\varphi_2(U_1(h), a_R), u = w_1(h)h/\varphi_2(h)) \in G_2.$$

Observe that $\varphi_2(h)$ satisfying the equation

$$G(h) := F(h; U_1(h)) = 2g\varphi_2^3(h) + (2g(a_R - a_L - h) - w_1^2(h))\varphi_2(h)^2 + h^2 w_1^2(h) = 0. \quad (3.3)$$

Lemma 3.1. *Let $U_L \in G_1 \cup \mathcal{C}_+ \cup G_2^+, a_L > a_R$, and let $U_1(h) = (h, u = w_1(U_L; h)) \in \mathcal{W}_1(U_L) \cap G_2$, where w_1 is defined by (2.1), and $(h_\pm, u_\pm) \in \mathcal{W}_1(U_L) \cap \mathcal{C}_\pm$. Then, the function $\varphi_2(h) := \varphi_2(U_1(h), a_R)$ is strictly increasing for $h \in [h_+, h_-]$.*

Proof. Differentiating (3.3) with respect to h, we get

$$\begin{aligned} 0 = G'(h) &= 6g\varphi_2^2(h)\varphi_2'(h) + 2(2g(a_R - a_L - h) - w_1^2(h))\varphi_2(h)\varphi_2'(h) \\ &\quad + \varphi_2^2(h)(-2g - 2w_1(h)w_1'(h)) + 2hw_1^2 + 2h^2 w_1(h)w_1'(h) \\ &= \varphi_2'(h)\varphi_2(h)[6g\varphi_2(h) + 2(2g(a_R - a_L - h) - w_1^2(h))] \\ &\quad + 2(hw_1^2(h) - g\varphi_2^2(h)) + 2w_1(h)w_1'(h)(h^2 - \varphi_2^2(h)). \end{aligned}$$

Then,

$$0 = \varphi_2'(h)A + B, \quad (3.4)$$

where

$$\begin{aligned} A &= \varphi_2^2(h)[3g\varphi_2(h) + 2g(a_R - a_L - h) - w_1^2(h)], \\ B &= \varphi_2(h)[(hw_1^2(h) - g\varphi_2^2(h)) + w_1(h)w_1'(h)(h^2 - \varphi_2^2(h))]. \end{aligned}$$

From (3.4), to prove that $\varphi_2'(h) > 0$, we need to show that

$$A > 0, \quad B < 0.$$

Indeed, it holds that

$$\begin{aligned} A &= 3g\varphi_2^3(h) - (2g\varphi_2^3(h) + h^2 w_1^2(h)) \\ &= g\varphi_2^3(h) - h^2 w_1^2(h) \\ &> gh^3 - h^2 w_1^2(h) = h^2(gh - w_1^2(h)) \geq 0, \end{aligned}$$

since $U_1(h) = (h, w_1(h)) \in G_2$. This gives

$$A > 0.$$

To prove that $B < 0$, we need to show that

$$\theta(h) := w_1(h)w_1'(h) \geq -g, \quad h_+ \leq h \leq h_-. \tag{3.5}$$

Indeed, if $w_1(h) < 0$, then

$$\theta(h) = w_1(h)w_1'(h) > 0 > -g.$$

So, we remain to prove (3.5) for the case $w_1(h) \geq 0$, i.e., $h_+ \leq h \leq h_\natural$, where $h_\natural$ is the h-intercept of the shock curve $\mathcal{W}_1(U_L)$. It holds that

$$\frac{d^2}{dh^2}(w_1^2(h)) = \frac{d}{dh}\,(2w_1(h)w_1'(h)) = 2(w_1'^2 + w_1(h)w_1''(h)) > 0,$$

for $h_+ \leq h \leq h_\natural$, since the curve $\mathcal{W}_1(U_L)$ is strictly convex. This implies that the function $\theta(h)$ is increasing. So,

$$\theta(h) \geq \theta(h_+) = \sqrt{gh_+}w_1'(h_+), \quad h_+ \leq h \leq h_\natural. \tag{3.6}$$

We will show that

$$w_1'(h_+) \geq -\sqrt{\frac{g}{h_+}}.$$

Indeed, since the function $w_1(h)$ is strictly convex, $w_1"(h) > 0$, so $w_1'(h)$ is increasing. That gives

$$w_1'(h) > w_1'(h'), \quad h > h'.$$

If $h < h_L$, then $w_1(h) = u_L - 2\sqrt{g}\Big(\sqrt{h} - \sqrt{h_L}\Big)$. Thus,

$$w_1'(h) = -\sqrt{\frac{g}{h}}, \quad h < h_L. \tag{3.7}$$

If $h \geq h_L$, then $w_1(h) = u_L - \sqrt{\frac{g}{2}}(h - h_L)\sqrt{\Big(\frac{1}{h} + \frac{1}{h_L}\Big)}$. Set

$$\Phi(h_L, h) = w_1(h) - u_L = -\sqrt{\frac{g}{2}}(h - h_L)\sqrt{\Big(\frac{1}{h} + \frac{1}{h_L}\Big)}.$$

Then, it holds that

$$\Phi(h_L, h) = -\Phi(h, h_L).$$

This yields

$$0 < -w_1'(h) = -\Phi_h(h_L, h) = \Phi_{h_L}(h, h_L),$$

so that

$$\Phi_{h_L}(h, h_L) \le \Phi_{h_L}(h, h) = \sqrt{\frac{g}{h}}, \quad h_L \le h.$$

The last inequality yields

$$w_1'(h) \ge -\sqrt{\frac{g}{h}}, \quad h \ge h_L. \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$w_1'(h) \ge -\sqrt{\frac{g}{h}}, \quad h_+ \le h \le h_\natural.$$

Substituting the last inequality into (3.6) for $h = h_+$, we get

$$\theta(h) := w_1(h)w_1'(h) \ge \theta(h_+) = -\sqrt{gh_+}\sqrt{\frac{g}{h_+}} = -g, \quad h_+ \le h \le h_\natural,$$

which establishes (3.5). Thus,

$$\begin{aligned} B &= \varphi_2(h)[hw_1^2(h) - g\varphi_2^2(h) + w_1(h)w_1'(h)(h^2 - \varphi_2^2(h))] \\ &< \varphi_2(h)[hw_1^2(h) - g\varphi_2^2(h) + g(\varphi_2^2(h) - h^2)] \\ &= \varphi_2(h)h(w_1^2(h) - gh) < 0, \end{aligned}$$

by (3.1) and $U_1(h) = (h, w_1(h)) \in G_2$. Thus, $A > 0, B < 0$ and therefore

$$\varphi_2'(h) = \frac{d}{dh}\varphi_2(U_1(h), a_R) > 0.$$

This terminates the proof of Lemma 3.1. □

For the u-component of the composite wave curve, we have the following result.

Lemma 3.2. *Consider the function $u(h) = w_1(U_L; h)h/\varphi_2(U_1(h), a_R)$, where $U_1(h) = (h, u = w_1(U_L; h)) \in \mathcal{W}_1(U_L) \cap G_2$.*

(a) If $U_L \in G_1 \cup \mathcal{C}_+$ and $a_L > a_R$, then the function $u(h)$ is strictly decreasing for $h \in [h_L^{\#}, h_-]$.

(b) If $U_L \in G_2^+$ and $a_L > a_R$, then the function $u(h)$ is strictly decreasing for $h \in [h_+, h_-]$.

Proof. First, consider the case $w_1(h) \geq 0$, or $h \in (h_L^{\#}, h_\natural)$, where $h_\natural$ is the h-intercept of the shock curve $\mathcal{W}_1(U_L)$. We will show that *if* $U_L \in G_1 \cup \mathcal{C}_+$, *then the function* $f_1(h) = w_1(h)h$ *is strictly decreasing for* $h \in [h_L^{\#}, h_-]$. Actually, we have

$$f_1'(h) = u_L - \frac{\sqrt{2g}\,(h_L h + 4h^2 - h_L^2)}{4hh_L\sqrt{\frac{1}{h} + \frac{1}{h_L}}}. \tag{3.9}$$

It holds that

$$f_1''(h) = -\frac{\sqrt{g/2}\,(8h^3 + 12h^2 h_L + 3hh_L + h_L^3)}{4h^3 h_L^3\left(\sqrt{\frac{1}{h} + \frac{1}{h_L}}\right)^3} \tag{3.10}$$

which is negative for $h > 0\dot{\text{S}}$o, the function f_1' in (3.10) is strictly decreasing on the interval $h \in (h_L^{\#}, h_\natural)$. The value $f_1'(h^{\#})$ can be evaluated as follows

$$f_1'\left(h_L^{\#}\right) = u_L - \frac{g}{8}\frac{h_L + 16u_L^2/g - 3\sqrt{h_L^2 + 8h_L u_L^2/g}}{u_L}.$$

We will show that $f_1'(h_L^{\#}) \leq 0$. Indeed, the condition

$$f_1'\left(h_L^{\#}\right) \leq 0$$

is equivalent to

$$u_L^2 \leq gh_L + 16u_L^2 - 3g\sqrt{h_L^2 + 8h_L u_L^2/g},$$

or

$$3g\sqrt{h_L^2 + 8h_L u_L^2/g} \leq \frac{g}{h_L}\left(h_L^2 + 8h_L u_L^2/g\right).$$

Simplifying the last inequality, we obtain

$${u_L}^2 \geq gh_L.$$

This shows that $f_1'(h) < 0, h_L^{\#} < h < h_\natural$ and so f_1 is strictly decreasing for $h_L^{\#} < h < h_\natural$.

Similarly, we will show that *if* $U_L \in G_2$, *then the function* $f_1(h) = w_1(h)h$ *is strictly decreasing for* $h \in [h_+, h_\natural]$, where $h_\natural$ is the h-intercept of the shock curve $\mathcal{W}_1(U_L)$. Indeed, if $U_L \in G_2$, then $w_1'(h_+) = -\sqrt{\frac{g}{h_+}}$, therefore

$$\begin{aligned} f_1'(h_+) &= w_1'(h_+)h_+ + w_1(h_+) \\ &= -\sqrt{\frac{g}{h_+}}h_+ + \sqrt{gh_+} = 0. \end{aligned}$$

Since $f_1''(h) < 0$, it implies that

$$f_1'(h) < 0, \quad h_+ < h < h_\natural.$$

Thus, f_1 is strictly decreasing function for $h_+ < h < h_\natural$.

In both cases $U_L \in G_1 \cup \mathcal{C}_+$ and $U_L \in G_2$,

$$u(h) = w_1(h)h\frac{1}{\varphi_2(U_1(h), h)} \tag{3.11}$$

is the product of two positive and strictly decreasing functions of $h_L^{\#} < h < h_\natural$ and $h_+ < h < h_\natural$, respectively, which is also strictly decreasing.

Second, we consider the case $w_1(h) < 0$. Observe that $u(h)$ satisfying the equation

$$a_L - a_R + \frac{1}{2g}(w_1^2(h) - u^2(h)) + h - \varphi_2(h) = 0.$$

Differentiating that equation with respect to h, we get

$$\frac{1}{g}(w_1(h)w_1'(h) - u(h)u'(h)) + 1 - \varphi_2'(h) = 0,$$

or

$$u(h)u'(h) = w_1(h)w_1'(h) + g - g\varphi_2'(h).$$

Multiplying both sides of the last equation by A and using (3.5), we get

$$\begin{aligned}
u(h)u'(h)A &= (w_1(h)w_1'(h) + g)A - gA\varphi_2'(h) \\
&= (w_1(h)w_1'(h) + g)A + gB \\
&= (w_1(h)w_1'(h) + g)(g\varphi_2^3(h) - h^2w_1^2(h)) \\
&\quad + g\varphi_2(h)[hw_1^2(h) - g\varphi_2^2(h) + w_1(h)w_1'(h)(h^2 - \varphi_2^2(h))] \\
&= w_1(h)w_1'(h)h^2(g\varphi_2(h) - w_1^2(h)) + ghw_1^2(h)(\varphi_2(h) - h) \\
&> 0,
\end{aligned}$$

since $w_1(h) < 0, \varphi_2(h) > h$ and $U_1(h) = (h, w_1(h)) \in G_2$. Since $A > 0$ and $u(h) = w_1(h)h/\varphi_2(h) < 0$, the last inequality implies that $u'(h) < 0$ by (3.11). This terminates the proof of Lemma 3.2. □

Lemmas 3.1 and 3.2 yield the monotone property of the composite wave curve $\mathcal{W}_{1\to3}(U_L)$ as in the following theorem.

Theorem 3.3. *If $U_L \in G_1 \cup \mathcal{C}_+$ and $a_L > a_R$, then the composite curve $\mathcal{W}_{1\to3}(U_L)$ can be parameterized by h-component in the form $u = \tilde{u}(h), h_L^{\#} \le h \le h_-$, where $u = \tilde{u}(h)$ is strictly decreasing function of $h_L^{\#} \le h \le h_-$.*

3.2. Case : $U_L \in G_2^+$ and $a_L > a_R$

From U_L, the solution begins with a 1-rarefaction wave to a state $U_+(h_+, u_+) = \mathcal{W}_1(U_L) \cap \mathcal{C}_+$. The solution is continued with a stationary wave from $U_+(a_L)$ to the state $U_+^1(a_R) \in G_1$ using $\varphi_1(U_+, a_R)$. There is one state $U_+^{1\#} \in \mathcal{W}_1(U_+^1)$ such that

$$\overline{\lambda}_1(U_+^1, U_+^{1\#}) = 0.$$

The solution is continued by a 1-wave from U_+^1 to state U such that $0 \le h \le h_+^{1\#}$. The set of U form the composite curve $\mathcal{W}_{1\to3\to1}(U_L)$. The composite curve $\mathcal{W}_{1\to3\to1}(U_L)$

is a part of the curve $\mathcal{W}_1(U_+^1)$. The curve $\mathcal{W}_1(U_+^1)$ intersects axis $\{h = 0\}$ at the point $I = \Big(0, u_{up} := u_+^1 + 2\sqrt{gh_+^1}\Big)$. So, I and $U_+^{1\#}$ are two endpoints of the curve $\mathcal{W}_{1\to3\to1}(U_L)$.

Again, let us denote by $U_\pm = (h_\pm, u_\pm)$ the state at which the wave $\mathcal{W}_1(U_L)$ intersects with the curves $\mathcal{C}_\pm$, respectively. Then, from U_L the Riemann solution can begin with a 1-wave to state $U \in \mathcal{W}_1(U_L) \cap G_2$. This 1-wave is followed by a stationary contact wave from state $U(a_L)$ to a state $U^0(a_R) \in G_2$ using $\varphi_2(U, a_R)$. The set of such a states U^0 forms a curve composite, denoted by $\mathcal{W}_{1\to3}(U_L)$. Let U_+^0 and U_-^0 be the two endpoints of this curve, i.e.,

$$U_\pm^0 = (h_\pm^0, u_\pm^0) = (\varphi_2(U_\pm, a_R), w_3(U_\pm, \varphi_2(U_\pm, a_R)))\,. \tag{3.12}$$

Besides, each level $a \in [a_R, a_L]$, from U_L the Riemann solution can begin with the 1-rarefaction wave to $U_+ = \mathcal{W}_1(U_L) \cap \mathcal{C}_+$. This rarefaction wave can be followed by a stationary contact wave from $U_+(a_L)$ to a state $M = M(a) \in G_1 \cup \mathcal{C}_+$ using $\varphi_1(U_+, a)$. The solution is continued with the 1-shock wave from the state M to the state $M^\# \in \mathcal{S}_1(M) \cap (G_2^+ \cup \mathcal{C}_+)$ such that

$$\overline{\lambda}_1(M, M^\#) = 0.$$

Then, the solution is continued with the stationary contact wave from the state $M^\#(a)$ to the state $M^{\#0}(a_R) \in G_2$ using $\varphi_2(M^\#, a_R)$. The set of $M^{\#0}$ forms a curve composite, denoted by $\mathcal{W}_{1\to3\to1\to3}(U_L)$. If $a = a_R$, then $M = U_+^1$ and $M^\# = M^{\#0} = U_+^{1\#}$. If $a = a_L$, then $M = M^\# = U_+$ and $M^{\#0} = U_+^0$. So, $U_+^{1\#}$ and U_+^0 are two endpoints of the curve composite $\mathcal{W}_{1\to3\to1\to3}(U_L)$.

Let us define the composite wave curve

$$\Lambda(U_L) = \mathcal{W}_{1\to3\to1}(U_L) \cup \mathcal{W}_{1\to3}(U_L) \cup \mathcal{W}_{1\to3\to1\to3}(U_L). \tag{3.13}$$

Note that I and U_-^0 are the two endpoints of the curve $\Lambda(U_L)$.

Lemmas 3.1 and 3.2 provide us with the monotone property of the composite wave curve $\mathcal{W}_{1\to3}(U_L)$, as seen in the following theorem.

Theorem 3.4. *If $U_L \in G_2^+$ and $a_L > a_R$, then the composite curve $\mathcal{W}_{1\to3}(U_L)$ can be parameterized by h-component in the form $u = \tilde{u}(h), h_+ \le h \le h_-$, where $u = \tilde{u}(h)$ is strictly decreasing function of $h_+ \le h \le h_-$.*

We have studied the curves of composite waves above in the case $a_R < a_L$ above, where composite waves are found in the form of combinations of 1-waves and stationary waves. When $a_R > a_L$, the construction will be relied on the backward wave curve $\mathcal{W}_2^B(U_R)$. So, composite waves will be found as a combination of stationary waves and backward 2-waves. The curve of these composite waves will be investigated below.

3.3. Case : $U_R \in G_1 \cup \mathcal{C}_+ \cup G_2^+$ and $a_R > a_L$

At the right-hand state U_R, the Riemann solution can arrive by a 2-wave from a state $U \in \mathcal{W}_2^B(U_R) \cap (G_2 \cap \mathcal{C}_\pm)$. Then, the Riemann solution is proceeded by a stationary wave to $U(a_R)$ from $U^0(a_L) \in G_2$ using $\varphi_2(U, a_L)$. The set of such states U^0 forms the

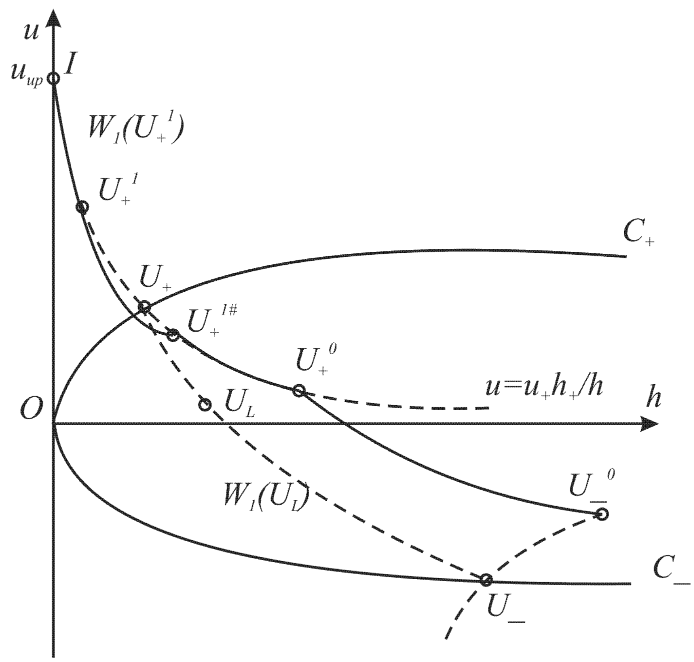

Figure 2: The composite curve $\Lambda(U_L)$.

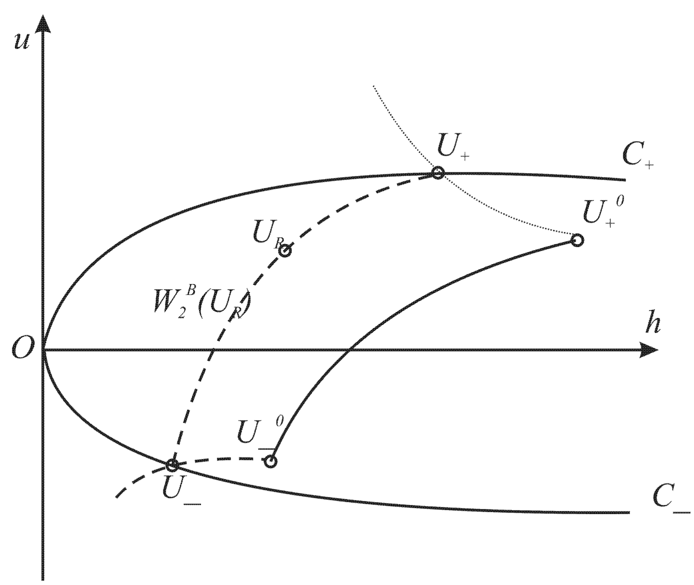

Figure 3: The composite curve $\mathcal{W}^B_{2\leftarrow 3}(U_R)$.

backward composite curve $\mathcal{W}^B_{2\leftarrow 3}(U_R)$. Let U^0_+ and U^0_- denote the two end-points of this curve, where

$$\begin{aligned} U_\pm &= (h_\pm, u_\pm) = \mathcal{W}^B_2(U_R) \cap \mathcal{C}_\pm, \\ U^0_\pm &= (h^0_\pm, u^0_\pm) = (\varphi_2(U_\pm, a_L), w_3(U_\pm, \varphi_2(U_\pm, a_L)))\,. \end{aligned} \tag{3.14}$$

Similar to Lemma 3.1, we have following lemma.

Lemma 3.5. *Let $U_R \in G_1 \cup \mathcal{C}_+ \cup G_2^+$, $a_R > a_L$, and let $U_2(h) = (h, u = w^B_2(U_R; h)) \in \mathcal{W}^B_2(U_L) \cap G_2$, where w^B_2 is defined by (2.1) and $(h_\pm, u_\pm) = \mathcal{W}_1(U_L) \cap \mathcal{C}_\pm$. Then, the function $\varphi_2(h) = \varphi_2(U_2(h), a_L), h \in [h_-, h_+]$ is strictly increasing.*

Proof. The function $\varphi_2(h)$ satisfies the equation

$$G(h) := F(h; U_1(h)) = 2g\varphi_2^3(h) + (2g(a_L - a_R - h) - w_2^{B^2}(h))\varphi_2(h)^2 + h^2 w_2^{B^2}(h) = 0. \tag{3.15}$$

Therefore,

$$0 = \varphi_2'(h)A + B, \tag{3.16}$$

where

$$\begin{aligned} A &= \varphi_2^2(h)[3g\varphi_2(h) + 2g(a_L - a_R - h) - w_2^{B^2}(h)], \\ B &= \varphi_2(h)[(hw_2^{B^2}(h) - g\varphi_2^2(h)) + w_2^B(h)w_2^{B'}(h)(h^2 - \varphi_2^2(h))]. \end{aligned}$$

To establish the monotony of φ_2, we will show that

$$A > 0, \quad B < 0.$$

Indeed, it holds that

$$\begin{aligned} A &= 3g\varphi_2^3(h) - (2g\varphi_2^3(h) + h^2 w_2^{B^2}(h)) \\ &= g\varphi_2^3(h) - h^2 w_2^{B^2}(h) \\ &> gh^3 - h^2 w_2^{B^2}(h) = h^2(gh - w_2^{B^2}(h)) \geq 0, \end{aligned}$$

since $U_2(h) = (h, w^B_2(h)) \in G_2$.

Next, to prove that $B < 0$, we need to show that

$$\theta(h) := w_2^B(h)w_2^{B'}(h) \geq -g, \quad h_- \leq h \leq h_+. \tag{3.17}$$

Indeed, if $w^B_2(h) > 0$, then

$$\theta(h) = w_2^B(h)w_2^{B'}(h) > 0 > -g.$$

So, we remain to prove (3.17) for the case $w^B_2(h) \leq 0$. This means that $h_- \leq h \leq h_\natural$, where $h_\natural$ is the h-intercept of the backward curve $\mathcal{W}^B_2(U_R)$. It holds that

$$\frac{d}{dh}\left(w_2^B(h)w_2^{B'}(h)\right) = w_2^{B'2} + w_2^B(h)w_2^{B''}(h) > 0,$$

for $h_- \leq h \leq h_\natural$, since the curve $\mathcal{W}^B_2(U_R)$ is strictly concave. This implies that the function $\theta(h)$ is increasing. So,

$$\theta(h) \geq \theta(h_-) = -\sqrt{gh_-}w_2^{B'}(h_-) = -\sqrt{gh_-}\sqrt{\frac{g}{h_-}} = -g, \quad h_- \leq h \leq h_\natural. \tag{3.18}$$

which establishes (3.17). Thus,

$$\begin{aligned} B &= \varphi_2(h)[hw_2^{B^2}(h) - g\varphi_2^2(h) + w_2^B(h)w_2^{B'}(h)(h^2 - \varphi_2^2(h))] \\ &< \varphi_2(h)[hw_2^{B^2}(h) - g\varphi_2^2(h) + g(\varphi_2^2(h) - h^2)] \\ &= \varphi_2(h)h(w_2^{B^2}(h) - gh) < 0, \end{aligned}$$

by (3.17) and $U_2(h) = (h, w_2^B(h)) \in G_2$. Thus, $A > 0, B < 0$ and therefore

$$\varphi_2'(h) = \frac{d}{dh}\varphi_2(U_2(h), a_L) > 0.$$

This terminates the proof of Lemma 3.5. □

Lemma 3.6. *If $U_R \in G_1 \cup \mathcal{C}_+ \cup G_2^+$ and $a_R > a_L$, the function $u(h) = w_2^B(h)h/\varphi_2(U_2(h), a_L)$ is strictly increasing with respect to $h \in [h_-, h_+]$, where $U_2(h) = (h, u = w_2^B(U_R; h)) \in \mathcal{W}_2^B(U_R) \cap G_2$ and w_2^B is defined by (2.1).*

Proof. First, consider the case $w_2^B(h) \le 0$. We will show that *the function $f_2(h) = w_2^B(h)h$ is strictly increasing with respect to $h \in [h_-, h_\natural]$*. Actually, since $w_2^B(h) = u_R + 2\sqrt{g}\Big(\sqrt{h} - \sqrt{h_R}\Big)$, we have

$$f_2'(h) = w_2^{B'}(h)h + w_2^B(h) = \sqrt{\frac{g}{h}}h + w_2^B(h) = \sqrt{gh} + w_2^B(h). \tag{3.19}$$

It holds that

$$f_2''(h) = \frac{1}{2}\sqrt{\frac{g}{h}} + \sqrt{\frac{g}{h}} = \frac{3}{2}\sqrt{\frac{g}{h}},$$

which is positive for $h > 0$. So, the function f_2' is strictly increasing on the interval $h \in (h_-, h_\natural)$, where $h_\natural$ is the h-intercept of the backward curve $\mathcal{W}_2^B(U_R)$. Moreover,

$$f_2'(h_-) = \sqrt{gh_-} + w_2^B(h_-) = \sqrt{gh_-} - \sqrt{gh_-} = 0.$$

From (3.19), the last inequality shows that $f_2'(h) > 0, h_- < h < h_\natural$ and so f_2 is strictly increasing for $h_- \le h \le h_\natural$. Let us now consider the function

$$u(h) = \frac{w_2^B(h)h}{\varphi_2(U_1(h), a_L)} = \frac{f_2(h)}{\varphi_2(h)}, \quad h_- \le h \le h_\natural. \tag{3.20}$$

It holds that

$$u'(h) = \frac{f_2'(h)\varphi_2(h) - f_2(h)\varphi_2'(h)}{\varphi_2^2(h)} > 0,$$

since $\varphi_2'(h) > 0$, $f_2'(h) > 0$ and $f_2(h) < 0$.

Next, we consider the case $w_2^B(h) > 0$. Observe that $u(h)$ satisfying the equation

$$a_R - a_L + \frac{1}{2g}(w_2^{B^2}(h) - u^2(h)) + h - \varphi_2(h) = 0.$$

Differentiating that equation with respect to h, we get

$$\frac{1}{g}(w_2^B(h)w_2^{B\prime}(h) - u(h)u'(h)) + 1 - \varphi_2'(h) = 0,$$

or

$$u(h)u'(h) = w_2^B(h)w_2^{B\prime}(h) + g - g\varphi_2'(h).$$

Multiplying both sides of the last equation by A and using (3.17), we get

$$\begin{aligned}
u(h)u'(h)A &= (w_2^B(h)w_2^{B\prime}(h) + g)A - gA\varphi_2'(h) \\
&= (w_2^B(h)w_2^{B\prime}(h) + g)A + gB \\
&= (w_2^B(h)w_2^{B\prime}(h) + g)(g\varphi_2^3(h) - h^2 w_2^{B^2}(h)) \\
&+ g\varphi_2(h)[hw_2^{B^2}(h) - g\varphi_2^2(h) + w_2^B(h)w_2^{B\prime}(h)(h^2 - \varphi_2^2(h))] \\
&= w_2^B(h)w_2^{B\prime}(h)h^2(g\varphi_2(h) - w_2^{B^2}(h)) + ghw_2^{B^2}(h)(\varphi_2(h) - h) \\
&> 0,
\end{aligned}$$

since $w_2^B(h) > 0, \varphi_2(h) > h$ and $U_2(h) = (h, w_2^B(h)) \in G_2$. Since $A > 0$ and $u(h) = w_2^B(h)h/\varphi_2(h) > 0$ by (3.20), the last inequality implies that $u'(h) > 0$. This terminates the proof of Lemma 3.6. □

From Lemmas 3.5 and 3.6, we obtain the monotonicity of the composite wave curve $\mathcal{W}_{2\leftarrow 3}^B(U_R)$ as follows.

Theorem 3.7. *If $U_R \in G_1 \cup \mathcal{C}_+ \cup G_2^+$ and $a_R > a_L$, then the composite curve $\mathcal{W}_{2\leftarrow 3}^B(U_R)$ can be parameterized by h-component in the form $u = \tilde{u}(h), h_- \le h \le h_+$, where $u = \tilde{u}(h)$ is strictly increasing function of $h_- \le h \le h_+$.*

4. Deterministic existence for the Riemann problem

4.1. Case : $U_L \in G_1 \cup \mathcal{C}_+$ and $a_L > a_R$

Let us denoted by $J = \big(0, w_2^B(U_R, 0)\big)$ the intersection point of the curve $\mathcal{W}_2^B(U_R)$: $u = w_2^B(U_R, h)$ and the u-axis. If $w_2^B(U_R, 0) < u_{up}$, then $I = (0, u_{up})$ is located above the curve $\mathcal{W}_2^B(U_R)$. If $u_-^0 < w_2^B(U_R, h_-^0)$ then $U_-^0 = (h_-^0, u_-^0)$ is located below the curve $\mathcal{W}_2^B(U_R)$. Whenever these two conditions are met, the backward wave curve $\mathcal{W}_2^B(U_R)$ intersects the composite curve $\Gamma(U_L)$. This leads to the existence of solutions of the Riemann problem, as stated in the following theorem.

Theorem 4.1. *Let w_2^B and w_3 be given by (2.1) and (2.13), respectively. Assume that the left-hand state $U_L \in G_1 \cup \mathcal{C}_+$, $a_L > a_R$, and the right-hand state U_R satisfies*

$$\begin{aligned}
&w_2^B(U_R, 0) < u_{up}, \\
&w_2^B(U_R, h_-^0) > u_-^0,
\end{aligned} \tag{4.1}$$

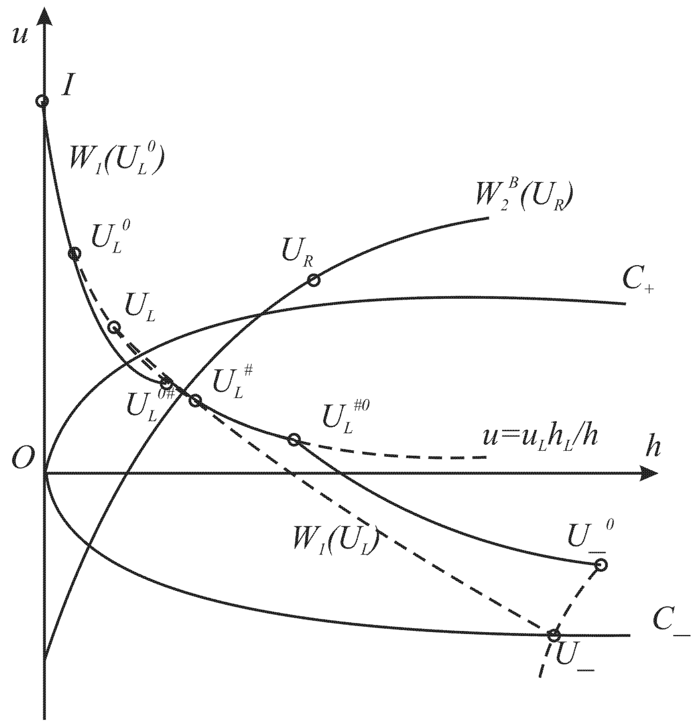

Figure 4: The composite curve $\Gamma(U_L)$ intersects the backward curve $\mathcal{W}_2^B(U_R)$.

where

$$
\begin{aligned}
&u_{up} = u_L^0 + 2\sqrt{gh_L^0},\\
&U_L^0 = (h_L^0, u_L^0) = (\varphi_1(U_L, a_R), w_3(U_L, \varphi_1(U_L, a_R))),\\
&U_-(h_-, u_-) = \mathcal{W}_1(U_L) \cap \mathcal{C}_-,\\
&U_-^0 = (h_-^0, u_-^0) = (\varphi_2(U_-, a_R), w_3(U_-, \varphi_2(U_-, a_R))).
\end{aligned}
\tag{4.2}
$$

Then, the Riemann problem (1.1)-(1.2) has a solution.

Proof. Let the curve $\Gamma(U_L)$ be parameterized as

$$h = h(m), \quad u = u(m),$$

for m in some interval I. Since I, U_-^0 are the two-endpoints of $\Gamma(U_L)$, we can assume without loss of generality that $I = [m_1, m_2]$, where the values m_1, m_2 are such that

$$I = (h(m_1), u(m_1)), \quad \text{and} \quad U_-^0 = (h(m_2), u(m_2)).$$

We define a function

$$G(m) := u(m) - w_2^B(U_R, h(m)), \quad m \in I.$$

it is easy to see that the function $G(m)$ is continuous on $I = [m_1, m_2]$. Moreover, it holds that

$$
\begin{aligned}
G(m_1) &= u(m_1) - w_2^B(U_R, h(m_1))\\
&= u_{up} - w_2^B(U_R, 0) > 0,
\end{aligned}
$$

and

$$\begin{aligned} G(m_2) &= u(m_2) - w_2^B(U_R, h(m_2)) \\ &= u_-^0 - w_2^B(U_R, h_-^0) < 0. \end{aligned}$$

Therefore, there exists a value $m_0 \in (m_1, m_2)$ such that $G(m_0) = 0$, by Intermediate-Value Theorem. This means that the curve $\Gamma(U_L)$ intersects the curve $\mathcal{W}_2^B(U_R)$ at a point corresponding to m_0, see Figure 4. First, if the curve $\Gamma(U_L)$ intersects the part $\mathcal{W}_{3\to1}(U_L)$, then the Riemann problem has a solution of the form

$$W_3(U_L, U_L^0) \oplus W_1(U_L^0, U) \oplus W_2(U, U_R).$$

Second, if the curve $\Gamma(U_L)$ intersects the part $\mathcal{W}_{3\to1\to3}(U_L)$, then the Riemann problem has a solution of the form

$$W_3(U_L, B) \oplus W_1(B, B^{\#}) \oplus W_3(B^{\#}, B^{\#0}) \oplus W_2(B^{\#0}, U_R).$$

Finally, if the curve $\Gamma(U_L)$ intersects the part $\mathcal{W}_{1\to3}(U_L)$, then the Riemann problem has a solution of the form

$$W_1(U_L, A) \oplus W_3(A, A^0) \oplus W_2(A^0, U_R).$$

This completes the proof of Theorem 4.1. □

Consider the forward wave curve $\mathcal{W}_2(U_0)$. A point is located above or below the curve $\mathcal{W}_2(U_0)$ can be characterized as in the following lemma.

Lemma 4.2. *Given two states $U_i = (h_i, u_i), i = 1, 2$. Then, $u_1 < w_2^B(U_2, h_1)$ if and only if $u_2 > w_2(U_1, h_2)$. This means that a state U_1 is located below the curve $\mathcal{W}_2^B(U_2)$ if and only if U_2 is located above the curve $\mathcal{W}_2(U_1)$ in the (h, u)-plane.*

Proof. The state U_1 is located below the curve $\mathcal{W}_2^B(U_2)$ means that

$$u_1 < w_2^B\left(U_2, h_1\right),$$

or

$$u_1 < \begin{cases} u_2 + 2\sqrt{g}\left(\sqrt{h_1} - \sqrt{h_2}\right), & h_1 \le h_2, \\ u_2 + \sqrt{\frac{g}{2}}\left(h_1 - h_2\right)\sqrt{\frac{1}{h_1} + \frac{1}{h_2}}, & h_1 > h_2. \end{cases}$$

The last inequalities can be equivalently rewritten as

$$u_2 > \begin{cases} u_1 + 2\sqrt{g}\left(\sqrt{h_2} - \sqrt{h_1}\right), & h_2 \ge h_1, \\ u_1 + \sqrt{\frac{g}{2}}\left(h_2 - h_1\right)\sqrt{\frac{1}{h_2} + \frac{1}{h_1}}, & h_2 < h_1, \end{cases}$$

which means that

$$u_2 > w_2(U_1, h_2).$$

This completes the proof of Lemma 4.2. □

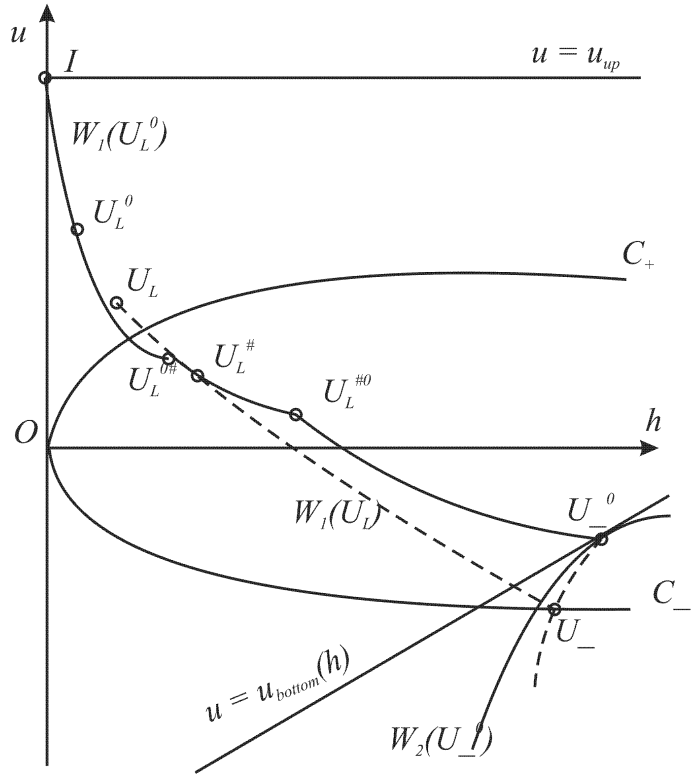

Figure 5: The triangle T.

As seen by Theorem 4.1, a large neighborhood of the left-hand state U_L in which the right-hand state U_R can be chosen is obtained for the existence. It is interesting that this region contains also large regions with simpler geometry such as triangles and rectangles. This enables us to see more clearly the existence domain. Let us now describe these regions. Since the curve $\mathcal{W}_2(U_-^0)$ is strictly concave, then the tangent d of the curve $\mathcal{W}_2(U_-^0)$ at U_-^0 is always above the curve $\mathcal{W}_2(U_-^0)$. The tangent d is given by

$$u = u_{bottom}(h) := u_-^0 + \sqrt{\frac{g}{h_-^0}}(h - h_-^0). \tag{4.3}$$

The tangent d intersects the line $u = u_{up}$, where u_{up} is given by (4.2), in the (h, u)-plane at a point with

$$h = h_{end} := h_-^0 + \sqrt{\frac{h_-^0}{g}}\left(u_{up} - u_-^0\right). \tag{4.4}$$

We define a triangle T as follows

$$T = \{(h, u) |\ 0 < h < h_{end}, u_{bottom}(h) < u < u_{up}\}, \tag{4.5}$$

see Figure 5.

Corollary 4.3. *Assume that $U_L \in G_1 \cup \mathcal{C}_+$, $a_L > a_R$. Then, $U_L \in T$ defined by (4.5), and whenever $U_R \in T$, the Riemann problem (1.1)-(1.2) has a solution.*

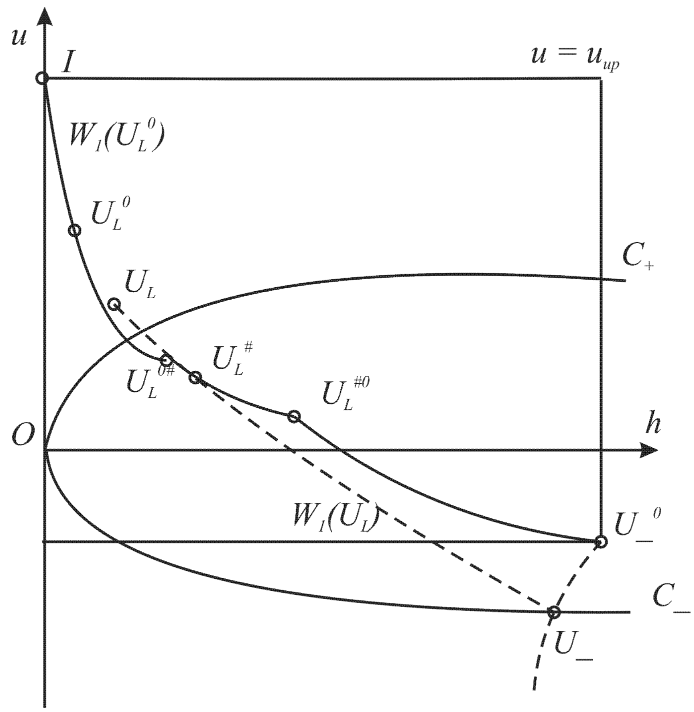

Figure 6: The rectangle R.

Proof. Since $U_R \in T$, then U_R is located above the tangent d, therefore, above the forward curve $\mathcal{W}_2(U_-^0)$. According to lemma 4.2, U_-^0 is located below the backward curve $\mathcal{W}_2^B(U_R)$, therefore, $u_-^0 < w_2^B(U_R, h_-^0)$. Furthermore, $w_2^B(0) = u_R - 2\sqrt{gh_R} < u_R < u_{up}$. According to theorem 4.1, the Riemann problem has a solution.
Now, we proof $U_L \in T$. Since $w_1(U_L, h)$ is strictly decreasing function and $U_L \in G_1, U_- \in G_2$, then

$$0 < h_L < h_- < \varphi_2(U_-, a_R) = h_-^0 < h_{end}.$$

Other hand, since $h_L^0 = \varphi_1(U_L, a_R) < h_L$, then

$$u_{bottom}(h_L) = u_-^0 + \sqrt{\frac{g}{h_-^0}}(h_L - h_-^0) < 0 < u_L < \frac{u_L h_L}{h_L^0} = u_L^0 < u_L^0 + 2\sqrt{gh_L^0} = u_{up}.$$

Thus, $U_L \in T$. □

We define the rectangle R as follows

$$R = \{(h, u)|\ 0 < h < h_-^0, u_-^0 < u < u_{up}\}, \tag{4.6}$$

see Figure 6. Then, it holds for every $(h, u) \in R$ that

$$0 < h < h_-^0 < h_{end},$$
$$u_{bottom}(h) = u_-^0 + \sqrt{\frac{g}{h_-^0}}(h - h_-^0) < u_-^0 < u < u_{up}.$$

This implies that $R \subset T$. Moreover, we can check easily that $U_L \in R$. This establishes the following corollary.

Corollary 4.4. *Consider the case where the left-hand state $U_L \in G_1 \cup \mathcal{C}_+$, $a_L > a_R$. Then, $U_L \in R$ defined by (4.6). Whenever $U_R \in R$, i.e,*

$$
\begin{aligned}
&0 < h_R < h_-^0,\\
&u_-^0 < u_R < u_{up},
\end{aligned}
$$

the Riemann problem (1.1)-(1.2) admits a solution.

From the proof of Theorem 4.1, the monotone property of the wave curves and the monotonicity of the curves of composite waves as seen in Section 3, we obtain the following results on the existence and uniqueness of the Riemann solutions.

Corollary 4.5. *Assume that $U_L \in G_1 \cup \mathcal{C}_+$, $a_L > a_R$, and $U_L^\# \in \mathcal{W}_1(U_L) \cap (G_2^+ \cup \mathcal{C}_+)$ is the state such that $\overline{\lambda}_1(U_L, U_L^\#) = 0$. Let $U_L^{\#0}$ be the state obtained from $U_L^\#$ by an admissible stationary jump from the level a_L to a_R, i.e.,*

$$U_L^{\#0} = (h_L^{\#0}, u_L^{\#0}) = \Big(\varphi_2(U_L^\#, a_R), w_3(U_L^\#, \varphi_2(U_L^\#, a_R))\Big).$$

(a) If the right-hand state U_R satisfies

$$
\begin{aligned}
&w_2^B(U_R, h_L^{\#0}) < u_L^{\#0},\\
&w_2^B(h_-^0) > u_-^0,
\end{aligned}
$$

the Riemann problem for (1.1)-(1.2) has a unique solution form

$$W_1(U_L, A) \oplus W_3(A, A^0) \oplus W_2(A^0, U_R).$$

(b) If the right-hand state U_R satisfies

$$
\begin{aligned}
&w_2^B(U_R, 0) < u_{up},\\
&w_2^B(h_L^{0\#}) > u_L^{0\#},
\end{aligned}
$$

where u_{up} is defined by (4.2), then, the Riemann problem for (1.1)-(1.2) has a unique solution form

$$W_3(U_L, U_L^0) \oplus W_1(U_L^0, U) \oplus W_2(U, U_R).$$

Corollary 4.6. *Assume that $U_L \in G_1 \cup \mathcal{C}_+$, $a_L > a_R$.*

(a) If the right-hand state U_R satisfies

$$
\begin{aligned}
&h_L^{\#0} < h_R < h_-^0,\\
&u_-^0 < u_R < u_L^{\#0},
\end{aligned}
$$

where $U_- \in \mathcal{W}_1(U_L) \cap \mathcal{C}_-$, and $U_-^0 = (h_-^0, u_-^0) \in G_2$ is the state obtained by a stationary jump from U_-, then, the Riemann problem for (1.1)-(1.2) has a unique solution form

$$W_1(U_L, A) \oplus W_3(A, A^0) \oplus W_2(A^0, U_R).$$

(b) If the right-hand state U_R satisfies

$$\begin{aligned} 0 &< h_R < h_L^{0\#}, \\ u_L^{0\#} &< u_R < u_{up}, \end{aligned}$$

where u_{up} is defined by (4.2), then, the Riemann problem for (1.1)-(1.2) has a unique solution form

$$W_3(U_L, U_L^0) \oplus W_1(U_L^0, U) \oplus W_2(U, U_R).$$

Proof. (a) Since $h_L^{\#0} < h_R$, then

$$u_2(h_L^{\#0}) = u_R + 2\sqrt{g}(h_L^{\#0} - h_R) < u_R < u_L^{\#0},$$

Moreover, since $h_R < h_-^0$, then

$$u_2(h_-^0) = u_R + \sqrt{\frac{g}{2}}(h_-^0 - h_R)\sqrt{\frac{1}{h_-^0} + \frac{1}{h_R}} > u_R > u_-^0.$$

According to corollary corollary 4.5, the Riemann problem for (1.1)-(1.2) has a unique solution form

$$W_1(U_L, A) \oplus W_3(A, A^0) \oplus W_2(A^0, U_R).$$

(b) Similar to (a). □

4.2. Case : $U_L \in G_2^+$ and $a_L > a_R$

The above argument gives us the following theorem, whose proof is omitted, since the proof is similar to the one of Theorem 4.1. The illustration for the intersection of the curve of composite waves $\Lambda(U_L)$ with the backward curve $\mathcal{W}_2^B(U_R)$ is given by Figure 7.

Theorem 4.7. *Let w_2^B and w_3 be given by (2.1) and (2.13), respectively. Assume that the left-hand state $U_L \in G_2^+$, $a_L > a_R$, and the right-hand state U_R satisfies*

$$\begin{aligned} w_2^B(U_R, 0) &< u_{up}, \\ w_2^B(U_R, h_-^0) &> u_-^0, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} U_\pm &= (h_\pm, u_\pm) = \mathcal{W}_1(U_L) \cap \mathcal{C}_\pm, \\ u_{up} &= u_+^1 + 2\sqrt{g h_+^1}, \\ U_+^1 &= (h_+^1, u_+^1) = (\varphi_1(U_+, a_R), w_3(U_+, \varphi_1(U_+, a_R))), \\ U_-^0 &= (h_-^0, u_-^0) = (\varphi_2(U_-, a_R), w_3(U_-, \varphi_2(U_-, a_R))). \end{aligned} \tag{4.8}$$

Then, the Riemann problem (1.1)-(1.2) admits a solution.

From Theorem 4.7 and the monotonicity of the curves $\mathcal{W}_{1\to 3}(U_L)$, $\mathcal{W}_{1\to 3\to 1}(U_L)$, $\mathcal{W}_2^B(U_R)$, we deduce the following results.

Corollary 4.8. *Assume that $U_L \in G_2^+$, $a_L > a_R$.*

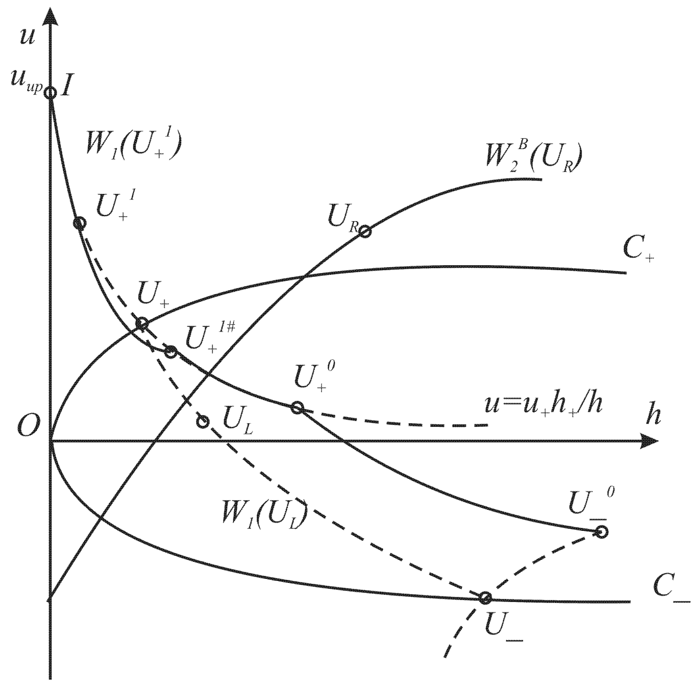

Figure 7: The composite curve $\Lambda(U_L)$ intersects the backward curve $\mathcal{W}_2^B(U_R)$.

(a) *If the right-hand state* U_R *satisfies*

$$w_2^B(U_R, h_+^0) < u_+^0,$$
$$w_2^B(U_R, h_-^0) > u_-^0,$$

where $U_\pm^0 = (h_\pm^0, u_\pm^0)$ *is given by (4.8), then, the Riemann problem has a unique solution form*

$$W_1(U_L, U) \oplus W_3(U, U^0) \oplus W_2(U^0, U_R).$$

(b) *Let* U_+^1 *and* u_{up} *be given by (4.8), and let the right-hand state* U_R *satisfy*

$$w_2^B(U_R, 0) < u_{up},$$
$$w_2^B(h_+^{1\#}) > u_+^{1\#},$$

where $U_+^{1\#} \in \mathcal{W}_1(U_+^1)$ *is the state such that* $\overline{\lambda}_1(U_+^1, U_+^{1\#}) = 0$. *Then, the Riemann problem has a unique solution of the form*

$$W_1(U_L, U_+) \oplus W_3(U_+, U_+^1) \oplus W_1(U_+^1, U) \oplus W_2(U, U_R),$$

Corollary 4.9. *Assume that* $U_L \in G_2^+$, $a_L > a_R$. *Using the same notations as in Corollary 4.8.*

(a) *If the right-hand state* U_R *satisfies*

$$h_+^0 < h_R < h_-^0,$$
$$u_-^0 < u_R < u_+^0,$$

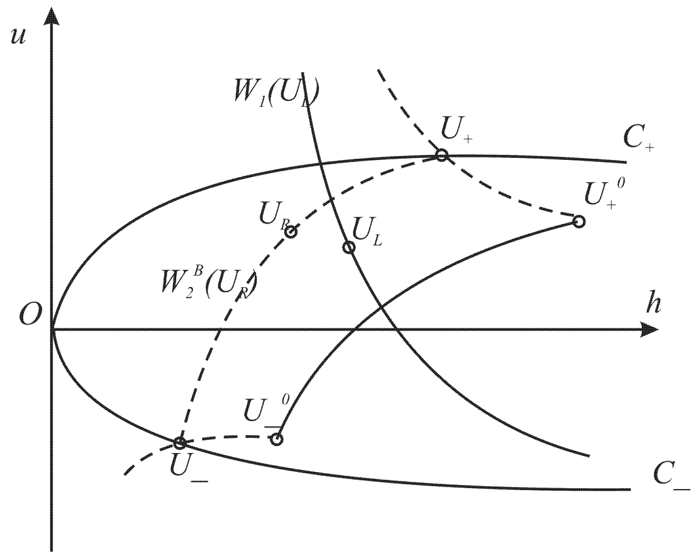

Figure 8: The composite curve $\mathcal{W}^B_{2\leftarrow 3}(U_R)$ intersects the curve $\mathcal{W}_1(U_L)$.

the Riemann problem has a unique solution form

$$W_1(U_L, U) \oplus W_3(U, U^0) \oplus W_2(U^0, U_R).$$

(b) If the right-hand state U_R satisfies

$$\begin{aligned} &0 < h_R < h_+^{1\#}, \\ &u_+^{1\#} < u_R < u_{up}, \end{aligned}$$

then, the Riemann problem has a unique solution form

$$W_1(U_L, U_+) \oplus W_3(U_+, U_+^1) \oplus W_1(U_+^1, U) \oplus W_2(U, U_R),$$

4.3. Case : $U_R \in G_1 \cup \mathcal{C}_+ \cup G_2^+$ and $a_R > a_L$

As seen before, the curve of 1-waves $\mathcal{W}_1(U_L)$ is strictly decreasing. Moreover, Lemma 3.6 also provides us with the monotonicity property of the backward curve of composite waves $\mathcal{W}^B_{2\leftarrow 3}(U_R)$. Thus, we can arrive at the following theorem.

Theorem 4.10. *Let w_1 and w_3 be defined as in (2.1) and (2.13), respectively. Assume that $U_R \in G_1 \cup \mathcal{C}_+ \cup G_2^+$, $a_R > a_L$, and the left-hand state U_L satisfies*

$$\begin{aligned} &w_1(U_L, h_+^0) < u_+^0, \\ &w_1(U_L, h_-^0) > u_-^0, \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} &U_\pm = (h_\pm, u_\pm) = \mathcal{W}_2^B(U_R) \cap \mathcal{C}_\pm, \\ &U_\pm^0 = (h_\pm^0, u_\pm^0) = (\varphi_2(U_\pm, a_L), w_3(U_\pm, \varphi_2(U_\pm, a_L)))\,. \end{aligned} \tag{4.10}$$

Then, the Riemann problem has a unique solution form

$$W_1(U_L, U) \oplus W_3(U, U^0) \oplus W_2(U^0, U_R).$$

The illustration for the intersection of the curve of composite waves $\mathcal{W}^B_{2\leftarrow 3}(U_R)$ and the curve $\mathcal{W}_1(U_L)$ is given by Figure 8.

Corollary 4.11. *Assume that $U_R \in G_1 \cup \mathcal{C}_+ \cup G_2^+$, $a_R > a_L$, and the left-hand state U_L satisfies*

$$h^0_- < h_L < h^0_+$$
$$u^0_- < u_L < u^0_+,$$

where $U^0_\pm = (h^0_\pm, u^0_\pm)$ is given by (4.10). Then, the Riemann problem has a unique solution form

$$W_1(U_L, U) \oplus W_3(U, U^0) \oplus W_2(U^0, U_R).$$

5. Conclusions

Hyperbolic systems of balance laws in nonconservative form possess very interesting but rather complicated phenomena. In particular, characteristic speeds may coincide and the order in Riemann solutions of elementary waves such as shocks, rarefaction waves, and stationary contacts may be changed from one region to another. This can be dealt with when solving the Riemann problem by building up the curves of composite waves. In this paper we first establish the monotonicity property of these curves of composite waves. Hence, together with earlier works, the monotonicity property of all the wave curves involved in the solving of the Riemann problem for the shallow water equations with discontinuous topography are observed. Second, we determine *explicitly* the domains for the existence and the uniqueness of the Riemann problem.

The future works in this direction may be involved in the monotonicity properties of the composite wave curves in the more complicated models of hyperbolic systems of balance laws in nonconservative form such as the model of fluid flows in a nozzle with variable cross-section, multi-phase flow models, and the use of these results in the Godunov-type numerical methods for these models, etc.

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