A NOTE ON DIRECT PRODUCTS AND EXT[1] CONTRAVARIANT FUNCTORS

FLAVIU POP AND CLAUDIU VALCULESCU

ABSTRACT. In this paper we prove, using inequalities between infinite cardinals, that, if R is an hereditary ring, the contravariant derived functor $\mathrm{Ext}^1_R(-,G)$ commutes with direct products if and only if G is an injective R-module.

1. Introduction

Commuting properties of Hom and Ext functors with respect to direct sums and direct products are very important in Module Theory. It is well known that, if G is an R-module (in this paper all modules are right R-modules), the covariant Hom-functor

$$\mathrm{Hom}_R(G,-) : \text{Mod-}R \to \text{Mod-}\mathbb{Z}$$

preserves direct products, i.e. the natural homomorphism

$$\phi_{\mathcal{F}} : \mathrm{Hom}_R(G, \prod_{i\in I} N_i) \to \prod_{i\in I} \mathrm{Hom}_R(G, N_i)$$

induced by the family $\mathrm{Hom}_R(G, p_i)$, where $p_i : \prod_{j\in I} N_j \to N_i$ are the canonical projections associated to direct product, is an isomorphism, for any family of R-modules $\mathcal{F} = (N_i)_{i\in I}$. On the other hand, the contravariant Hom-functor

$$\mathrm{Hom}_R(-,G) : \text{Mod-}R \to \text{Mod-}\mathbb{Z}$$

does not preserve, in general, the direct products. More precisely, if $\mathcal{F} = (M_i)_{i\in I}$ is a family of R-modules, then the natural homomorphism

$$\psi_{\mathcal{F}} : \mathrm{Hom}_R(\prod_{i\in I} M_i, G) \to \prod_{i\in I} \mathrm{Hom}_R(M_i, G)$$

induced by the family $\mathrm{Hom}_R(u_i, G)$, where $u_i : M_i \to \prod_{j\in I} M_j$ are the canonical injections associated to the direct product, is not, in general, an isomorphism. We note that if we replace direct products $\prod_{i\in I} M_i$ by direct sums $\oplus_{i\in I} M_i$, a similar discussion about commuting properties leads to important notions in Module Theory: small and self-small modules, respectively slender and self-slender modules, and the structure of these modules can be very complicated (see [3], [9], [11]).

The same arguments are valid for the covariant(respectively, the contravariant) derived functors $\mathrm{Ext}^n_R(G,-) : \text{Mod-}R \to \text{Mod-}\mathbb{Z}$ (respectively, $\mathrm{Ext}^n_R(-,G) : \text{Mod-}R \to \text{Mod-}\mathbb{Z}$). In the case of hereditary rings, the modules G such that $\mathrm{Ext}^1_R(G,-)$ commutes naturally with direct sums are studied in [6] and [17]. For

Date: September 11, 2013.
2010 *Mathematics Subject Classification.* 16D90, 16E30.
Key words and phrases. direct product, derived functor, Ext functor, extensions, hereditary ring, injective module, cotorsion group.

commuting properties with respect to some limits and colimits we refer to [7] and [13, Section 6.1 and Section 6.3].

A survey of recent results about the behavior of these functors is presented in [1] and, although the authors treat the general case, they are focused on the context of abelian groups.

In [5], Breaz showed that $\psi_{\mathcal{F}}$ is an isomorphism for all families $\mathcal{F}$ if and only if G is the trivial module. Following a similar idea, in this paper, we will prove that, if R is an hereditary ring, the $\mathbb{Z}$-modules $\mathrm{Ext}^1_R(\prod_{i\in I} M_i, G)$ and $\prod_{i\in I}\mathrm{Ext}^1_R(M_i, G)$ are naturally isomorphic if and only if G is injective.

2. Main Theorem

Throughout this paper, by a ring R we will understand a unital associative ring, an R-module is a right R-module and we will denote by Mod-R the category of all right R-modules. Also, we will use $\mathrm{Ext}^n_R(-, G)$ for the n-th contravariant derived functor, where n is a positive integer and G is an R-module. If $f : M \to N$ is an R-homomorphism, then we denote the induced homomorphism $\mathrm{Ext}^n_R(f, G)$(respectively, $\mathrm{Hom}_R(f, G)$) by f^n(respectively, by f^0).

If M is an R-module and λ is a cardinal, then M^λ(respectively, $M^{(\lambda)}$) represents the direct product(respectively, the direct sum) of λ copies of M. Moreover, if $\mathcal{F} = (M_i)_{i\in I}$ is a family of R-modules then, for all positive integers n, we consider the natural isomorphisms induced by the universal property of direct product

$$\chi^n_{\mathcal{F}} : \mathrm{Ext}^n_R(\bigoplus_{i\in I} M_i, G) \to \prod_{i\in I}\mathrm{Ext}^n_R(M_i, G)$$

respectively, the natural homomorphisms induced by the universal property of direct product

$$\psi^n_{\mathcal{F}} : \mathrm{Ext}^n_R(\prod_{i\in I} M_i, G) \to \prod_{i\in I}\mathrm{Ext}^n_R(M_i, G).$$

If $\mathcal{F} = (M)_{\kappa\le\lambda}$, then we denote $\chi^n_{\mathcal{F}}$(respectively, $\psi^n_{\mathcal{F}}$) by $\chi^n_{M,\lambda}$ (respectively, by $\psi^n_{M,\lambda}$).

A ring R is called *hereditary* if all submodules of projective R-modules are projective. We note that, R is hereditary if and only if $\mathrm{Ext}^k_R(M, N) = 0$, for all R-modules M and N and for all integers $k \ge 2$.

In order to prove our main result, we need some cardinal arguments. The first lemma is a well known result(and it was proven, for example, in [4]), and the second lemma is also well known(see, for instance, [15, Lemma 3.1]). Moreover, these results were generalized by Šťovíček in [16]. Let X be a set and let $\mathcal{A}$ be a family of subsets of X. We say that $\mathcal{A}$ is *almost disjoint* if the intersection of any two distinct elements of $\mathcal{A}$ is finite.

Lemma 2.1. *Let λ be an infinite cardinal. Then there is a family of $\lambda^{\aleph_0}$ countable almost disjoint subsets of λ.*

Lemma 2.2. *For any cardinal σ, there is an infinite cardinal $\lambda \ge \sigma$ such that $\lambda^{\aleph_0} = 2^\lambda$. Consequently, for every cardinal σ, there is an infinite cardinal λ such that $\sigma \le \lambda < \lambda^{\aleph_0}$.*

We have the following lemma, which is very important for the proof of the main result.

Lemma 2.3. *Let R be a ring and let G be an R-module. Let λ be a cardinal and let $\mathcal{F} = (M_\kappa)_{\kappa \leq \lambda}$ be a family of R-modules. Suppose that*

$$\psi^1_{\mathcal{F}} : \operatorname{Ext}^1_R(\prod_{\kappa\leq\lambda} M_\kappa, G) \to \prod_{\kappa\leq\lambda} \operatorname{Ext}^1_R(M_\kappa, G)$$

is an isomorphism. If

$$i : \bigoplus_{\kappa\leq\lambda} M_\kappa \to \prod_{\kappa\leq\lambda} M_\kappa$$

is the natural homomorphism then the induced homomorphism

$$i^1 : \operatorname{Ext}^1_R(\prod_{\kappa\leq\lambda} M_\kappa, G) \to \operatorname{Ext}^1_R(\bigoplus_{\kappa\leq\lambda} M_\kappa, G)$$

is an isomorphism.

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \operatorname{Ext}^1_R(\prod_{\kappa\leq\lambda} M_\kappa, G) & \xrightarrow{i^1} & \operatorname{Ext}^1_R(\bigoplus_{\kappa\leq\lambda} M_\kappa, G) \\ {\scriptstyle \psi^1_{\mathcal{F}}}\downarrow & & \downarrow{\scriptstyle \chi^1_{\mathcal{F}}} \\ \prod_{\kappa\leq\lambda} \operatorname{Ext}^1_R(M_\kappa, G) & \xrightarrow{id} & \prod_{\kappa\leq\lambda} \operatorname{Ext}^1_R(M_\kappa, G). \end{array}$$

By hypothesis, $\psi^1_{\mathcal{F}}$ is an isomorphism and, since $\chi^1_{\mathcal{F}}$ is also an isomorphism, it follows that i^1 is an isomorphism too. □

We also need the following useful lemma.

Lemma 2.4. *Let R be an hereditary ring and let G be an R-module. If $\operatorname{Ext}^1_R(M, G) \neq 0$ then $\operatorname{Ext}^1_R(M^\lambda/M^{(\lambda)}, G) \neq 0$, for any R-module M and for any infinite cardinal λ.*

Proof. Let M be an R-module with $\operatorname{Ext}^1_R(M, G) \neq 0$ and let λ be an infinite cardinal. The R-homomorphism

$$\theta : M \to M^\lambda/M^{(\lambda)}$$

defined by

$$\theta(x) = (x)_{\kappa\leq\lambda} + M^{(\lambda)} = (x, x, x, \cdots) + M^{(\lambda)}$$

is a monomorphism. Then, we have the following short exact sequence

$$0 \to M \xrightarrow{\theta} M^\lambda/M^{(\lambda)} \longrightarrow \operatorname{Coker}(\theta) \to 0.$$

Since R is hereditary, the induced sequence

$$0 \to \operatorname{Hom}_R(\operatorname{Coker}(\theta), G) \longrightarrow \operatorname{Hom}_R(M^\lambda/M^{(\lambda)}, G) \xrightarrow{\theta^0} \operatorname{Hom}_R(M, G) \xrightarrow{\Delta^0}$$

$$\xrightarrow{\Delta^0} \operatorname{Ext}^1_R(\operatorname{Coker}(\theta), G) \longrightarrow \operatorname{Ext}^1_R(M^\lambda/M^{(\lambda)}, G) \xrightarrow{\theta^1} \operatorname{Ext}^1_R(M, G) \xrightarrow{\Delta^1} 0$$

is exact, so that $\theta^1 : \operatorname{Ext}^1_R(M^\lambda/M^{(\lambda)}, G) \to \operatorname{Ext}^1_R(M, G)$ is an epimorphism. Since $\operatorname{Ext}^1_R(M, G) \neq 0$, it follows that $\operatorname{Ext}^1_R(M^\lambda/M^{(\lambda)}, G) \neq 0$. □

Now we give the main result of the paper.

Theorem 2.5. *Let R be an hereditary ring and let G be an R-module. If M is an R-module, then the following statements are equivalent:*

(a) $\psi^1_{M,\lambda} : \operatorname{Ext}^1_R(M^\lambda, G) \to \operatorname{Ext}^1_R(M, G)^\lambda$ *is an isomorphism, for any cardinal λ;*

(b) $\mathrm{Ext}^1_R(M^\mu, G) = 0$, *for any cardinal* μ.

Proof. Let M be an R-module.

$(a) \Rightarrow (b)$ Suppose that there exists a cardinal μ such that $\mathrm{Ext}^1_R(M^\mu, G) \neq 0$. Since $\psi^1_{M,\mu}$ is an isomorphism, it follows that $\mathrm{Ext}^1_R(M, G) \neq 0$.

We denote by X the R-module $M^{\aleph_0}/M^{(\aleph_0)}$. By Lemma 2.2, there is an infinite cardinal $\lambda \geq |\mathrm{Hom}_R(M, G)|$ such that $\lambda^{\aleph_0} = 2^\lambda$. We have the following short exact sequence

$$0 \to M^{(\lambda)} \xrightarrow{i} M^\lambda \xrightarrow{p} M^\lambda/M^{(\lambda)} \to 0,$$

hence the induced sequence

$$0 \to \mathrm{Hom}_R(M^\lambda/M^{(\lambda)}, G) \xrightarrow{p^0} \mathrm{Hom}_R(M^\lambda, G) \xrightarrow{i^0} \mathrm{Hom}_R(M^{(\lambda)}, G) \xrightarrow{\Delta^0}$$

$$\xrightarrow{\Delta^0} \mathrm{Ext}^1_R(M^\lambda/M^{(\lambda)}, G) \xrightarrow{p^1} \mathrm{Ext}^1_R(M^\lambda, G) \xrightarrow{i^1} \mathrm{Ext}^1_R(M^{(\lambda)}, G) \xrightarrow{\Delta^1} 0$$

is also exact. From Lemma 2.3, the natural homomorphism i^1 is an isomorphism, so that $\mathrm{Im}(p^1) = \mathrm{Ker}(i^1) = 0$. It follows that

$$\mathrm{Hom}_R(M^{(\lambda)}, G) \xrightarrow{\Delta^0} \mathrm{Ext}^1_R(M^\lambda/M^{(\lambda)}, G) \to 0$$

is exact, hence we have the inequality

$$|\mathrm{Hom}_R(M, G)|^\lambda \geq \left|\mathrm{Ext}^1_R(M^\lambda/M^{(\lambda)}, G)\right|.$$

By Lemma 2.4 we know that $\mathrm{Ext}^1_R(M^\lambda/M^{(\lambda)}, G) \neq 0$, hence $\mathrm{Hom}_R(M, G) \neq 0$.

By Lemma 2.1, there is a family $(A_\alpha)_{\alpha \leq \lambda^{\aleph_0}}$ of $\lambda^{\aleph_0}$ countable almost disjoint subsets of λ. We view M^{A_α} as embedded in M^λ and we consider $p_\alpha : M^{A_\alpha} \to M^\lambda/M^{(\lambda)}$ be the restriction to M^{A_α} of the canonical projection $p : M^\lambda \to M^\lambda/M^{(\lambda)}$, for all $\alpha \leq \lambda^{\aleph_0}$. We note that $\mathrm{Ker}(p_\alpha) = M^{(A_\alpha)}$ and $\mathrm{Im}(p_\alpha) \cong X$, for all $\alpha \leq \lambda^{\aleph_0}$. Moreover, Bazzoni proved in [4] that the sum $\sum_{\alpha \leq \lambda^{\aleph_0}} \mathrm{Im}(p_\alpha)$ in $M^\lambda/M^{(\lambda)}$ is a direct sum.

Since $\bigoplus_{\alpha \leq \lambda^{\aleph_0}} \mathrm{Im}(p_\alpha)$ is a submodule of $M^\lambda/M^{(\lambda)}$, there is a submodule $M^{(\lambda)} \leq V \leq M^\lambda$ such that

$$\overline{V} = V/M^{(\lambda)} \cong X^{(\lambda^{\aleph_0})}.$$

From the fact that $\overline{V}$ is a submodule of the quotient $M^\lambda/M^{(\lambda)}$, we have the following short exact sequence

$$0 \to \overline{V} \xrightarrow{j} M^\lambda/M^{(\lambda)} \xrightarrow{q} \mathrm{Coker}(j) \to 0.$$

By hypothesis, R is hereditary, hence we have the epimorphism

$$\mathrm{Ext}^1_R(M^\lambda/M^{(\lambda)}, G) \xrightarrow{j^1} \mathrm{Ext}^1_R(\overline{V}, G) \to 0,$$

and then we obtain

$$\left|\mathrm{Ext}^1_R(M^\lambda/M^{(\lambda)}, G)\right| \geq \left|\mathrm{Ext}^1_R(X, G)\right|^{\lambda^{\aleph_0}}.$$

It follows that

$$|\mathrm{Hom}_R(M, G)|^\lambda \geq \left|\mathrm{Ext}^1_R(X, G)\right|^{\lambda^{\aleph_0}}.$$

By Lemma 2.4, $\mathrm{Ext}^1_R(X, G) \neq 0$, so that $\left|\mathrm{Ext}^1_R(X, G)\right| \geq 2$, hence we have

$$\left|\mathrm{Ext}^1_R(X, G)\right|^{\lambda^{\aleph_0}} \geq 2^{\lambda^{\aleph_0}} = 2^{2^\lambda}.$$

Due to the fact that $\lambda \geq |\mathrm{Hom}_R(M,G)| \geq 2$, it follows that

$$2^{\lambda} = |\mathrm{Hom}_R(M,G)|^{\lambda} \geq \left|\mathrm{Ext}^1_R(X,G)\right|^{\lambda^{\aleph_0}} \geq 2^{2^{\lambda}}$$

which is a contradiction. Therefore $\mathrm{Ext}^1_R(M^{\mu},G)=0$.

$(b) \Rightarrow (a)$ Let λ be an arbitrary cardinal. We observe that M is a direct summand of M^{λ}. Hence $\mathrm{Ext}^1_R(M,G)=0$ since it is a direct summand of $\mathrm{Ext}^1_R(M^{\lambda},G)$. It follows that both modules, $\mathrm{Ext}^1_R(M^{\lambda},G)$ and $\mathrm{Ext}^1_R(M,G)^{\lambda}$ are the trivial modules, and therefore $\psi^1_{M,\lambda}$ is an isomorphism. □

Remark 2.6. The condition (b) cannot be replaced by $\mathrm{Ext}^1_R(M,G)=0$. For instance, if M is projective but M^{λ} is not projective (e.g. $M=R=\mathbb{Z}$ and $\lambda \geq \aleph_0$, cf.[10, Theorem 8.4]) there exists a module G such that $\mathrm{Ext}^1_R(M,G)^{\lambda}=0$, but $\mathrm{Ext}^1_R(M^{\lambda},G)\neq 0$.

Using the main result we obtain the promised characterization:

Corollary 2.7. *Let R be an hereditary ring and let G be an R-module. The following statements are equivalent:*

(a) $\psi^1_{\mathcal{F}} : \mathrm{Ext}^1_R(\prod_{i\in I} M_i, G) \to \prod_{i\in I}\mathrm{Ext}^1_R(M_i,G)$ *is an isomorphism, for any family of R-modules $\mathcal{F}=(M_i)_{i\in I}$;*
(b) $\psi^1_{M,\lambda} : \mathrm{Ext}^1_R(M^{\lambda},G) \to \mathrm{Ext}^1_R(M,G)^{\lambda}$ *is an isomorphism, for any R-module M and for any cardinal λ;*
(c) *G is injective.*

Remark 2.8. The condition on R assumed in the hypothesis, i.e. R is hereditary, could be replaced by the condition $id(G)\leq 1$, where $id(G)$ is the injective dimension of the R-module G.

Remark 2.9. We note that we can replace the condition that "the natural homomorphism is an isomorphism" by the weak condition "there exists an isomorphism". In this case the problems discussed here can be more complicated (see [8], [12]), and the solutions can be dependent by some set theoretic axioms (see [2, Section 3] or [14]).

By the same Theorem 2.5, we obtain some simpler versions for some results proved in [12]. Recall that an abelian group G is a cotorsion group if $\mathrm{Ext}^1(\mathbb{Q},G)=0$.

Theorem 2.10. *Let G be an abelian group. The following statements are equivalent:*

(a) $\psi^1_{\mathbb{Q},\lambda} : \mathrm{Ext}^1(\mathbb{Q}^{\lambda},G) \to \mathrm{Ext}^1(\mathbb{Q},G)^{\lambda}$ *is an isomorphism, for any cardinal λ;*
(b) *G is cotorsion.*

Proof. This follows from that fact that $\mathbb{Q}^{\lambda}$ is a divisible torsion-free group, hence it is a direct sum of copies of $\mathbb{Q}$ (cf. [10, Theorem 2.3]). □

Aknowledgements Thanks are due to referees, for their valuable suggestions which improved our study.

The authors are supported by the CNCS-UEFISCDI grant PN-II-RU-TE-2011-3-0065.

References

[1] U. Albrecht, S. Breaz, P. Schultz, *Functorial properties of* Hom *and* Ext, in: L.Strungmann(ed) et al., Groups and model theory. In honor of Rudiger Gobel's 70th birthday. Proceedings of the conference, conference center 'Die Wolfsburg', Muhl an der Ruhr, Germany, May 30 - June 3, 2011, Providence RI:American Mathematical Society(AMS). Contemporary Mathematics 576, 1-15(2012).

[2] U. Albrecht, S. Breaz, P. Schultz, *The* Ext *functor and self-sums*, Forum Math. (to appear), doi:10.1515/forum-2011-0141.

[3] U. Albrecht, S. Breaz, W. Wickless, *Self-small abelian groups*, Bull. Aust. Math. Soc., 80(2) (2009), 205-216.

[4] S. Bazzoni, *Cotilting modules are pure-injective*, Proc. Amer. Math. Soc. 131(12)(2003), 3665-3672.

[5] S. Breaz, *Direct products and the contravariant hom functor*, Bull. Lond. Math. Soc., 44(1) (2012), 136-138.

[6] S. Breaz, *Modules M such that* $\mathrm{Ext}^1_R(M,-)$ *commutes with direct limits*, Algebr. Represent. Theory (to appear).

[7] S. Breaz, *When* Ext^1 *commutes with direct unions*, preprint.

[8] S. Breaz, P. Schultz, *When* Ext *commutes with direct sums*, J. Algebra Appl., 11(5)(2012), doi:10.1142/S0219498812501538.

[9] S. Breaz, J.Žemlička, *When every self-small module is finitely generated*, J. Algebra, 315(2)(2007), 885-893.

[10] G. Călugăreanu, S. Breaz, C. Modoi, C. Pelea, D. Vălcan, *Exercises in Abelian Group Theory*, Kluwer Academic Publishers, 2003.

[11] R. Göbel, B. Goldsmith, O. Kolman, *On modules which are self–slender*, Houston J. Math., 35(3) (2009), 725-736.

[12] R. Göbel, R. Prelle, *Solution of two problems on cotorsion Abelian groups*, Arch. Math., 31(1)(1978), 423-431.

[13] R. Göbel, J. Trlifaj, *Approximations and Endomorphism Algebras of Modules*, vol 1, Approximations, Walter de Gruyter, Berlin, 2012.

[14] B. Goldsmith, O. Kolman, *On cosmall Abelian groups*, J. Algebra, 317(2) (2007), 510-518.

[15] P. Griffith, *On a subfunctor of Ext*, Arch. Math. 21(1970), 17-22.

[16] J. Štoviček, *All n-cotilting modules are pure-injective*, Proc. Amer. Math. Soc. 134(7)(2006), 1891-1897.

[17] R. Strebel, *A homological finiteness criterion*, Math. Z., 151 (1976), 263-275.

Flaviu Pop: "Babeş-Bolyai" University, Faculty of Economics and Business Administration, str. T. Mihali, nr. 58-60, 400591, Cluj-Napoca, Romania

E-mail address: `flaviu.v@gmail.com; flaviu.pop@econ.ubbcluj.ro`

Claudiu Valculescu: "Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, Str. Mihail Kogălniceanu 1, 400084, Cluj-Napoca, Romania

E-mail address: `claudio040788@gmail.com`