ON THE INVARIANT SUBSPACE PROBLEM

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Abstract. In an attempt to solve the Invariant Subspace Problem, we introduce a certain orthonormal basis of Hilbert spaces, and prove that a bounded linear operator on a Hilbert space must have an invariant subspace once this basis fulfills certain conditions. Ultimately, this basis is used to show that every bounded linear operator on a Hilbert space is the sum of a shift and an upper triangular operators, each of which having an invariant subspace.

1. Introduction

Let H be a complex Hilbert space, and $L(H, H)$ be the space of all bounded linear operators acting on H. A closed subspace W of H is called invariant or T-invariant, of $T \in L(H, H)$, if $T(W) \subset W$. The invariant subspace problem is the long standing simple question:"Does every bounded linear operator $T \in L(H, H)$ have a non-trivial invariant subspace?". Non-trivial subspace means a closed subspace different from $\{0\}$ and H. This problem is open for more than half a century, and mathematicians have been trying to attack this problem producing many significant contributions with a huge variety of techniques, making this challenging problem a target. However the solution seems to be nowhere in sight. It is unknown who first posed this question. It appeared in 1949-1950, years after Von Neumann's unpublished result (1930's) , in which he proved that every compact operator T on a Hilbert space has nontrivial invariant subspace.

In this paper we shall discuss, briefly, the most popular known results, and introduce our new technique toward solving this problem.

It is clear that if H is finite-dimensional then every $T \in L(H, H)$ has an eigenvalue, say λ, which yields an invariant subspace of T, namely the kernel of $T - \lambda I$. So the problem is solved for finite dimensional case. Now, Suppose H is infinite-dimensional but not separable. It is easy to see that for any $T \in L(H, H)$ and any non-zero vector $x \in H$, the closed subspace generated by $\{x, Tx, T^2x, \dots\}$ is a non-trivial invariant subspace under T. Therefore, every bounded linear operator on a non-separable infinite-dimensional Hilbert space has a non-trivial invariant subspace.

So what remains to be answered is: "Does every bounded linear operator on an infinite-dimensional separable Hilbert space have a non-trivial invariant subspace?"

This question has a negative answer if we replace the Hilbert space H by a general Banach space. The first counterexample was announced by the Swedish mathematician Per Enflo in 1975-1976, but his paper [5] was published later in

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1987. In [7], C.J. Read also constructed a bounded linear operator on the Banach space ℓ^1 without non-trivial invariant subspaces.

Sometime during the 1930's, John von Neumann proved that compact operators have non-trivial invariant subspaces, but did not publish it. The proof was rediscovered and finally published by N. Aronszajn and K. T. Smith [1] in 1954. While von Neumann's original proof uses orthogonal projections, and therefore applies only to Hilbert spaces, Aronszajn and Smith also included an alternative proof that extends to general Banach spaces.

von Neumann's Theorem resisted generalizations for more than a decade after the Aronszajn and Smith paper, and not for lack of interest. In 1966 Bernstein and Robinson [2] extended the result to the slightly larger class of polynomially compact operators (A linear operator T on a Banach space is said to be polynomially compact if there is a non-zero polynomial p such that $p(T)$ is compact). Later on, some further generalizations were also discovered. We refer the reader to [8] and [4] for more on history and development of the problem.

While the techniques of von Neumann and subsequent generalizations yielded many interesting and surprising theorems during the 1950's and 60's, their effectiveness was reaching its limit by the 70's. Just as this was occurring, a young mathematician named Victor Lomonosov introduced a new and powerful technique [6]. His celebrated result is stated as: If X is a Banach space, and $T \in L(X, X)$ commutes with a non-zero compact operator K, then T has a non-trivial invariant subspace.

Having said all of this, we turn now to our work in this article. We introduce what seems to be an interesting orthonormal basis (e_n) of Hilbert spaces. Although the construction of this basis was not meant to be a basis, but it happens that H trivially has an invariant subspace when (e_n) is not a basis. This is why the study is reduced to the case when (e_n) is a basis.

We begin our results by showing that T has an invariant subspace when this basis fulfils certain properties. However, we conclude our results by showing that any $T \in L(H, H)$ can be broken down to two operators, each of which having an invariant subspace.

This paper is self contained that no other readings are needed to understand the shown results.

2. Main Results

In this section, we present our new construction; where a basis of certain properties is introduced. First, an orthonormal set is to be introduced, and then it shall be seen that this set must be a basis in order to study the invariant subspace problem.

In the rest of this paper, H represents a Hilbert space, $T \in L(H, H)$, and for $x_1, x_2, x_3, \cdots \in H$, $[x_1, x_2, x_3, \cdots]$ will denote the span of the set $\{x_1, x_2, x_3, \cdots\}$.

Lemma 2.1. *There exists an orthonormal sequence $(e_n) \subset H$ such that*

$$Te_n = \sum_{k=1}^{n+1} a_{nk} e_k,$$

where $\{a_{nk}\}$ are scalars satisfying

$$\sum_{k=1}^{n+1} |a_{nk}|^2 \leq \|T\|^2,$$

for each $n \in \mathbb{N}$.

Proof. Let $e_1 \in H$ be such that $\|e_1\| = 1$, then $Te_1 =< Te_1, e_1 > e_1 + u$ for some $u \in H$. Let $e_2 = \frac{u}{\|u\|}$, then clearly $e_2 \perp e_1$ and $\|e_2\| = 1$. Hence, $Te_1 = a_{11}e_1 + a_{12}e_2$ where $a_{11} =< Te_1, e_1 >$ and $a_{12} = \|u\|$.
Now, let $E = [e_1, e_2]$, then $Te_2 = u_1 + u_2$ where $u_1 \in E$ and $u_2 \in E^{\perp}$. Let $e_3 = \frac{u_2}{\|u_2\|}$, then clearly $Te_2 = a_{21}e_1 + a_{22}e_2 + a_{23}e_3$ for some scalars a_{21}, a_{22} and a_{23}.
If we continue this construction, we get the first result of the Lemma.
As for the other result, observe that

$$\|Te_n\|^2 = \sum_{k=1}^{n+1} |a_{nk}|^2 \leq \|T\|^2 \|e_n\|^2 = \|T\|^2,$$

as required. □

Henceforth, the sequence (e_n) is implicitly understood to be that of Lemma 2.1.
In the following results, we use the set (e_n) extensively to show some cases that guarantee the existence of an invariant subspace of T.

Lemma 2.2. *Let (a_{nk}) be the scalars of Lemma 2.1. If $a_{n,n+1} = 0$ for any $n \in \mathbb{N}$, then T has an invariant subspace.*

Proof. It is clear that the space $\overline{[e_1, e_2, ..., e_n]}$ is an invariant subspace of T when $a_{n,n+1} = 0$. □

An immediate consequence of this Lemma is:

Corollary 2.3. *Let $T \in L(H, H)$, then T has a finite dimensional invariant subspace if and only if $a_{n,n+1} = 0$ for some $n \in \mathbb{N}$.*

We know that if the orbit, $\{T^n x\}$, of any element $x \in H$ is not dense, then T has an invariant subspace. In the following result, we show that the dependence of this orbit, for a certain element, is sufficient for T to have an invariant subspace.

Lemma 2.4. *If the set $\{Te_1, T^2e_1, ...\}$ is linearly dependent, then T has an invariant subspace.*

Proof. Let N be the first index such that

$$T^N e_1 = \alpha_1 Te_1 + \alpha_2 T^2 e_1 + ... + \alpha_{N-1} T^{N-1} e_1. \tag{2.1}$$

Observe first that

$$T^N e_1 = \left(\sum_{k=1}^{N} A_k e_k \right) + \left(\prod_{k=1}^{N} a_{k,k+1} \right) e_{N+1}, \; N \in \mathbb{N}; A_k \text{ are scalars.}$$

Since the sequence (e_n) is linearly independent and because of the truth of equation(2.1), we must have

$$\prod_{k=1}^{N} a_{k,k+1} = 0.$$

But then, $a_{k,k+1} = 0$ for some $1 \leq k \leq N$, which implies the existence of an invariant subspace by Lemma 2.2. □

Lemma 2.5. *If the set $\{Te_1, Te_2, ...\}$ is linearly dependent, then T does have an invariant subspace.*

Proof. Suppose that $\{Te_1, Te_2, ...\}$ is linearly dependent, then for some $j \in \mathbb{N}$ we have

$$Te_j = \sum_{k=1}^{j-1} b_k Te_k$$

for some choice of nonzero scalars (b_k). But then $\overline{[e_1, ..., e_j]}$ is an invariant subspace. Indeed, $Te_k \in \overline{[e_1, ..., e_j]}$ for every $1 \leq k \leq j-1$ by construction. Moreover, $Te_j \in \overline{[e_1, ..., e_j]}$ by the definition of Te_j. This shows that $T\left(\overline{[e_1, ..., e_j]}\right) \subset \overline{[e_1, ..., e_j]}$; that is $\overline{[e_1, ..., e_j]}$ is an invariant subspace. □

Observe also that if $\overline{[Te_1, Te_2, ...]} \neq H$, then $\overline{[Te_1, Te_2, ...]}$ is an invariant subspace.

Lemma 2.6. *If (e_n) is not a basis of H, then T does have an invariant subspace.*

Proof. If (e_n) is not a basis, then clearly $\overline{[e_1, e_2, ...]}$ is an invariant subspace. □

Therefore, by Lemmas 2.4, 2.5 and 2.6 we can assume, without loss of generality, that

i) (e_n) is a basis for H.
ii) $\{Te_1, Te_2, ...\}$ is linearly independent and $\overline{[Te_1, Te_2, ...]} = H$.
iii) $\{Te_1, T^2e_1, ...\}$ is linearly independent and $\overline{[Te_1, T^2e_1, ...]} = H$.

These conditions are assumed to be true in the sequel.
In the following two results, we show how the behavior of this basis, (e_n), is related to the existence of an invariant subspace.

Theorem 2.7. *If for some $n \in \mathbb{N}$, $e_n \in [T^n e_1, T^{n+1} e_1, ...]$ then T has an invariant subspace.*

Proof. Recall that

$$T^N e_1 = \left(\sum_{k=1}^{N} A_k e_k\right) + \left(\prod_{k=1}^{N} a_{k,k+1}\right) e_{N+1},\ N \in \mathbb{N}; A_k \text{ are scalars.}$$

Now if $e_n \in [T^n e_1, T^{n+1} e_1, ...]$, then for some choice of scalars $\alpha_k; n \leq\ k \leq N$, we have

$$e_n = \alpha_n T^n e_1 + \alpha_{n+1} T^{n+1} e_1 + ... + \alpha_N T^N e_1;\ \alpha_N \neq 0$$

where N is the first such index. But in this representation, we observe that e_{N+1} appears once only as a result of the term $\alpha_N T^N e_1$. The coefficient of e_{N+1} in this representation is

$$B_N := \alpha_N \left(\prod_{k=1}^{N} a_{k,k+1} \right).$$

But since e_1 and e_2 are orthogonal, we must have $B_N = 0$. For B_N to be zero, we have either $a_{k,k+1} = 0$ for some $1 \leq k \leq N$ or $\alpha_N = 0$. If $\alpha_N = 0$, this contradicts the assumption that N is the first index for which

$$e_n = \alpha_n T^n e_1 + \alpha_{n+1} T^{n+1} e_1 + ... + \alpha_N T^N e_1;\ \alpha_N \neq 0.$$

Therefore, $a_{k,k+1} = 0$ for some $1 \leq k \leq N$. But this implies the existence of an invariant subspace for T according to Lemma 2.2. This completes the proof. $\square$

Theorem 2.8. *If* $[e_2, e_3, ...] \subset [Te_1, T^2e_1, ...]$, *then* T *does have an invariant subspace.*

Proof. Suppose $[e_2, e_3, ...] \subset [Te_1, T^2e_1, ...] := E$. Since $e_2 \in E$, we have

$$e_2 = \sum_{k=1}^{N} \alpha_k T^k e_1; \alpha_N \neq 0.$$

Observe that if $N \geq 2$, then T has an invariant subspace. This follows from Theorem 2.7. Therefore, N can be assumed to be 1. That is,

$$e_2 = \alpha_1 T e_1 = \alpha_1 (a_{11} e_1 + a_{12} e_2) \Rightarrow a_{11} = 0.$$

Thus, $Te_1 = b_{12} e_2$ for some scalar b_{12}.
Now, since $e_3 \in E$, we have, similarly,

$$\begin{aligned} e_3 &= \sum_{k=1}^{2} \alpha_k T^k e_1 \\ &= \alpha_1 T e_1 + \alpha_2 T^2 e_1 \\ &= \alpha_1 b_{12} e_2 + \alpha_2 b_{12} (a_{21} e_1 + a_{22} e_2 + a_{23} e_3). \end{aligned}$$

Since (e_n) is linearly independent, we must have $a_{21} = 0$. Consequently, $Te_2 = a_{22} e_2 + a_{23} e_3$. Continuing this way, we see that $Te_n \in [e_2, e_3, ..., e_{n+1}], n \geq 1$. But then $\overline{[e_2, e_3, ...]}$ is an invariant subspace. $\square$

As promised in the introduction, we now show that any $T \in L(H, H)$ can be broken down into two operators; a shift and an upper triangular, each of which having an invariant subspace. In this result, we shall use the assumption that (e_n) is a basis.

Theorem 2.9. *Let* $T \in L(H, H)$ *then* T *can be written as a sum of a shift operator* T_1 *and an upper triangular operator* T_2*; each of which does have an invariant subspace.*

Proof. Let a_{nk} be the scalars of Lemma 2.1, and define two operators T_1 and T_2 as follows:

$$T_1 e_n = a_{n,n+1} e_{n+1} \text{ and } T_2 e_n = \sum_{k=1}^{n} a_{nk} e_k$$

extended in the natural way to $[e_1, e_2, ...]$ linearly. Then clearly $T = T_1 + T_2$. Moreover, $T_1 \in L(H, H)$. Indeed, for $x \in [e_1, e_2, ...]$,

$$\begin{aligned} \|T_1 x\|^2 &= \left\| T_1 \left(\sum_{n=1}^{N} < x, e_n > e_n \right) \right\|^2 ; \text{ for some } N \in \mathbb{N} \\ &= \left\| \sum_{n=1}^{N} < x, e_n > T_1 e_n \right\|^2 \\ &= \left\| \sum_{n=1}^{N} < x, e_n > a_{n,n+1} e_{n+1} \right\|^2 \\ &= \sum_{n=1}^{N} | < x, e_n > |^2 \, |a_{n,n+1}|^2 \\ &\leq \|T\|^2 \sum_{n=1}^{N} | < x, e_n > |^2; \text{ by Lemma 2.1} \\ &\leq \|T\|^2 \|x\|^2, \end{aligned}$$

which means that T_1 is bounded on $[e_1, e_2, ...]$, and $\|T_1\| \leq \|T\|$. But then T_2 is bounded on $[e_1, e_2, ...]$ because $T_2 = T - T_1$. Now, extend T_1 and T_2 to H in the natural way, and observe that this is possible because $\overline{[e_1, e_2, ...]} = H$.

We assert that both T_1 and T_2 do have invariant subspaces. Indeed, if $E_k = \overline{[e_k, e_{k+1}, ...]}$ for any $k \geq 2$, then E_k is an invariant subspace for T_1. Observe that $E_k \neq H$ because $e_1 \notin E_k$ and that (e_n) is a basis for H.

On the other hand, $F_k = \overline{[e_1, e_2, ..., e_k]}$ is an invariant subspace for T_2 for any $k \in \mathbb{N}$. Again, $F_k \neq H$ because (e_n) is a basis for H.

Consequently, both T_1 and T_2 have invariant subspaces. This completes the proof. □

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