Azumaya semigroup algebras *

Xiaojiang Guo
Department of Mathematics, Jiangxi Normal University, Nanchang,
Jiangxi 330022, P.R. China
Email: xjguo@jxnu.edu.cn; xjguo1967@sohu.com

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Abstract

A semigroup is called almost idempotent-free if it has exactly one nonzero $\mathcal{J}$-class containing idempotents. It is mainly investigated when almost idempotent-free semigroup algebras are Azumaya algebras. Necessary and sufficient conditions for semigroup algebras of an almost idempotent-free semigroup with certain properties to be Azumaya are obtained.
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1 Introduction

Recall that an algebra $\mathfrak{A}$ with unity over a commutative ring R is called *separable* if $\mathfrak{A}$ is a projective left $\mathfrak{A} \otimes_R \mathfrak{A}^o$-module under the action given by $(\sum_i x_i \otimes y_i)a = \sum_i x_i a y_i$ for all $a, x_i \in \mathfrak{A}, y_i \in \mathfrak{A}^o$, where $\mathfrak{A}^o$ is the opposite algebra of $\mathfrak{A}$; $\mathfrak{A}$ is called an *Azumaya algebra* if it is separable over its center $Z(\mathfrak{A})$. If R is a commutative ring with unity 1_R and G a finite group of order n, then the group ring $R[G]$ is separable if and only if $n1_R$ is invertible in R. This is a well-known generalization of Maschke's theorem. It is worthy to point out that if $n1_R$ is invertible in R, then $R[G]$ is an Azumaya algebra. So, $R[G]$ is separable if and only if $R[G]$ is Azumaya. On the other hand, Azumaya algebras form a class of very well behaved PI-algebras that allow development of the Brauer group classification of commutative rings (see [6, 17]). They also prove to be very useful as an efficient tool in the study of general algebras satisfying polynomial identities. We refer to [2] for an important application of this type.

Z-separability of the integral semigroup ring $Z[S]$ of an arbitrary finite semigroup S was first studied by Shapiro in [19]. Then Cheng [3] obtained a description of R-separable ring $R[S]$ for an arbitrary commutative coefficient ring R. DeMeyer and Hardy independently considered this problem in [7]. An extension of these results to the class

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of so-called excellent extensions was given by Okniński in [12]. In [13], Okniński and Van Oystaeyen gave some necessary and sufficient conditions for a cancellative monoid algebra to be an Azumaya algebra. In a sequence of papers, Van Oystaeyen studied group-graded Azumaya algebras (see, [14, 15]).

A semigroup S is called *idempotent-free* if S has no idempotents except possibly identity and zero elements; S is called *almost idempotent-free* if it has exactly one nonzero $\mathcal{J}$-class containing idempotents. Okniński pointed out that for a finitely generated semigroup S and a field K, $K[S]$ is an Azumaya algebra if and only if $K[S] \cong \oplus_{i=1}^{n} K_0[S_i]$ and each $K_0[S_i]$ is an Azumaya algebra, where S_i are almost idempotent-free semigroups (see [11, Corollary 17, p.325]). So, it is important to probe when semigroup algebras of almost idempotent-free semigroups are Azumaya algebras. Because of this view, Okniński raised the problem ([11, Problem 28, p.332]):

Problem 1.1 *Characterize Azumaya algebras $K[S]$ of almost idempotent-free semigroups.*

He conjectured that the equivalences of [11, Proposition 8, p.318] can be extended to idempotent-free semigroups; that is,

Conjecture 1.2 *Let S be an idempotent-free monoid with the group of units U. Then the following statements are equivalent:*
(1) *$K[S]$ is an Azumaya algebra;*
(2) *$K[S] = Z(K[S])K[U]$ and $K[U]$ is an Azumaya algebra;*
(3) *$C_{K[S]}(U)$ (the centralizer of U) $= Z(K[S])$ and $K[U]$ is an Azumaya algebra.*

Furthermore, in his monograph, he raised [11, Problem 29, p.332]:

Problem 1.3 *Assume that S is a finite semigroup. Is an Azumaya algebra $K[S]$ always isomorphic to a finite direct product of matrix algebras over semigroup algebras of idempotent-free monoids?*

Our main aim is to give some partial answer to these problems.

We proceed as follows: after citing known results, we prove that for an almost idempotent-free semigroup S satisfying the regularity condition, if $K[S]$ is Azumaya, then $K[S] \cong M_n(K[G])$ where G is an idempotent-free monoid (Theorem 3.3). Based on this fact, we give a positive answer of Problem 1.3 when S satisfies the regularity condition (Corollary 3.6). In Section 4, we consider semigroup algebras of almost idempotent-free semigroups. We obtain a necessary and sufficient condition for the semigroup $K[S]$ of an almost idempotent-free semigroup S satisfying the regularity condition and min_J to be Azumaya (Theorems 4.4 and 4.6). It is proved that Conjecture 1.2 is true for the case when the idempotent-free semigroup satisfying min_J (Theorem 4.10).

2 Preliminaries

Throughout this note we use notations and terminologies from the monograph of Okniński [11] and the text book of Clifford and Preston [4]. Other undefined terms can be found in the textbook [9].

2.1 Semigroups

Let S be a semigroup. Assume that I is an ideal of S. Form the set $S/I = (S\backslash I) \cup \{0\}$ and define a multiplication $\circ$ on S/I by the rule that

$$a \circ b = \begin{cases} ab & \text{if } a, b, ab \in S\backslash I; \\ 0 & \text{otherwise,} \end{cases}$$

where ab is the product of a and b in the semigroup S. It is not difficult to check that $(S/I, \circ)$ is a semigroup. In what follows, we denote the above semigroup by S/I. In fact, S/I is the *Rees factor semigroup of S modulo I* (for detail, [4, p.17]).

Also, S is called 0-*simple* if $S^2 \neq 0$ and S has at most two ideals: 0 and S. In what follows, we always use $\mathcal{J}$ to stand for the usual Green's relation: $(a, b) \in \mathcal{J}$ if and only if there exist $x, y, u, v \in S^1$ such that $a = xby$ and $b = uav$. For $a \in S$, we use $J(a)$ to denote the smallest ideal $S^1 a S^1$ of S containing a, and J_a to denote the $\mathcal{J}$-class of S containing a. We shall call the $\mathcal{J}$-class containing regular elements the *regular $\mathcal{J}$-class* of S. Define

$$J_a \leq J_b \text{ if } J(a) \subseteq J(b).$$

It is evident that $\leq$ is a partial order on the set $S/\mathcal{J}$. When $J_a \leq J_b$ and $J_a \neq J_b$, we shall denote $J_a < J_b$. As in Howie [9], we say that S *satisfies the condition* min_J if the partially ordered set $S/\mathcal{J}$ satisfies the minimal condition. Note that $I(a) = \{x \in J(a) : J_x < J_a\}$ is an ideal of S. The Rees factor semigroup $J(a)/I(a) = J_a \cup \{0\}$ of $J(a)$ modulo $I(a)$ is either a 0-simple semigroup or a null semigroup. For convenience, we denote the zero element by 0. $J(a)/I(a)$ is called the *principal factor* of S determined by a. So, any principal factor is either a 0-simple semigroup or a null semigroup.

Let n be an integer. We say that S has *the property* $\mathfrak{P}_n$ if for any $x_1, x_2, ..., x_n \in S$, there exists a nontrivial permutation σ in the symmetric group $\mathfrak{S}_n$ such that $x_1 x_2 ... x_n = x_{\sigma(1)} x_{\sigma(2)} ... x_{\sigma(n)}$. It is easy to see that any subsemigroup and any homomorphic image of a semigroup having the property $\mathfrak{P}_n$ have the property $\mathfrak{P}_n$. For simplicity, we say that S has *the permutational property* if some $\mathfrak{P}_n$, $n \geq 2$, is satisfied in S. In [8], Domanov proved the following result (also see [11, Theorem 17, p.229]):

Lemma 2.1 *If T is a* 0*-simple semigroup satisfying the permutational property, then T is a completely* 0*-simple semigroup.*

2.2 Semigroup algebras

We always assume K is a field and S a semigroup. We denote by $K[S]$ the semigroup algebra of S over K. In general, if I is a subset of S, then $K[I]$ denotes the set of K-linear combinations of elements in I, that is, $K[I]$ is a vector space with I as a basis. So each element of $K[I]$ is a finite summation of the form $\sum_{x \in I} r_x x, r_x \in K, x \in I$. In particular, if I_1 and I_2 are subsets of S, then $K[I_1 \cap I_2] = K[I_1] \cap K[I_2]$. If S is a semigroup with zero θ, then $K[\theta]$ is an ideal of $K[S]$, and we define $K_0[S] = K[S]/K[\theta]$. This K-algebra $K_0[S]$ is called the *contracted semigroup algebra* of S over K. If S has no zero, then we have $K_0[S] = K[S]$. Clearly, an element a of $K_0[S]$ is a finite linear combination $a = \sum r_s s$ of elements $s \in S\backslash\{\theta\}$. The support of $a \in K_0[S]$, denoted by $\text{supp}(a)$, is the set $\{s \in S\backslash\{\theta\} \mid r_s \neq 0\}$.

Lemma 2.2 [11, Proposition 1, p.221] *Assume $K[S]$ satisfies a polynomial identity of degree n. Then S has the property $\mathfrak{P}_n$.*

We now recall some known results on Azumaya algebras (for example, see [11, Lemma 1, p.313]).

Lemma 2.3 *Assume that $\mathfrak{A}$ is an Azumaya algebra, and denote by $Z(\mathfrak{A})$ the center of $\mathfrak{A}$. Then*

(1) *For every ideal $\mathfrak{J}$ of $\mathfrak{A}$ and $\mathfrak{I}$ of $Z(\mathfrak{A})$, we have $(\mathfrak{J}\cap Z(\mathfrak{A}))\mathfrak{A} = \mathfrak{J}$ and $\mathfrak{I}\mathfrak{A}\cap Z(\mathfrak{A}) = \mathfrak{I}$.*

(2) *For every ideal $\mathfrak{J}$ of $\mathfrak{A}$, $Z(\mathfrak{A}/\mathfrak{J}) = (Z(\mathfrak{A}) + \mathfrak{J})/\mathfrak{J}$ and $\mathfrak{A}/\mathfrak{J}$ is an Azumaya algebra.*

(3) *$\mathfrak{A}$ is a finitely generated projective module over its center $Z(\mathfrak{A})$, and $Z(\mathfrak{A})$ is a direct summand of this module.*

In [1], Adjamagbo pointed out that

Lemma 2.4 [1, Recall 1.1 (8), p.92] *For any commutative algebra $\mathfrak{A}$ and commutative $\mathfrak{A}$-algebra $\mathfrak{C}$, if $\mathfrak{B}$ is separable over $\mathfrak{A}$, then $\mathfrak{B}\otimes_{\mathfrak{A}}\mathfrak{C}$ is separable over $\mathfrak{C}$. Conversely, if $\mathfrak{B}\otimes_{\mathfrak{A}}\mathfrak{C}$ is separable over $\mathfrak{C}$ and if, in addition, $\mathfrak{B}$ is a finitely generated $\mathfrak{A}$-module or $\mathfrak{A}$-algebra and $\mathfrak{C}$ a faithfully flat $\mathfrak{A}$-module, then $\mathfrak{B}$ is separable over $\mathfrak{A}$.*

The following lemma follows immediately from [11, Proposition 13, p.323].

Lemma 2.5 *Let $K[S]$ be an Azumaya algebra and $t \in S$. If the principal factor of S determined by t is 0-simple, then $K[StS]$ is a ring direct summand of $K[S]$.*

By the proof of [11, Proposition 13, p.323], Lemma 2.5 is true for $K_0[S]$.
The following observation is trivial.

Lemma 2.6 *Let S be a semigroup. Then $K[S]$ is an Azumaya algebra if and only if $K_0[S]$ is an Azumaya algebra.*

We need the following well known result (see [18, Proposition 25B.8] or [6]).

Lemma 2.7 *Let S be a monoid. Then $M_n(K_0[S])$ is Azumaya if and only if $K_0[S]$ is Azumaya.*

3 The regularity condition

A semigroup is said to *satisfy the regularity condition* if all of its regular elements form a subsemigroup. Obviously, cancellative monoids satisfy the regularity condition. In this section we study when semigroup algebras of almost idempotent-free semigroups satisfying the regularity are Azumaya algebras.

Lemma 3.1 *Let S be an almost idempotent-free semigroup. If $K[S]$ is an Azumaya algebra, then the following statements are true:*

(1) *For any $x \in S$, x is not regular in S if and only if $J(x)/I(x)$ is a null semigroup. Moreover, every element of the $\mathcal{J}$-class of S containing a nonzero regular element is regular.*

(2) *The nonzero regular $\mathcal{J}$-class is the greatest element in the set $S/\mathcal{J}$ with the above order $\leq$. Moreover, $V := (S\backslash Reg(S)) \cup \{0\}$ is an ideal of S.*

(3) *S/V is a completely 0-simple semigroup. Moreover, for any nonzero regular elements a, b of S, $a\mathcal{D}b$.*

Proof. (1) Assume x is not regular in S. Then $J(x)/I(x)$ is a null semigroup; if not, then $J(x)/I(x)$ is 0-simple, but since $K[S]$ is Azumaya, now $K[S]$ is a PI-algebra, so by Lemmas 2.1 and 2.2, $J(x)/I(x)$ is a completely 0-simple semigroup. So, the assertions follow.

(2) By hypothesis, $K[S]$ has an identity. Let e be the identity of $K[S]$ and J a maximum element of the set $A := \{J_a : a \in supp(e)\}$. If $b \in J \cap supp(e)$, then by $be = b$, $bu = b$ for some $u \in supp(e)$. Since $J = J_b = J_{bu} \leq J_u$ and by the maximality of J, we have $J = J_u$. That is, $bu, u \in J$. This shows that $J(b)/I(b) = J \cup \{0\}$ is 0-simple. Thus by Lemmas 2.1 and 2.2, $J(b)/I(b)$ is a completely 0-simple semigroup, so b is regular in S, thereby J is a nonzero regular $\mathcal{J}$-class of S. For any $w \in S$, since $we = w$, there exists $z \in supp(e)$ such that $w = wz$. It follows that $J_w \leq J_z \leq J$ because J is the greatest element of A. Note that S has only a nonzero regular $\mathcal{J}$-class. Therefore the nonzero regular $\mathcal{J}$-class is the greatest element in the set $S/\mathcal{J}$.

For any $a, b \in S$, if $ab \in Reg(S)\backslash\{0\}$, then $J_{ab} \leq J_a, J_b$ and $J_{ab} = J_a = J_b$ since the nonzero regular $\mathcal{J}$-class is maximal, so $a, b \in Reg(S)\backslash\{0\}$, whence V is an ideal of S.

(3) Because S is an almost idempotent-free semigroup and by (2), $S/V = J_a \cup \{0\}$ where a is regular. Since a is regular, we know that $J(a)/I(a)$ is 0-simple, that is, S/V is a 0-simple semigroup. On the other hand, since $K[S]$ is Azumaya, S/V satisfies the permutational property. Now by Lemma 2.1, S/V is a completely 0-simple semigroup. The rest is trivial. □

Proposition 3.2 *Let S be an almost idempotent-free semigroup. If $K[S]$ is an Azumaya algebra, then S has finitely many idempotents.*

Proof. By Lemma 3.1 (2), $K[V]$ is an ideal of $K[S]$, so that by Lemma 2.3 (2), $K_0[S/V] \cong K[S]/K[V]$ is Azumaya. It follows that $K_0[S/V]$ has an identity. Now by [11, Proposition 2.5, P.59], S/V has finitely many $\mathcal{L}$-classes and $\mathcal{R}$-classes. Therefore as S/V is a completely 0-simple semigroup, S has finitely many idempotents. □

Theorem 3.3 *Let S be an almost idempotent-free semigroup. If S satisfies the regularity condition, then $K_0[S]$ is an Azumaya algebra if and only if $K_0[S] \cong M_n(K_0[G])$ where G is an idempotent-free monoid and $K_0[G]$ is an Azumaya algebra.*

Proof. Note that $K_0[S] = K_0[S^0]$. So, we may assume that S has zero element. By Lemma 2.7, we need only to verify the necessity. Now, assume that $K_0[S]$ is an Azumaya algebra. Then $K_0[S]$ has an identity, say E. Let $E = F + D$ with $supp(F) \subseteq Reg(S)$ and $supp(D) \subseteq V$ where $V = (S\backslash Reg(S)) \cup \{0\}$. For any $a \in Reg(S)$, we have $a = aE = aF + aD$ and $a = Ea = Fa + Da$. By Lemma 3.1(2), V is an ideal of S, and $supp(aD), supp(Da) \subseteq V$. Thus $a = Fa = aF$, in other words, F is an identity of $K_0[Reg(S)]$.

Compute

$$F = FE = F^2 + FD = EF = F^2 + DF.$$

But $DF, FD \in K[V]$ and $F^2 \in K[Reg(S)]$, so $F^2 = F$ and $FD = 0 = DF$. Since

$$\begin{aligned} F + D &= E = E^2 = (F + D)^2 = F^2 + FD + DF + D^2 \\ &= F^2 + D^2 \end{aligned}$$

and by $D^2 \in K[V]$ and $F^2 \in K[Reg(S)]$, we observe that $D = D^2$. Let $x \in supp(D)$ such that J_x is the maximum element of the set $\{J_u : u \in supp(D)\}$. The fact that $D = D^2$ derives that $x = yz$ for some $y, z \in supp(D)$. It follows that $J_x \leq J_y, J_z$. Thus $J_x = J_y = J_z$ by the maximality of J_x. Now, $x = yz$ can show that $J(x)/I(x)$ is 0-simple, and further by Lemma 3.1(1), x is a regular element, contradicting to the fact that $supp(D) \subseteq V$. Consequently, $D = 0$ and whence $E = F$ and is the identity of $K_0[S]$.

By Lemma 3.1 (3), S/V is isomorphic to some Rees matrix semigroup $\mathcal{M}^0(H, I, \Lambda; P)$ over a group H. Again by Lemma 3.1 (2), $S/V \cong Reg(S)$, so that $K_0[S/V] = K_0[Reg(S)]$. For convenience, *we identify* $Reg(S)$ *with* $\mathcal{M}^0(H, I, \Lambda; P)$. By [11, Lemma 1, p.48], the mapping ψ linearly spanned by the the mapping defined by

$$K_0[Reg(S)] \to \mathfrak{M}(K[H], I, \Lambda; P); (a, i, m) \mapsto (a_{jn})$$

is an algebra isomorphism, where $\mathfrak{M}(K[H], I, \Lambda; P)$ is an algebra of matrix type and (a_{jn}) the $I \times \Lambda$ matrix with entry $a_{jn} = a$ if $j = i, n = m$ and, otherwise, $a_{jn} = 0$. Since, by the forgoing proof, $K_0[Reg(S)]$ has an identity, and by [11, Proposition 25, p.59], this shows that $|I| = |\Lambda| = n < +\infty$ and P is an invertible matrix in $M_n(K[H])$. It is not difficult to check that P^{-1} is the identity of $\mathfrak{M}(K[H], I, \Lambda; P)$ and the mapping defined by $\varphi: \ A \mapsto AP$ is an isomorphism of $\mathfrak{M}(K[H], I, \Lambda; P)$ onto $M_n(K[H])$.

Denote by ε_i the $n \times n$ matrix with unity of $K[H]$ at the row i, column i position and the zero 0 of $K[H]$ in all other entries. So, $F = \sum_{i=1}^n f_i$ where $f_i = \psi^{-1}\varphi^{-1}(\varepsilon_i)$. Put

$$M_{im} = \{(a, i, m) : a \in H\}.$$

It is not difficult to know that each f_i is an idempotent, supp $(f_i) \subseteq M_{ii}$ and $f_i f_m \neq 0$ only if $i = m$.

Lemma 3.4 *Let* S *satisfy the conditions of Theorem 3.3. Then* $K_0[S] = \bigoplus_{i=1}^n e_i K_0[S]$, *where* $e_i = (p_{m_i,i}^{-1}, i, m_i)$ $(m_i \in \Lambda)$ *and* $e_i K_0[S] \cong e_j K_0[S]$ *for any* i, j.

Proof. By the properties of Rees matrix semigroups, $e_i x = x = x e_i$ for all $x \in M_{ii}$. It follows that

$$f_i \cdot K_0[S] \subseteq \sum_{u \in supp\,(f_i)} u \cdot K_0[S] \subseteq e_i \cdot K_0[S].$$

Because $supp\,(f_j e_i) \subseteq M_{ji}$, we have

$$supp\left(\sum_{j=1}^{i-1} f_j e_i + \sum_{j=i+1}^{n} f_j e_i\right) \bigcap M_{ii} = \emptyset$$

and $supp\,(f_i e_i) \subseteq M_{ii}$. But

$$e_i = f_i e_i + \sum_{j=1}^{i-1} f_j e_i + \sum_{j=i+1}^{n} f_j e_i,$$

now $e_i = f_i e_i$. Therefore $e_i \cdot K_0[S] \subseteq f_i \cdot K_0[S]$ and whence $e_i \cdot K_0[S] = f_i \cdot K_0[S]$.

Note that $F = \sum_{i=1}^n f_i$ is the identity of $K_0[S]$ and f_i's are orthogonal. We observe that $K_0[S] = \bigoplus_{i=1}^n f_i \cdot K_0[S] = \bigoplus_{i=1}^n e_i \cdot K_0[S]$. In addition, e_i's are all nonzero, further by Lemma 3.1 (3), e_is are related by $\mathcal{D}$. Therefore by [10, Proposition (21.20), p.315], $e_iK_0[S] \cong e_jK_0[S]$ for any i, j. □

Now, we have

$$\begin{aligned} K_0[S] &= \bigoplus_{i=1}^{n} e_i \cdot K_0[S] \cong End_{K_0[S]}(ne_1 \cdot K_0[S]) \cong M_n(e_1K_0[S]e_1) \\ &\cong M_n(K_0[e_1Se_1]). \end{aligned}$$

On the other hand, since S is an almost idempotent-free semigroup, it is easy to see that $E(e_1Se_1) = E(e_1Reg(S)e_1)$. But $S/V \cong Reg(S)$ is a completely 0-simple semigroup, we have $E(e_1Reg(S)e_1) = \{e_1, 0\}$. However e_1Se_1 is an idempotent-free monoid.

We have now proved that $K_0[S] \cong M_n(K_0[e_1Se_1])$, so that $M_n(Z(K_0[e_1Se_1]))$ is an Azumaya algebra. By Lemma 2.7, $K_0[e_1Se_1]$ is an Azumaya algebra. The proof is finished. □

Let us turn back to the proof of Theorem 3.3. If $S\backslash V$ is a subsemigroup of S, then P has no zero entries. But P is invertible in $M_n(K[H])$, so the cardinality of the sets I and Λ must be equal to 1. In other words, $n = 1$. Thus S is an idempotent-free monoid. Note that $S\backslash V$ is a subsemigroup of S is equivalent to that the set of nonzero regular elements of S forms a subsemigroup of S, we have the following corollary.

Corollary 3.5 *Let S be an almost idempotent-free semigroup whose set of nonzero regular elements forms a subsemigroup. If $K[S]$ is Azumaya, then S is an idempotent-free monoid.*

By applying the result [11, Corollary 17, p.325], we have that for any finite semigroup S, $K_0[S]$ is an Azumaya algebra if and only if $K_0[S] \cong \oplus_{i=1}^{m} K_0[S_i]$ where each S_i is an almost idempotent-free semigroup and $K_0[S_i]$ an Azumaya algebra. Furthermore, by Theorem 3.2, the following corollary is immediate. This corollary confirms Problem 1.3 when the semigroup satisfies the regularity condition.

Corollary 3.6 *Let S be a finite semigroup. If S satisfies the regularity condition, then $K_0[S]$ is an Azumaya algebra if and only if $K_0[S] \cong \bigoplus_{i=1}^{m} M_{n_i}(K_0[G_i])$ where each G_i is an idempotent-free monoid and $K_0[G_i]$ an Azumaya algebra.*

4 The condition min_J

The aim of this section is to determine when semigroup algebras of almost idempotent-free semigroups satisfying the condition min_J are Azumaya algebras.

Lemma 4.1 *Let S be an almost idempotent-free semigroup satisfying the condition min_J. Assume that $K[S]$ is an Azumaya algebra. If I and J are $\mathcal{J}$-classes of S that are not regular, then for any $a \in I, b \in J$, we have $J_{ab} < I, J$ and $J_{ba} < I, J$.*

Proof. For any $a \in I, b \in J$, we have $J_{ab} \leq J_b = J$. If $J_{ab} = J$, then there exist $x, y \in S^1$ such that $b = xaby$. Thus $b = (xa)^n by^n$ for all positive integer n. Obviously, $(xa)^n \neq 0$. Note that

$$\cdots \leq J_{(xa)^3} \leq J_{(xa)^2} \leq J_{xa}.$$

But S satisfies the condition min_J, so there exists positive integer m such that

$$J_{(xa)^m} = J_{(xa)^{m+1}} = \cdots .$$

It follows that $(xa)^m(xa)^m \in J_{(xa)^m}$, in other words, $J_{(xa)^m}$ is 0-simple. Thus by Lemma 3.1(1), $(xa)^m$ is regular, whence by Lemma 3.1(2), I is the unique nonzero regular $\mathcal{J}$-class of S since $J_{(xa)^m} \leq J_a = I$, giving $J_{(xa)^m} = I$. This is a contradiction. Consequently, $J_{ab} < J$; similarly $J_{ab} < I$. The rest can be similarly proved. □

Lemma 4.2 *Let S be an almost idempotent-free semigroup satisfying the condition min_J. If $K[S]$ is an Azumaya algebra, then for any $a \in S$, either a is regular or $a^n = 0$ for some positive integer n if S has zero element 0.*

Proof. Without loss of generality, we assume that S has zero element 0. Now let a be not a regular element of S. Obviously,

$$J_a \geq J_{a^2} \geq \cdots \geq J_{a^n} \geq \cdots .$$

But S satisfies the condition min_J, so there exists positive integer m such that

$$J_{a^m} = J_{a^{m+1}} = \cdots .$$

It follows that $J_{a^m} = J_{a^{2m}}$, whence $a^m \circ a^m \in J_{a^m}$. Thus $a^m = 0$ or $J(a^m)/I(a^m)$ is 0-simple. If the second case holds, then by Lemma 3.1(1), a^m is regular. Note that $J_{a^m} \leq J_a$. By Lemma 3.1(2), $J_{a^m} = J_a$ and whence a is regular, contrary to the hypothesis. Therefore $a^m = 0$ and the result follows. □

Recall that a semigroup S is called an *ideal extension of nil semigroups by a completely 0-simple semigroup* if S has a nil ideal I such that S/I is a completely 0-simple semigroup. The following corollary is immediate from Lemmas 3.1 and 4.2.

Corollary 4.3 *Let S be an almost idempotent-free semigroup satisfying min_J. If $K_0[S]$ is Azumaya, then S is an ideal extension of nil semigroups by a completely 0-simple semigroup.*

Based on Lemma 4.2, we have the following theorem.

Theorem 4.4 *Let S be an almost idempotent-free semigroup satisfying the condition min_J, and assume that S has no zero elements. If $K[S]$ is an Azumaya algebra, then S is a group.*

Proof. Assume $K[S]$ is an Azumaya algebra. By Lemma 4.2, all elements of S are regular, in other words, S is a regular semigroup. By Corollary 3.5, S is an idempotent-free monoid. Thus S is a group. □

We shall simply denote

$$Z(K[S])K[Reg(S)] = \left\{ \sum_{i=1}^{m} z_i y_i : z_i \in Z(K[S]), y_i \in K[Reg(S)], m \in N \right\},$$

where N is the set of positive integers.

Lemma 4.5 *Let S be an almost idempotent-free semigroup satisfying the condition min_J. If $K[S]$ is an Azumaya algebra, then $K[S] = Z(K[S])K[Reg(S)]$.*

Proof. Assume that $K[S]$ is an Azumaya algebra. By Theorem 4.4, the theorem is clearly true for the case if S has no zero elements. So, we now let S have a zero element. Assume, on the contrary, that there exists $\alpha \notin Z(K[S])K[Reg(S)]$. It follows that $a_1 \notin Z(K[S])K[Reg(S)]$ for some $a_1 \in supp(\alpha)$. Obviously, $a_1 \in S \backslash Reg(S)$. By Lemma 2.3(1), we have

$$(K[J(a_1)] \cap Z(K[S]))K[S] = K[J(a_1)] \quad (1)$$

and $a_1 = \alpha_1(\beta_1 + \beta_1') = \alpha_1\beta_1 + \alpha_1\beta_1'$ for some $\alpha_1 \in K[J(a_1)] \cap Z(K[S])$, $\beta_1 \in K[Reg(S)]$ and $\beta_1' \in K[V]$. But $a_1 \notin Z(K[S])K[Reg(S)]$, so $\alpha_1\beta_1' \notin Z(K[S])K[Reg(S)]$ and $\alpha_1\beta_1' \neq 0$. Since $J(a_1) = J_{a_1} \cup I(a_1)$, we know that J_{a_1} is the greatest element of the set $\{J_x : x \in J(a_1)\}$, and whence J_{a_1} is bigger than the greatest element of the set

$$B := \{J_x : x \in supp(u), u \in K[J(a_1)] \cap Z(K[S])\}.$$

In particular, $J_{a_1} \geq J_x$ for any $x \in supp(\alpha_1)$. Now, by Lemma 4.1, $J_{a_1} > J_y$ for any $y \in supp(\alpha_1\beta_1')$. Because $\alpha_1\beta_1' \notin Z(K[S])K[Reg(S)]$, we have $a_2 \in supp(\alpha_1\beta_1')$ such that $a_2 \notin Z(K[S])K[Reg(S)]$. By the forgoing proof, $J_{a_1} > J_{a_2}$.

By applying the same arguments to a_2, we have a_3 such that $a_3 \notin Z(K[S])K[Reg(S)]$ and $J_{a_2} > J_{a_3}$. Continuing this process, we may obtain the $\mathcal{J}$-classes:

$$J_{a_1} > J_{a_2} > \cdots > J_{a_n} > \cdots,$$

contrary to the hypothesis that S satisfies the condition min_J. Consequently, $K[S] = Z(K[S])K[Reg(S)]$. □

We now arrive at the main result of this section.

Theorem 4.6 *Let S be an almost idempotent-free semigroup satisfying min_J. If, in addition, S satisfies the regularity condition, then $K[S]$ is an Azumaya algebra if and only if $K[S] = Z(K[S])K[Reg(S)]$ and $K[Reg(S)]$ is an Azumaya algebra.*

Proof. Assume that $K[S]$ is an Azumaya algebra. By Lemma 4.5, we need only to verify that $K[Reg(S)]$ is an Azumaya algebra. In fact, by Lemma 3.1 (2), V is an ideal of S, so by Lemma 3.1 (2), $K_0[Reg(S)] \cong K_0[S/V] \cong K[S]/K[V]$ and further is an Azumaya algebra by Lemma 2.3. However by Lemma 2.6, $K[Reg(S)]$ is an Azumaya algebra.

Conversely, suppose that $K[S] = Z(K[S])K[Reg(S)]$ and $K[Reg(S)]$ is an Azumaya algebra. Since $K[S] = Z(K[S])K[Reg(S)]$, we know that the center $Z(K[S])$ of $K[S]$ is equal to

$$\{x \in K[S] : xu = ux \text{ for all } u \in K[Reg(S)]\}.$$

This means that the center $Z(K[Reg(S)])$ of $K[Reg(S)]$ is contained in $Z(K[S])$. By Lemma 2.4, we observe that $Z(K[S]) \otimes_{Z(K[Reg(S)])} K[Reg(S)]$ is an Azumaya algebra. It is easy to see that $K[S]$ is a homomorphic image of $Z(K[S]) \otimes_{Z(K[Reg(S)])} K[Reg(S)]$ since $K[S] = Z(K[S])K[Reg(S)]$. Now by Lemma 2.3 (2), $K[S]$ is an Azumaya algebra. □

It is well known that finite semigroups satisfy the condition min_J. Obviously, any infinite group satisfies the condition min_J. So, not all semigroups satisfying the condition min_J are finite. The next example illustrates that there exist Azumaya semigroup algebras of almost idempotent-free semigroups satisfying the condition min_J but not finite.

Example 4.7 Let U be a null semigroup with zero 0, and G a group with identity 1 and such that $K[G]$ is an Azumaya algebra. Assume that S is the disjoint union of U and G. Define a multiplication $*$ by:

$$x * y = \begin{cases} x & \text{if } y \in G \text{ but } x \in U \\ y & \text{if } x \in G \text{ but } y \in U \\ xy & \text{otherwise,} \end{cases}$$

where xy is the product of x and y in the semigroup U or the group G. By computation, $(S, *)$ is a monoid with identity 1. It is easy to check that the Green's relation $\mathcal{J}$ on S is equal to $(G \times G) \cup \Delta_U$ where Δ_U is the identity relation on U. This shows that in the semigroup S, maximal chains of $\mathcal{J}$-classes of S have the form: $J_0 < J_x < J_g$, where $x \in U\backslash\{0\}, g \in G$. So, S satisfies min_J. On the other hand, it is easy to see that $Z(K[S]) = K[U] + Z(K[G])$ and whence $K[S] = Z(K[S])K[G]$ since $1 \in Z(K[S])$. By Theorem 4.6, $K[S]$ is an Azumaya algebra. When U is infinite, S is clear infinite.

The following example illustrates the condition min_J is not necessary for a semigroup algebra $K[S]$ to be an Azumaya algebra.

Example 4.8 Let G be a group such that $K[G]$ is Azumaya. Let U be the ω-chain $\{e_0, e_1, e_2, \cdots\}$ with

$$e_0 > e_1 > e_2 > \cdots$$

(for detail, see [9, Example 4.6, p.144]). The semigroup S constructed as in Example 4.7 has the $\mathcal{J}$-class chain:

$$J_1 > J_{e_0} > J_{e_1} > J_{e_2} > \cdots .$$

This shows that S does not satisfy the condition min_J. Note that U is in the center of S. Thus $Z(K[S]) = Z(K[G]) + K[U]$, so that $K[S] = Z(K[S])K[G]$. It follows that $K[S]$ is a homomorphic image of $K[G] \otimes_{Z(K[G])} Z(K[S])$. But $K[G] \otimes_{Z(K[G])} Z(K[S])$ is Azumaya (by Lemma 2.4). Therefore $K[S]$ is an Azumaya algebra.

The remainder of this section is devoted to semigroup algebras of idempotent-free semigroups.

Lemma 4.9 *Let S be an idempotent-free semigroup. If $K[S]$ is an Azumaya algebra, then S is a monoid satisfying the regularity condition and $Reg(S)\backslash\{0\}$ is a subgroup of S. Moreover, if S has no zero elements, then S is a group.*

Proof. Because S is an idempotent-free semigroup and by Lemma 3.1 (3), S/V is a completely 0-simple semigroup without idempotents except possibly identity and zero element. It follows that S/V is a 0-group (that is, a group adjoining a zero). This shows that $Reg(S)\backslash\{0\}$ is a subgroup of S, and whence S satisfies the regularity condition. Now, by the proof of Theorem 3.3, the identity of $K[S]$ coincides with the one of $K[Reg(S)]$, so that S is a monoid. The rest is trivial. □

By Lemma 4.9 and Theorem 4.6, the following theorem is immediate. This theorem answers particularly Conjecture 1.2.

Theorem 4.10 *Let S be an idempotent-free monoid with the group of units U. If S satisfies the condition min_J, then $K[S]$ is an Azumaya algebra if and only if $K[S] = Z(K[S])K[U]$ and $K[U]$ is an Azumaya algebra.*

It is clear to see that cancellative monoids are idempotent-free semigroups. By Theorem 4.4, we have the following corollary.

Corollary 4.11 *Let S be a cancellative monoid satisfying the condition min_J. If $K[S]$ is an Azumaya algebra, then S is a group.*

Proof. By Theorem 4.4, S is isomorphic to some Rees matrix semigroup over a group, and further a regular semigroup. But any regular cancellative monoid is a group, now S is a group. □

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